CHEN–RUAN COHOMOLOGY OF SOME MODULI SPACES, II

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Abstract. Let X be a compact connected Riemann surface of genus at least two. Let r be a prime number and $\xi \longrightarrow X$ a holomorphic line bundle such that r is not a divisor of degree(ξ). Let $\mathcal{M}_\xi(r)$ denote the moduli space of stable vector bundles over X of rank r and determinant ξ. By Γ we will denote the group of line bundles L over X such that $L^{\otimes r}$ is trivial. This group Γ acts on $\mathcal{M}_\xi(r)$ by the rule $(E\, , L) \longmapsto E \bigotimes L$. We compute the Chen–Ruan cohomology of the corresponding orbifold.

1. Introduction

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. Let $\mathcal{M}_\eta(2)$ denote the moduli space of stable vector bundles E over X of rank two with $\bigwedge^2 E = \eta$, where $\eta \longrightarrow X$ is a fixed holomorphic line bundle of degree one. This $\mathcal{M}_\eta(2)$ is a smooth complex projective variety of dimension $3g-3$. Let $\Gamma_2 = \text{Pic}^0(X)_2$ be the group of holomorphic line bundles L over X such that $L^{\otimes 2}$ is trivial. Such a line bundle L defines an involution of $\mathcal{M}_\eta(2)$ by sending any E to $E \bigotimes L$. This defines a holomorphic action of Γ_2 on $\mathcal{M}_\eta(2)$. In [3], we computed the Chen–Ruan cohomology algebra of the corresponding orbifold $\mathcal{M}_\eta(2)/\Gamma_2$. (See [4], [5], [11], [1] for Chen–Ruan cohomology.) We note that $\mathcal{M}_\eta(2)/\Gamma_2$ is the moduli space of topologically nontrivial stable $\text{PSL}(2, \mathbb{C})$–bundles over X.

Our aim here is to extend the above result to the moduli spaces of vector bundles of rank r over X, where r is any prime number.

Fix a holomorphic line bundle $\xi \longrightarrow X$ of degree d such that d is not a multiple of the fixed prime number r. Let $\mathcal{M}_\xi(r)$ denote the moduli space of stable vector bundles $E \longrightarrow X$ of rank r with $\bigwedge^r E = \xi$. This moduli space $\mathcal{M}_\xi(r)$ is an irreducible smooth complex projective variety of dimension $(r^2-1)(g-1)$. Let Γ denote the group of holomorphic line bundles L over X such that $L^{\otimes r}$ is trivial. This group Γ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{\oplus 2g}$. Any line bundle $L \in \Gamma$ defines a holomorphic automorphism of $\mathcal{M}_\xi(r)$ by sending any E to $E \bigotimes L$. These automorphisms together define a holomorphic action of Γ on $\mathcal{M}_\xi(r)$.

The corresponding orbifold $\mathcal{M}_\xi(r)/\Gamma$ is the moduli space of stable $\text{PSL}(r, \mathbb{C})$–bundles F over X such that the second Stiefel–Whitney class

$$w_2(F) \in H^2(X, \mathbb{Z}/r\mathbb{Z}) = \mathbb{Z}/r\mathbb{Z}$$

coincides with the image of d.

We compute the Chen–Ruan cohomology algebra of the orbifold $\mathcal{M}_\xi(r)/\Gamma$.

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We note that the results of Section 2 and Section 3 are proved for all integers r. From Section 4 onwards we assume that r is a prime number.

2. Group action and cohomology of fixed point sets

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. Fix a holomorphic line bundle

$$\xi \longrightarrow X\,. \tag{2.1}$$

Let $d \in \mathbb{Z}$ be the degree of ξ. Fix an integer $r \geq 2$ which is coprime to d. Let $\mathcal{M}_\xi(r)$ denote the moduli space of stable vector bundles $E \longrightarrow X$ of rank r and $\det E := \bigwedge^r E = \xi$. This moduli space $\mathcal{M}_\xi(r)$ is an irreducible smooth complex projective variety of dimension $(r^2-1)(g-1)$. The group of all holomorphic automorphisms of $\mathcal{M}_\xi(r)$ will be denoted by $\mathrm{Aut}(\mathcal{M}_\xi(r))$; it is known to be a finite group.

Define

$$\Gamma := \{L \in \mathrm{Pic}^0(X) \mid L^{\otimes r} = \mathcal{O}_X\}\,. \tag{2.2}$$

It is a group under the operation of tensor product of line bundles. The order of Γ is r^{2g}. For any $L \in \Gamma$, let

$$\phi_L \in \mathrm{Aut}(\mathcal{M}_\xi(r)) \tag{2.3}$$

be the automorphism defined by $E \longmapsto E \bigotimes L$. Let

$$\phi : \Gamma \longrightarrow \mathrm{Aut}(\mathcal{M}_\xi(r)) \tag{2.4}$$

be the homomorphism defined by $L \longmapsto \phi_L$. We will describe the fixed point set

$$\mathcal{M}_\xi(r)^L := \mathcal{M}_\xi(r)^{\phi_L} \subset \mathcal{M}_\xi(r) \tag{2.5}$$

of the automorphism ϕ_L.

Take any nontrivial line bundle $L \in \Gamma \setminus \{\mathcal{O}_X\}$. Let ℓ denote the order of L. So ℓ is a divisor of r. Fix a nonzero holomorphic section

$$s : X \longrightarrow L^{\otimes \ell}\,.$$

Define

$$Y_L := \{z \in L \mid z^{\otimes \ell} \in \mathrm{image}(s)\} \subset L\,. \tag{2.6}$$

Let

$$\gamma_L : Y_L \longrightarrow X \tag{2.7}$$

be the restriction of the natural projection $L \longrightarrow X$. Consider the action of the multiplicative group $\mathbb{C}^*$ on the total space of L. The action of the subgroup

$$\mu_\ell := \{c \in \mathbb{C} \mid c^\ell = 1\} \subset \mathbb{C}^* \tag{2.8}$$

preserves the complex curve Y_L defined in Eq. (2.6). In fact, Y_L is a principal μ_ℓ–bundle over X. Since any two nonzero holomorphic sections of $L^{\otimes \ell}$ differ by multiplication with a nonzero constant scalar, the isomorphism class of the principal μ_ℓ–bundle $Y_L \longrightarrow X$ is independent of the choice of the section s. In particular, the isomorphism class of the complex curve Y_L depends only on L.

Since the order of L is exactly ℓ, the curve Y_L is irreducible.

Let $\mathcal{N}^{Y_L}(d)$ denote the moduli space of stable vector bundle bundles $F \longrightarrow Y_L$ of rank r/ℓ and degree d. We have a holomorphic submersion

$$\psi : \mathcal{N}^{Y_L}(d) \longrightarrow \mathrm{Pic}^d(X) \tag{2.9}$$

that sends any F to $\bigwedge^r \gamma_{L*}F$.

Let $\mathrm{Gal}(\gamma_L)$ be the Galois group of the covering γ_L in Eq. (2.7); so $\mathrm{Gal}(\gamma_L) = \mu_\ell$. For any $V \in \mathcal{N}^{Y_L}(d)$, clearly

$$\rho^* V \in \mathcal{N}^{Y_L}(d)$$

for all $\rho \in \mathrm{Gal}(\gamma_L)$. Therefore, $\mathrm{Gal}(\gamma_L)$ acts on $\mathcal{N}^{Y_L}(d)$; the action of $\rho \in \mathrm{Gal}(\gamma_L)$ sends any V to ρ^*V. We also note that $\psi(V) = \psi(\rho^*V)$, where ψ is the projection in Eq. (2.9). Therefore, the action of $\mathrm{Gal}(\gamma_L)$ on $\mathcal{N}^{Y_L}(d)$ preserves the subvariety

$$\psi^{-1}(\xi) \subset \mathcal{N}^{Y_L}(d)\,, \tag{2.10}$$

where ξ is the line bundle in Eq. (2.1).

Lemma 2.1. *Take any nontrivial line bundle $L \in \Gamma$. The fixed point set $\mathcal{M}_\xi(r)^L$ (see Eq. (2.5)) is identified with the quotient variety $\psi^{-1}(\xi)/\mathrm{Gal}(\gamma_L)$ (see Eq. (2.10)). The identification is defined by $F \longmapsto \gamma_{L*}F$.*

Proof. Let $0_X \subset L$ be the image of the zero section of $L \longrightarrow X$. Note that the pullback of L to the complement $L \setminus \{0_X\}$ has a tautological trivialization. Since $Y_L \subset L \setminus \{0_X\}$, the holomorphic line bundle $\gamma_L^* L$ is canonically trivialized. This trivialization

$$\gamma_L^* L = \mathcal{O}_{Y_L} \tag{2.11}$$

defines a holomorphic isomorphism

$$V \longrightarrow V \bigotimes \mathcal{O}_{Y_L} = V \bigotimes \gamma_L^* L \tag{2.12}$$

for any vector bundle $V \longrightarrow Y_L$; the above isomorphism $V \longrightarrow V \bigotimes \mathcal{O}_{Y_L}$ sends any v to $v \bigotimes 1$. Using projection formula, the isomorphism in Eq. (2.12) gives an isomorphism

$$\gamma_{L*}V \longrightarrow \gamma_{L*}(V \bigotimes (\gamma_L^* L)) = (\gamma_{L*}V) \bigotimes L\,. \tag{2.13}$$

Note that

$$\gamma_L^* \gamma_{L*} V = \bigoplus_{\rho \in \mathrm{Gal}(\gamma_L)} \rho^* V\,.$$

Hence $\gamma_L^*\gamma_{L*}V$ is semistable if V is so. This implies that if V is semistable, then the vector bundle $\gamma_{L*}V$ is also semistable. Since $d = \mathrm{degree}(\xi)$ is coprime to r, any semistable vector bundle over X of rank r and degree d is stable. Therefore, $\gamma_{L*}V \in \mathcal{M}_\xi(r)$ for each $V \in \psi^{-1}(\xi)$. In view of the isomorphism in Eq. (2.13) we conclude that

$$\gamma_{L*}V \in \mathcal{M}_\xi(r)^L$$

if $V \in \psi^{-1}(\xi)$.

Since $\gamma_{L*}F = \gamma_{L*}(\rho^*F)$, for all $\rho \in \mathrm{Gal}(\gamma_L)$, we get a morphism

$$\widehat{\gamma} : \psi^{-1}(\xi)/\mathrm{Gal}(\gamma_L) \longrightarrow \mathcal{M}_\xi(r)^L \tag{2.14}$$

defined by $V \longmapsto \gamma_{L*}V$.

To construct the inverse of the map $\widehat{\gamma}$, take any $E \in \mathcal{M}_\xi(r)^L$. Fix an isomorphism

$$\theta' : E \longrightarrow E \bigotimes L\,. \tag{2.15}$$

Since E is stable, it follows that E is simple; this means that all automorphisms of E are constant scalar multiplications. Therefore, any two isomorphisms between E and $E \bigotimes L$ differ by multiplication with a constant scalar. Let

$$\theta \in H^0(X,\, End(E) \bigotimes L) \tag{2.16}$$

be the section defined by θ' in Eq. (2.15).

Consider the pullback

$$\gamma_L^*\theta \in H^0(Y_L,\, \gamma_L^* End(E) \bigotimes \gamma_L^* L)$$

of the section in Eq. (2.16). Using the canonical trivialization of the line bundle $\gamma_L^* L$ (see Eq. (2.11)), this section $\gamma_L^*\theta$ defines a holomorphic section

$$\theta_0 \in H^0(Y_L,\, \gamma_L^* End(E)) = H^0(Y_L,\, End(\gamma_L^* E))\,. \tag{2.17}$$

Since Y_L is irreducible (this was noted earlier) it does not admit any nonconstant holomorphic functions, hence the characteristic polynomial of $\theta_0(x)$ is independent of the point $x \in Y_L$. Therefore, the set of eigenvalues of $\theta_0(x)$ does not change as x moves over Y_L. Similarly, the multiplicity of each eigenvalue of $\theta_0(x)$ is also independent of $x \in Y_L$. Therefore, for each eigenvalue λ of $\theta_0(x)$, we have the associated generalized eigenbundle

$$\gamma_L^* E \supset E^\lambda \longrightarrow Y_L \tag{2.18}$$

whose fiber over any $y \in Y_L$ is the generalized eigenspace of $\theta_0(y) \in \mathrm{End}((\gamma_L^* E)_y)$ for the eigenvalue λ.

Recall that Y_L was constructed by fixing a section s of $L^{\otimes \ell}$ (see Eq. (2.6)); it was also noted that the isomorphism class of the covering γ_L is independent of the choice of s. We choose s such that the ℓ–fold composition

$$(\theta')^\ell := \overbrace{\theta' \circ \cdots \circ \theta'}^{\ell\text{-times}} : E \longrightarrow E \bigotimes L^{\otimes \ell} \tag{2.19}$$

coincides with $\mathrm{Id}_E \bigotimes s$, where θ' is the homomorphism in Eq. (2.15). Since the vector bundle E is simple, there is exactly one such section s. In fact,

$$s = \mathrm{trace}((\theta')^\ell)/r\,.$$

We construct Y_L using this s.

With this construction of Y_L, we have

$$(\theta_0)^\ell := \overbrace{\theta_0 \circ \cdots \circ \theta_0}^{\ell\text{-times}} = \mathrm{Id}_{\gamma_L^* E}\,,$$

where θ_0 is constructed in Eq. (2.17). Consequently, the set of eigenvalues of $\theta_0(x)$ is contained in μ_ℓ (defined in Eq. (2.8)); we noted earlier that the set of eigenvalues of $\theta_0(x)$ along with their multiplicities are independent of $x \in Y_L$.

Since Y_L is a principal μ_ℓ–bundle over X, the Galois group $\mathrm{Gal}(\gamma_L)$ is identified with μ_ℓ. Note that $\mathrm{Gal}(\gamma_L)$ has a natural action on the vector bundle $\gamma_L^* E$ which is a lift of the action of $\text{Gal}(\gamma_L)$ on Y_L. Examining the construction of θ_0 (see Eq. (2.17)) from θ', it follows that the action of any

$$\rho \in \text{Gal}(\gamma_L) = \mu_\ell$$

on $\gamma_L^* E$ takes the eigenbundle E^λ (see Eq. (2.18)) to the eigenbundle $E^{\lambda\cdot\rho}$. This immediately implies that each element of μ_ℓ is an eigenvalue of $\theta_0(x)$, and the multiplicities of the eigenvalues of $\theta_0(x)$ coincide. Hence, the multiplicity of each eigenvalue of $\theta_0(x)$ is r/ℓ.

Consider

$$E^1 \longrightarrow Y_L\,, \tag{2.20}$$

which is the eigenbundle for the eigenvalue $1 \in \mu_\ell$. Define

$$\widetilde{E}^1 := \bigoplus_{\rho\in\text{Gal}(\gamma_L)} \rho^* E^1\,.$$

There is a natural action of $\text{Gal}(\gamma_L) = \mu_\ell$ on $\widetilde{E}^1$. Since the action of any $\rho \in \mu_\ell$ on $\gamma_L^* E$ takes the eigenbundle E^λ to the eigenbundle $E^{\lambda\cdot\rho}$, it follows immediately that we have a $\text{Gal}(\gamma_L)$–equivariant identification

$$\gamma_L^* E = \widetilde{E}^1 := \bigoplus_{\rho\in\text{Gal}(\gamma_L)} \rho^* E^1\,. \tag{2.21}$$

In view of this $\text{Gal}(\gamma_L)$–equivariant isomorphism we conclude that the composition

$$E \longrightarrow \gamma_{L*}\gamma_L^* E \stackrel{\sim}{\longrightarrow} \gamma_{L*}\widetilde{E}^1 = \gamma_{L*}(\bigoplus_{\rho\in\text{Gal}(\gamma_L)} \rho^* E^1) \longrightarrow \gamma_{L*}E^1$$

is an isomorphism; here

$$E \longrightarrow \gamma_{L*}\gamma_L^* E$$

is the natural homomorphism and $\bigoplus_{\rho\in\text{Gal}(\gamma_L)} \rho^* E^1 \longrightarrow E^1$ is the projection to the direct summand corresponding to $\rho = 1$.

Since E is stable, and $\gamma_{L*}E^1 = E$, it follows that E^1 is stable. Indeed, if a subbundle $F \subset E^1$ violates the stability condition for E^1, then the subbundle

$$\gamma_{L*}F \subset \gamma_{L*}E^1 = E$$

violates the stability for E. Let

$$\Phi : \mathcal{M}_\xi(r)^L \longrightarrow \psi^{-1}(\xi)/\text{Gal}(\gamma_L) \tag{2.22}$$

be the morphism that sends any E to E^1. Since E is isomorphic to $\gamma_{L*}E^1$, it follows that

$$\widehat{\gamma}\circ\Phi = \text{Id}_{\mathcal{M}_\xi(r)^L}\,,$$

where $\widehat{\gamma}$ is constructed in Eq. (2.14).

Therefore, to complete the proof of the lemma it suffices to show that for $F, F' \in \psi^{-1}(\xi)$, if

$$\gamma_{L*}F = \gamma_{L*}F'\,, \tag{2.23}$$

then

$$F' = \tau^* F \tag{2.24}$$

holds for some $\tau \in \text{Gal}(\gamma_L)$.

Take $F\,, F' \in \psi^{-1}(\xi)$ such that Eq. (2.23) holds. Note that

$$\bigoplus_{\tau,\eta\in \mathrm{Gal}(\gamma_L)} Hom(\eta^*F\,,\tau^*F') = \bigoplus_{\tau\in \mathrm{Gal}(\gamma_L)} Hom(F\,,\tau^*F')^{\oplus \ell} = Hom(\gamma_L^*\gamma_{L*}F\,,\gamma_L^*\gamma_{L*}F')\,. \tag{2.25}$$

Since $\gamma_{L*}F = \gamma_{L*}F'$,

$$H^0(X,\, Hom(\gamma_{L*}F\,,\gamma_{L*}F')) \neq 0\,.$$

Consequently,

$$H^0(Y_L,\, Hom(\gamma_L^*\gamma_{L*}F\,,\gamma_L^*\gamma_{L*}F')) \neq 0\,.$$

Therefore, from Eq. (2.25) we conclude that there is some $\tau \in \mathrm{Gal}(\gamma_L)$ such that

$$H^0(Y_L,\, Hom(F\,,\tau^*F')) \neq 0\,. \tag{2.26}$$

Since both F and τ^*F' are stable vector bundles with

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} = \frac{\mathrm{degree}(\tau^*F')}{\mathrm{rank}(\tau^*F')}\,,$$

from Eq. (2.26) we conclude that the vector bundle F is isomorphic to τ^*F'. In other words, Eq. (2.24) holds. This completes the proof of the lemma. □

In Lemma 3.5 we will show that the action of $\mathrm{Gal}(\gamma_L)$ on $\psi^{-1}(\xi)$ is free.

In the next section we will investigate the action of the isotropy subgroups on the tangent bundle for the action of Γ of $\mathcal{M}_\xi(r)$.

3. Action on the tangent bundle

Fix any $L \in \Gamma \setminus \{\mathcal{O}_X\}$. Take any

$$E \in \mathcal{M}_\xi(r)^L$$

(see Eq. (2.5)). Fix an isomorphism θ' as in Eq. (2.15). Let s be the unique section of $L^{\otimes \ell}$ such that the homomorphism $(\theta')^\ell$ in Eq. (2.19) coincides with $\mathrm{Id}_E \bigotimes s$. As we noted earlier, $s = \mathrm{trace}((\theta')^\ell)/r$. Let

$$\gamma_L\,:\, Y_L \longrightarrow X$$

be the covering Y_L constructed as in Eq. (2.6) using this section s. Recall that $\mathrm{Gal}(\gamma_L) = \mu_\ell$, where ℓ is the order of L.

Let

$$F := E^1 \longrightarrow Y_L \tag{3.1}$$

be the vector bundle constructed in Eq. (2.20). The decomposition of γ_L^*E in Eq. (2.21) yields the following decomposition of the pullback $\gamma_L^*End(E) = End(\gamma_L^*E)$:

$$\gamma_L^*End(E) = \bigoplus_{t\in \mathrm{Gal}(\gamma_L)}\bigoplus_{u\in \mathrm{Gal}(\gamma_L)} Hom(u^*F\,,(ut)^*F) = \bigoplus_{t\in \mathrm{Gal}(\gamma_L)}\bigoplus_{u\in \mathrm{Gal}(\gamma_L)} (u^*F)^\vee \bigotimes (ut)^*F \tag{3.2}$$

(see also Eq. (2.25)).

Note that for each $t \in \mathrm{Gal}(\gamma_L)$, the vector bundle

$$\mathcal{E}_t := \bigoplus_{u\in \mathrm{Gal}(\gamma_L)} Hom(u^*F\,,(ut)^*F) \longrightarrow Y_L \tag{3.3}$$

in Eq. (3.2) is left invariant by the natural action of $\text{Gal}(\gamma_L)$ on the vector bundle $\gamma_L^* End(E)$. Therefore, $\mathcal{E}_t$ descends to X. Let

$$\mathcal{F}_t \longrightarrow X \tag{3.4}$$

be the descent of $\mathcal{E}_t$. So

$$\mathcal{F}_t = \gamma_{L*} Hom(F\,, t^*F)\,, \tag{3.5}$$

and

$$\gamma_L^* \mathcal{F}_t = \mathcal{E}_t\,.$$

The decomposition

$$\gamma_L^* End(E) = \bigoplus_{t \in \text{Gal}(\gamma_L)} \mathcal{E}_t$$

in Eq. (3.2) is preserved by the action of $\text{Gal}(\gamma_L)$. Therefore, this decomposition descends to the following decomposition:

$$End(E) = \bigoplus_{t \in \text{Gal}(\gamma_L)} \mathcal{F}_t\,. \tag{3.6}$$

We will now describe the differential $d\phi_L(E)$, where ϕ_L is the automorphism in Eq. (2.3), and $E \in \mathcal{M}_\xi(r)^L$.

Recall that we fixed an isomorphism $\theta' : E \longrightarrow E \bigotimes L$ as in Eq. (2.15). This isomorphism θ' induces an isomorphism of the endomorphism bundle $End(E)$ with $End(E \bigotimes L)$. Since

$$End(E) = E \bigotimes E^\vee = (E \bigotimes L) \bigotimes (E \bigotimes L)^\vee = End(E \bigotimes L)\,,$$

the isomorphism of $End(E)$ with $End(E \bigotimes L)$ defined by θ' gives an automorphism of $End(E)$. Let

$$\widehat{\theta} : End(E) \longrightarrow End(E) \tag{3.7}$$

be this automorphism constructed from θ'. Since any two isomorphisms between E and $E \bigotimes L$ differ by a constant scalar, the automorphism $\widehat{\theta}$ is independent of the choice of θ'.

Let

$$\text{ad}(E) \subset End(E)$$

be the holomorphic subbundle of corank one given by the sheaf of trace zero endomorphisms. Clearly,

$$\widehat{\theta}(\text{ad}(E)) \subset \text{ad}(E)\,.$$

We note that $T_E \mathcal{M}_\xi(r) = H^1(X, \text{ad}(E))$; here T denotes the holomorphic tangent bundle. Let

$$\overline{\theta} : H^1(X, \text{ad}(E)) \longrightarrow H^1(X, \text{ad}(E)) \tag{3.8}$$

be the automorphism induced by $\widehat{\theta}$.

From the construction of ϕ_L it follows that the differential

$$d\phi_L(E) : T_E \mathcal{M}_\xi(r) \longrightarrow T_E \mathcal{M}_\xi(r) \tag{3.9}$$

coincides with $\overline{\theta}$ constructed in Eq. (3.8).

In the proof of Lemma 2.1 we observed that the pullback to Y_L of the isomorphism θ' coincides with the isomorphism

$$\gamma_L^* E \longrightarrow \gamma_L^* E \bigotimes \gamma_L^* L$$

obtained by tensoring with the tautological section of $\gamma_L^* L$ (see Eq. (2.11)). Consider the automorphism $\widehat{\theta}$ in Eq. (3.7) induced by θ'. From the above description of θ' it follows immediately that $\widehat{\theta}$ acts on the subbundle $\mathcal{E}_t$ (see Eq. (3.3)) as multiplication by

$$t \in \mu_\ell \subset \mathbb{C}^* .$$

Lemma 3.1. *Take any*

$$t \in \mu_\ell \setminus \{1\} . \tag{3.10}$$

(ℓ is the order of L; see Eq. (2.8) *for μ_ℓ). Consider $\mathcal{F}_t$ constructed in Eq.* (3.4) *(see also Eq.* (3.6)*). Then*

$$\mathcal{F}_t \subset \mathrm{ad}(E) . \tag{3.11}$$

Proof. We first note that $\dim H^0(X,\, End(E)) = 1$ because E is stable. On the other hand,

$$\dim H^0(X,\, \gamma_{L*} Hom(F\, , \mathrm{Id}^* F)) \geq 1 ,$$

where $\mathrm{Id} = 1 \in \mathrm{Gal}(\gamma_L) = \mu_\ell$ is the identity element. Therefore, from Eq. (3.5) and Eq. (3.6),

$$H^0(X,\, \mathcal{F}_t) = 0 , \tag{3.12}$$

where t is the element in Eq. (3.10).

Remark 3.2. Since the vector bundle E is stable, it admits a unique Hermitian–Einstein connection. The connection on $End(E)$ induced by a Hermitian–Einstein connection on E is also Hermitian–Einstein. Therefore, the vector bundle $End(E)$ is polystable of degree zero. □

Continuing with the proof of Lemma 3.1, since $End(E)$ is polystable of degree zero, and $\mathcal{F}_t$ is a direct summand of $End(E)$ (see Eq. (3.5) and Eq. (3.6)), it follows that $\mathcal{F}_t$ is also polystable of degree zero. Consider the trace map

$$End(E) \supset \mathcal{F}_t \stackrel{\text{trace}}{\longrightarrow} \mathcal{O}_X .$$

Since $\mathcal{F}_t$ is polystable of degree zero, if the above trace homomorphism on $\mathcal{F}_t$ is nonzero, then $\mathcal{O}_X$ is a direct summand of $\mathcal{F}_t$. If $\mathcal{O}_X$ is a direct summand of $\mathcal{F}_t$, then Eq. (3.12) is contradicted. Hence the trace map on $\mathcal{F}_t$ vanishes identically. This implies that Eq. (3.11) holds. This completes the proof of the lemma. □

Consider the tautological trivialization of the line bundle $\gamma_L^* L$ (see Eq. (2.11)). The action of any element $t \in \mathrm{Gal}(\gamma_L) = \mu_\ell$ takes the tautological section of $\gamma_L^* L$ to t^{-1}–times the tautological section. Using this it follows immediately that t acts on $\mathcal{E}_t$ in Eq. (3.3) as multiplication by t. Now from the construction of the automorphism $\widehat{\theta}$ in Eq. (3.7) it follows that $\widehat{\theta}$ acts on $\mathcal{F}_t$ as multiplication by t. In view of this and Eq. (3.11), we conclude that for all $t \in \mu_\ell \setminus \{1\}$, the automorphism $\overline{\theta}$ in Eq. (3.8) acts on the subspace

$$H^1(X,\, \mathcal{F}_t) \subset H^1(X,\, \mathrm{ad}(E)) = T_E \mathcal{M}_\xi(r) \tag{3.13}$$

as multiplication by t.

We will calculate the dimension of the subspace $H^1(X, \mathcal{F}_t)$ in Eq. (3.13), where $t \in \mu_\ell \setminus \{1\}$. Note that

$$\mathrm{rank}(\mathcal{F}_t) = r^2/\ell \quad \text{and} \quad \mathrm{degree}(\mathcal{F}_t) = 0\,.$$

Since $H^0(X, \mathrm{ad}(E)) = 0$, from Eq. (3.11) we have

$$H^0(X, \mathcal{F}_t) = 0\,. \tag{3.14}$$

Therefore, the Riemann–Roch theorem says that

$$\dim H^1(X, \mathcal{F}_t) = r^2(g-1)/\ell\,.$$

Hence, we have proved the following lemma:

Lemma 3.3. *Take any* $t \in \mu_\ell \setminus \{1\}$. *Then* $\mathcal{F}_t \subset \mathrm{ad}(E)$, *and the automorphism* $\overline{\theta}$ *in Eq.* (3.8) *acts on the subspace*

$$H^1(X, \mathcal{F}_t) \subset H^1(X, \mathrm{ad}(E)) = T_E\mathcal{M}_\xi(r)$$

as multiplication by t. *Also,*

$$\dim H^1(X, \mathcal{F}_t) = r^2(g-1)/\ell\,.$$

Now we set $t = 1 \in \mu_\ell = \mathrm{Gal}(\gamma_L)$. The automorphism $\widehat{\theta}$ of $End(E)$ acts trivially on the subbundle $\mathcal{F}_1 \subset End(E)$. Therefore, $\overline{\theta}$ acts trivially on the subspace

$$H^1(X, \mathcal{F}_1) \bigcap H^1(X, \mathrm{ad}(E)) \subset H^1(X, \mathrm{ad}(E)) = T_E\mathcal{M}_\xi(r)\,.$$

From Eq. (3.6) and Eq. (3.11),

$$\mathrm{ad}(E) = (\mathcal{F}_1 \bigcap \mathrm{ad}(E)) \bigoplus (\bigoplus_{\tau\in\mu_\ell\setminus\{1\}} \mathcal{F}_\tau)\,.$$

Consequently, the dimension of the subspace of $H^1(X, \mathrm{ad}(E))$ on which $\overline{\theta}$ in Eq. (3.8) acts as the identity map is

$$\dim H^1(X, \mathrm{ad}(E)) - \sum_{t\in\mu_\ell\setminus\{1\}} \dim H^1(X, \mathcal{F}_t)$$

$$= (r^2-1)(g-1) - \frac{(\ell-1)r^2(g-1)}{\ell} = \frac{r^2(g-1)}{\ell} - g + 1\,.$$

Combining this with Lemma 3.3 we get the following proposition.

Proposition 3.4. *The eigenvalues of the differential*

$$d\phi_L(E) : T_E\mathcal{M}_\xi(r) \longrightarrow T_E\mathcal{M}_\xi(r)$$

are μ_ℓ. *For any* $t \in \mu_\ell \setminus \{1\}$, *the multiplicity of the eigenvalue* t *is* $r^2(g-1)/\ell$. *The multiplicity of the eigenvalue* 1 *is* $1 - g + r^2(g-1)/\ell$.

Lemma 3.5. *The action of* $\mathrm{Gal}(\gamma_L)$ *on* $\psi^{-1}(\xi)$ *in Lemma 2.1 is free.*

Proof. In Eq. (3.14) we saw that $H^0(X, \mathcal{F}_t) = 0$ for all $t \in \text{Gal}(\gamma_L) \setminus \{1\}$. Now, from Eq. (3.5) it follows that

$$H^0(X, \mathcal{F}_t) = H^0(Y_L, Hom(F, t^*F)) = 0$$

for all $t \in \text{Gal}(\gamma_L) \setminus \{1\}$. Since $F = E^1$ (see Eq. (3.1)), this implies that for each $t \in \text{Gal}(\gamma_L) \setminus \{1\}$, the vector bundle E^1 is not isomorphic to t^*E^1. This completes the proof of the lemma. □

4. Intersection of fixed point sets

Henceforth, we will always assume that r is a prime number. As before, $d = \text{degree}(\xi)$ is assumed to be coprime to r.

We note that the group Γ (see Eq. (2.2)) is a vector space over the field $\mathbb{Z}/r\mathbb{Z}$. For any $J, L \in \Gamma$, and any $n \in \mathbb{Z}/r\mathbb{Z}$,

$$J + L := J \bigotimes L \quad \text{and} \quad nL := L^{\otimes n}.$$

For a line bundle L_0 over X, by $L_0^{\otimes 0}$ we will denote the trivial line bundle $\mathcal{O}_X$.

Take two linearly independent elements

$$J, L \in \Gamma. \tag{4.1}$$

We note that the line bundles $J^{\otimes i} \bigotimes L^{\otimes j}$, $i, j \in [1, r-1]$, are all distinct and nontrivial.

Take any

$$E \in \mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L$$

(see Eq. (2.5)), where J and L are, as in Eq. (4.1), linearly independent. Fix isomorphisms

$$\theta_1 : E \longrightarrow E \bigotimes J \quad \text{and} \quad \theta_2 : E \longrightarrow E \bigotimes L. \tag{4.2}$$

The isomorphism θ_1 (respectively, θ_2) gives an inclusion of the line bundle $J^\vee = J^{\otimes(r-1)}$ (respectively, $L^\vee = L^{\otimes(r-1)}$) in $End(E) = E \bigotimes E^\vee$. Note that since r is a prime number, any $J^{\otimes i}$ (respectively, $L^{\otimes i}$) is a tensor power of $J^\vee$ (respectively, $L^\vee$). Using the associative algebra structure of the fibers of $End(E)$ defined by composition of homomorphisms, the above homomorphisms

$$J^\vee \longrightarrow End(E) \quad \text{and } L^\vee \longrightarrow End(E)$$

give an inclusion of $(J^\vee)^{\otimes i} \bigotimes (L^\vee)^{\otimes j}$ in $End(E)$ for all $i, j \geq 0$. Therefore, we get an inclusion of $J^{\otimes i} \bigotimes L^{\otimes j}$ in $End(E)$ for all $i, j \geq 0$. The line bundle $J^{\otimes 0} \bigotimes L^{\otimes 0} = \mathcal{O}_X$ sits inside $End(E)$ as scalar multiplications.

Let

$$\Theta : \bigoplus_{i,j=0}^{r-1} J^{\otimes i} \bigotimes L^{\otimes j} \longrightarrow End(E) \tag{4.3}$$

be the homomorphism constructed as above.

Lemma 4.1. *The homomorphism Θ in Eq. (4.3) is a holomorphic isomorphism of vector bundles.*

Proof. Take any proper subset

$$S \subset [0, r-1] \times [0, r-1],$$

and also take any $(i_0, j_0) \in [0, r-1] \times [0, r-1] \setminus S$ in the complement. Assume that the restriction of the homomorphism Θ in Eq. (4.3) to

$$W_S := \bigoplus_{(i,j)\in S} J^{\otimes i} \bigotimes L^{\otimes j}$$

is injective. Note that W_S is polystable of degree zero, and in Remark 3.2 we observed that the vector bundle $End(E)$ is polystable. Since both W_S and $End(E)$ are polystable vector bundles of degree zero, it follows that $\Theta(W_S)$ is a subbundle of $End(E)$. Furthermore, there is a holomorphic subbundle

$$W'_S \subset End(E)$$

which is a direct summand of $\Theta(W_S)$. In particular,

$$End(E) = \Theta(W_S) \bigoplus W'_S. \tag{4.4}$$

We have

$$H^0(X, Hom(J^{\otimes i_0} \bigotimes L^{\otimes j_0}, W_S)) = 0$$

because $J^{\otimes i_0} \bigotimes L^{\otimes j_0}$ is distinct from $J^{\otimes i} \bigotimes L^{\otimes j}$ for all $(i, j) \in S$. Therefore, the projection of $J^{\otimes i_0} \bigotimes L^{\otimes j_0}$ to the direct summand $\Theta(W_S) \subset End(E)$ in Eq. (4.4) vanishes identically. This implies that

$$\Theta(J^{\otimes i_0} \bigotimes L^{\otimes j_0}) \subset W'_S.$$

Hence Θ makes $W_S \bigoplus (J^{\otimes i_0} \bigotimes L^{\otimes j_0})$ a subbundle of $End(E)$.

Now, using induction we conclude that Θ in Eq. (4.3) is a pointwise injective homomorphism of vector bundles. Since

$$r^2 = \operatorname{rank}(End(E)) = \operatorname{rank}(\bigoplus_{i,j=0}^{r-1} J^{\otimes i} \bigotimes L^{\otimes j}),$$

it follows that Θ is an isomorphism of vector bundles. □

Take two vector bundles $E, F \in \mathcal{M}_\xi(r)$. If the vector bundle $End(E)$ is isomorphic to $End(F)$, then at least one of the following two statements is valid:

(1) There is a line bundle $\zeta \longrightarrow X$ such that

$$E = F \bigotimes \zeta. \tag{4.5}$$

(2) There is a line bundle $\zeta \longrightarrow X$ such that

$$E = F^\vee \bigotimes \zeta. \tag{4.6}$$

Since $\bigwedge^r E = \bigwedge^r F$, taking the r–th exterior power of both sides of Eq. (4.5) it follows that $\zeta^{\otimes r} = \mathcal{O}_X$.

Similarly, since $\bigwedge^r E = \bigwedge^r F = \xi$, taking the r–th exterior power of both sides of Eq. (4.6) it follows that $\zeta^{\otimes r} = \xi^{\otimes 2}$. Recall that r is a prime number, and degree(ξ) is not a multiple of r. Hence, if Eq. (4.6) holds, then $r = 2$.

Therefore, Lemma 4.1 has the following corollary:

Corollary 4.2. *Take two linearly independent elements* $J\,,L\,\in\,\Gamma$*. If*

$$E\,,F\,\in\,\mathcal{M}_\xi(r)^J\bigcap\mathcal{M}_\xi(r)^L\,,$$

then at least one of the following two statements is valid:

(1) *There is a line bundle* $\zeta\,\in\,\Gamma$ *such that* $F\bigotimes\zeta$ *is holomorphically isomorphic to* E*.*
(2) *There is a line bundle* $\zeta\,\in\,\Gamma$ *such that* $F^\vee\bigotimes\zeta$ *is holomorphically isomorphic to* E*.*

If the second statement holds, then $r\,=\,2$*.*

Remark 4.3. Take a line bundle $L\,\in\,\Gamma$, and also take a vector bundle $E\,\in\,\mathcal{M}_\xi(r)^L$. Let

$$f_0\,:\,E\,\longrightarrow\,E\bigotimes L$$

be a holomorphic isomorphism.

- Take any $\zeta\,\in\,\Gamma$. Then

$$f_0\otimes\mathrm{Id}_\zeta\,:\,E\bigotimes\zeta\,\longrightarrow\,E\bigotimes L\bigotimes\zeta\,=\,E\bigotimes\zeta\bigotimes L$$

is also an isomorphism. Hence $E\bigotimes\zeta\,\in\,\mathcal{M}_\xi(r)^L$. Therefore, if

$$E\,\in\,\mathcal{M}_\xi(r)^J\bigcap\mathcal{M}_\xi(r)^L\,,$$

and $\zeta\,\in\,\Gamma$, then

$$E\bigotimes\zeta\,\in\,\mathcal{M}_\xi(r)^J\bigcap\mathcal{M}_\xi(r)^L\,.$$

- Consider the isomorphism

$$f_0^\vee\,:\,E^\vee\bigotimes L^\vee\,=\,(E\bigotimes L)^\vee\,\longrightarrow\,E^\vee\,. \tag{4.7}$$

For any holomorphic line bundle $\eta\,\longrightarrow\,X$, tensoring the isomorphism in Eq. (4.7) by

$$\mathrm{Id}_{L\otimes\eta}\,:\,L\bigotimes\eta\,\longrightarrow\,L\bigotimes\eta$$

we get an isomorphism

$$E^\vee\bigotimes\eta\,\longrightarrow\,E^\vee\bigotimes L\bigotimes\eta\,=\,E^\vee\bigotimes\eta\bigotimes L\,.$$

Therefore, if $\bigwedge^r(E^\vee\bigotimes\eta)\,=\,\xi$, then $E^\vee\bigotimes\eta\,\in\,\mathcal{M}_\xi(r)^L$. Now using the first part of the remark we conclude the following: if

$$E\,\in\,\mathcal{M}_\xi(r)^J\bigcap\mathcal{M}_\xi(r)^L\,,$$

and $\bigwedge^r(E^\vee\bigotimes\eta)\,=\,\xi$, then

$$E^\vee\bigotimes\eta\bigotimes\zeta\,\in\,\mathcal{M}_\xi(r)^J\bigcap\mathcal{M}_\xi(r)^L$$

for all $\zeta\,\in\,\Gamma$.

□

Remark 4.4. Assume that $r = 2$. Take any $E \in \mathcal{M}_\xi(2)$. Since $\bigwedge^2 E = \xi$, contracting both sides by $E^\vee$ it follows that $E = E^\vee \bigotimes \xi$. $\square$

As before, take two linearly independent elements $J, L \in \Gamma$. Consider the coverings

$$\gamma_J : Y_J \longrightarrow X \quad \text{and} \quad \gamma_L : Y_L \longrightarrow X$$

constructed as in Eq. (2.7) from J and L respectively. Since the Galois groups of both γ_J and γ_L are $\mathbb{Z}/r\mathbb{Z}$, we get surjective homomorphisms

$$\rho_J : H_1(X, \mathbb{Z}) \longrightarrow \mathbb{Z}/r\mathbb{Z} \tag{4.8}$$

and

$$\rho_L : H_1(X, \mathbb{Z}) \longrightarrow \mathbb{Z}/r\mathbb{Z}\,. \tag{4.9}$$

Using ρ_J and ρ_L, we will construct a homomorphism from $H_1(X, \mathbb{Z})$ to $\mathrm{PGL}(r, \mathbb{C})$.

Let D be the $r \times r$ diagonal matrix whose (j, j)–th entry is $\exp(2\pi\sqrt{-1}j/r)$. Let

$$\rho'_1 : \mathbb{Z}/r\mathbb{Z} \longrightarrow \mathrm{PGL}(r, \mathbb{C}) \tag{4.10}$$

be the homomorphism that sends any τ to the image of D^τ in $\mathrm{PGL}(r, \mathbb{C})$.

Let

$$R \in \mathrm{GL}(r, \mathbb{C})$$

be the matrix defined by $R(e_i) = e_{i+1}$ for all $i \in [1, r-1]$ and $R(e_r) = e_1$, where $\{e_j\}_{j=1}^r$ is the standard basis of $\mathbb{C}^r$. Let

$$\rho'_2 : \mathbb{Z}/r\mathbb{Z} \longrightarrow \mathrm{PGL}(r, \mathbb{C}) \tag{4.11}$$

be the homomorphism that sends any τ to the image of R^τ in $\mathrm{PGL}(r, \mathbb{C})$. Note that the image of R, in $\mathrm{PGL}(r, \mathbb{C})$, commutes with the image of the above defined diagonal matrix D. Consequently, the images of the two homomorphisms ρ'_1 and ρ'_2 commute.

Let

$$\Psi : H_1(X, \mathbb{Z}) \longrightarrow \mathrm{PGL}(r, \mathbb{C}) \tag{4.12}$$

be the homomorphism defined by $\gamma \longmapsto \rho'_1(\rho_J(\gamma))\rho'_2(\rho_L(\gamma))$, where ρ_J (respectively, ρ_L) is defined in Eq. (4.8) (respectively, Eq. (4.9)), and ρ'_1 (respectively, ρ'_2) is defined in Eq. (4.10) (respectively, Eq. (4.11)). Since the images of ρ'_1 and ρ'_2 commute, the map Ψ is indeed a homomorphism.

The group $H_1(X, \mathbb{Z})$ is a quotient of the fundamental group of X, and $\mathrm{PGL}(r, \mathbb{C})$ has the standard action on $\mathbb{CP}^{r-1}$. Hence any homomorphism from $H_1(X, \mathbb{Z})$ to $\mathrm{PGL}(r, \mathbb{C})$ defines a flat projective bundle over X of relative dimension $r - 1$. Let

$$\mathcal{P}_{J,L} \longrightarrow X \tag{4.13}$$

be the flat projective bundle given by the homomorphism Ψ in Eq. (4.12).

Let $\underline{\mathrm{GL}(r, \mathbb{C})}$ (respectively, $\underline{\mathrm{PGL}(r, \mathbb{C})}$) be the sheaf of locally constant functions on X with values in $\mathrm{GL}(r, \mathbb{C})$ (respectively, $\mathrm{PGL}(r, \mathbb{C})$). We have the short exact sequence of sheaves

$$e \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow \underline{\mathrm{GL}(r, \mathbb{C})} \longrightarrow \underline{\mathrm{PGL}(r, \mathbb{C})} \longrightarrow e \tag{4.14}$$

on X, where $\underline{\mathbb{Z}/r\mathbb{Z}}$ is the sheaf of locally constant functions with values in $\mathbb{Z}/r\mathbb{Z}$; for notational convenience, we will denote the sheaf $\underline{\mathbb{Z}/r\mathbb{Z}}$ by $\mathbb{Z}/r\mathbb{Z}$. Let

$$\chi \,:\, H^1(X,\, \underline{\mathrm{PGL}(r,\mathbb{C})}) \longrightarrow H^2(X,\, \mathbb{Z}/r\mathbb{Z}) \,=\, \mathbb{Z}/r\mathbb{Z} \tag{4.15}$$

be the homomorphism in the exact sequence of cohomologies associated to the short exact sequence in Eq. (4.14). The projective bundle $\mathcal{P}_{J,L}$ in Eq. (4.13) defines an element

$$c(\mathcal{P}_{J,L}) \,\in\, H^1(X,\, \underline{\mathrm{PGL}(r,\mathbb{C})})\,.$$

Let

$$A_{J,L} \,:=\, \chi(c(\mathcal{P}_{J,L})) \,\in\, H^2(X,\, \mathbb{Z}/r\mathbb{Z}) \,=\, \mathbb{Z}/r\mathbb{Z} \tag{4.16}$$

be the cohomology class, where χ is the homomorphism in Eq. (4.15).

A homomorphism $H_1(X,\, \mathbb{Z}) \longrightarrow \mathbb{Z}/r\mathbb{Z}$ defines a cohomology class in $H^1(X,\, \mathbb{Z}/r\mathbb{Z})$. Let

$$\overline{\rho}_J\,, \overline{\rho}_L \,\in\, H^1(X,\, \mathbb{Z}/r\mathbb{Z}) \tag{4.17}$$

be the cohomology classes corresponding to the homomorphisms ρ_J and ρ_L constructed in Eq. (4.8) and Eq. (4.9) respectively. Let

$$\overline{\rho}_J \cup \overline{\rho}_L \,\in\, H^2(X,\, \mathbb{Z}/r\mathbb{Z})$$

be the cup product. It can be checked that

$$A_{J,L} \,=\, \overline{\rho}_J \cup \overline{\rho}_L \,\in\, H^2(X,\, \mathbb{Z}/r\mathbb{Z})\,, \tag{4.18}$$

where $A_{J,L}$ is constructed in Eq. (4.16).

Given any holomorphic projective bundle $\mathbb{P}$ over X, there is a holomorphic vector bundle $\mathcal{V} \longrightarrow X$ such that $\mathbb{P}$ is isomorphic to the projective bundle over X parametrizing the lines in the fibers of $\mathcal{V}$. Let

$$\mathcal{W} \longrightarrow X \tag{4.19}$$

be a holomorphic vector bundle of rank r such that the holomorphic projective bundle over X parametrizing lines in the fibers of $\mathcal{W}$ is holomorphically isomorphic to the projective bundle $\mathcal{P}_{J,L}$ in Eq. (4.13).

Proposition 4.5. *The vector bundle $\mathcal{W}$ in Eq.* (4.19) *is stable.*

The image of $\mathrm{degree}(\mathcal{W}) \,\in\, \mathbb{Z}$ *in $\mathbb{Z}/r\mathbb{Z}$ coincides with $A_{J,L}$ in Eq.* (4.16).

Also,

$$End(\mathcal{W}) \,=\, \bigoplus_{i,j=0}^{r-1} J^{\otimes i} \bigotimes L^{\otimes j}\,. \tag{4.20}$$

In particular, $\mathcal{W}$ is isomorphic to both $\mathcal{W} \bigotimes J$ and $\mathcal{W} \bigotimes L$.

Proof. Consider the homomorphism Ψ in Eq. (4.12). Its image is a finite subgroup of $\mathrm{PGL}(r,\mathbb{C})$, and hence $\Psi(H_1(X,\, \mathbb{Z}))$ lies inside a maximal compact subgroup of $\mathrm{PGL}(r,\mathbb{C})$. Also, the subgroup

$$\Psi(H_1(X,\, \mathbb{Z})) \,\subset\, \mathrm{PGL}(r,\mathbb{C})$$

is irreducible in following sense. Consider the standard action of $\mathrm{PGL}(r, \mathbb{C})$ on the projective space $\mathbb{CP}^{r-1}$ that parametrizes all lines in $\mathbb{C}^r$. The action of the subgroup $\Psi(H_1(X, \mathbb{Z}))$ leaves invariant no proper linear subspace of $\mathbb{CP}^{r-1}$.

Since $\Psi(H_1(X, \mathbb{Z}))$ is an irreducible subgroup of $\mathrm{PGL}(r, \mathbb{C})$ lying inside a maximal compact subgroup, it follows that the principal $\mathrm{PGL}(r, \mathbb{C})$–bundle over X defined by the projective bundle $\mathcal{P}_{J,L}$ (see Eq. (4.13)) is stable [10, p. 146, Theorem 7.1]. Consequently, the corresponding vector bundle $\mathcal{W}$ in Eq. (4.19) is stable.

From the definition of $A_{J,L}$ in Eq. (4.16) it follows that

$$\mathrm{degree}(\mathcal{W}) \equiv A_{J,L} \mod r\,.$$

To construct the isomorphism in Eq. (4.20), consider the homomorphism

$$h : \mathbb{Z}/r\mathbb{Z} \longrightarrow \mathbb{C}^*$$

defined by $n \longmapsto \exp(2\pi\sqrt{-1}n/r)$. We note that $h \circ \rho_J$ is a character of $H_1(X, \mathbb{Z})$, where ρ_J is the homomorphism in Eq. (4.8). Any character of $H_1(X, \mathbb{Z})$ defines a flat complex line bundle over X (since $H_1(X, \mathbb{Z})$ is a quotient the fundamental group of X, a character of $H_1(X, \mathbb{Z})$ is also a character of the fundamental group, and hence any character of $H_1(X, \mathbb{Z})$ defines a flat complex line bundle over X). The holomorphic line bundle corresponding to the character $h \circ \rho_J$ is J itself. Similarly, the holomorphic line bundle over X corresponding to the character $h \circ \rho_L$ of $H_1(X, \mathbb{Z})$, where ρ_L is the homomorphism in Eq. (4.9), is identified with L.

Let $\mathfrak{m}(J)$ (respectively, $\mathfrak{m}(L)$) be the one–dimensional complex $H_1(X, \mathbb{Z})$–module defined by the character $h \circ \rho_J$ (respectively, $h \circ \rho_L$) of $H_1(X, \mathbb{Z})$. The holomorphic line bundle over X associated to the $H_1(X, \mathbb{Z})$–module $\mathfrak{m}(J)$ (respectively, $\mathfrak{m}(L)$) coincides with the holomorphic line bundle corresponding to the character $h \circ \rho_J$ (respectively, $h \circ \rho_L$) of $H_1(X, \mathbb{Z})$. Therefore, the holomorphic line bundle over X associated to the $H_1(X, \mathbb{Z})$–module $\mathfrak{m}(J)$ (respectively, $\mathfrak{m}(L)$) coincides with J (respectively, L).

On the other hand, consider the adjoint action of $\mathrm{PGL}(r, \mathbb{C})$ on the vector space $\mathrm{M}(r, \mathbb{C})$ of $r \times r$–matrices with entries in $\mathbb{C}$. Using this action, and the homomorphism Ψ constructed in Eq. (4.12), the vector space $\mathrm{M}(r, \mathbb{C})$ becomes a $H_1(X, \mathbb{Z})$–module. This $H_1(X, \mathbb{Z})$–module $\mathrm{M}(r, \mathbb{C})$ has the following decomposition:

$$\mathrm{M}(r, \mathbb{C}) = \bigoplus_{i=0}^{r-1}\bigoplus_{j=0}^{r-1} \mathfrak{m}(J)^{\otimes i} \bigotimes \mathfrak{m}(L)^{\otimes j} \tag{4.21}$$

where $\mathfrak{m}(J)$ and $\mathfrak{m}(L)$ are the one–dimensional $H_1(X, \mathbb{Z})$–modules defined above.

The holomorphic vector bundle over X associated to the above mentioned $H_1(X, \mathbb{Z})$–module $\mathrm{M}(r, \mathbb{C})$ is identified with the vector bundle $End(\mathcal{W})$, where $\mathcal{W}$ is the vector bundle in Eq. (4.19). On the other hand, we noted earlier that the holomorphic line bundle over X associated to the $H_1(X, \mathbb{Z})$–module $\mathfrak{m}(J)$ (respectively, $\mathfrak{m}(L)$) is J (respectively, L). Therefore, fixing an isomorphism as in Eq. (4.21) we obtain an isomorphism as in Eq. (4.20).

A nonzero holomorphic homomorphism

$$p : End(\mathcal{W}) \longrightarrow J \tag{4.22}$$

gives a nonzero holomorphic section of $End(\mathcal{W})^\vee \bigotimes J = End(\mathcal{W}) \bigotimes J$. Hence p gives a nonzero holomorphic homomorphism of vector bundles

$$\mathcal{W} \longrightarrow \mathcal{W} \bigotimes J .$$

Since the vector bundle $\mathcal{W}$ is stable, and degree$(J) = 0$, any nonzero homomorphism $\mathcal{W} \longrightarrow \mathcal{W} \bigotimes J$ must be an isomorphism.

The decomposition in Eq. (4.20) ensures that a nonzero homomorphism p as in Eq. (4.22) exists. Hence we conclude that $\mathcal{W}$ is isomorphic to $\mathcal{W} \bigotimes J$. Similarly, the vector bundle $\mathcal{W} \bigotimes L$ is isomorphic to $\mathcal{W}$. This completes the proof of the proposition. □

Recall that $\mathcal{M}_\xi(r)^L$ is the fixed point set defined in Eq. (2.5).

Lemma 4.6. *Take two linearly independent elements $J , L \in \Gamma$. Let*

$$\overline{\rho}_J , \overline{\rho}_L \in H^1(X, \mathbb{Z}/r\mathbb{Z})$$

be the corresponding cohomology classes constructed as in Eq. (4.17)*. Then*

$$\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L = \emptyset$$

if and only the cup product $\overline{\rho}_J \cup \overline{\rho}_L \in H^2(X, \mathbb{Z}/r\mathbb{Z})$ vanishes.

If $\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L \neq \emptyset$, then

$$\#(\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L) = r^{2g-2} . \tag{4.23}$$

Proof. First assume that

$$\overline{\rho}_J \cup \overline{\rho}_L \equiv d := \text{degree}(\xi) \mod r . \tag{4.24}$$

Then from the second part of Proposition 4.5 we know that

$$\text{degree}(\mathcal{W}) = ar + d$$

for some integer a, where $\mathcal{W}$ is the vector bundle in Eq. (4.19). Since $\mathcal{W}$ is also stable (see the first part of Proposition 4.5), there is a holomorphic line bundle

$$\mathcal{L}_0 \longrightarrow X$$

of degree $-a$ such that

$$\mathcal{W}_0 := \mathcal{W} \bigotimes \mathcal{L}_0 \in \mathcal{M}_\xi(r) .$$

The vector bundle $\mathcal{W} \longrightarrow X$ is isomorphic to both $\mathcal{W} \bigotimes J$ and $\mathcal{W} \bigotimes L$ (see Proposition 4.5). Hence $\mathcal{W}_0$ is isomorphic to both $\mathcal{W}_0 \bigotimes J$ and $\mathcal{W}_0 \bigotimes L$. Indeed, for any isomorphism

$$f_0 : \mathcal{W} \longrightarrow \mathcal{W} \bigotimes J ,$$

the homomorphism

$$f_0 \bigotimes \text{Id}_{\mathcal{L}_0} : \mathcal{W}_0 := \mathcal{W} \bigotimes \mathcal{L}_0 \longrightarrow \mathcal{W} \bigotimes J \bigotimes \mathcal{L}_0 = \mathcal{W} \bigotimes \mathcal{L}_0 \bigotimes J = \mathcal{W}_0 \bigotimes J$$

is an isomorphism; similarly, $\mathcal{W}_0$ is isomorphic to $\mathcal{W}_0 \bigotimes L$. In other words, we have

$$\mathcal{W}_0 \in \mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L . \tag{4.25}$$

Now assume that

$$\delta := \overline{\rho}_J \cup \overline{\rho}_L \neq 0 , \tag{4.26}$$

where $\overline{\rho}_J$ and $\overline{\rho}_L$ are as in Eq. (4.24). Fix a positive integer n_0 such that

$$d \equiv n_0\delta \mod r\,. \tag{4.27}$$

We note that such an integer n_0 exists because r is a prime number, $d \not\equiv 0 \bmod r$, and $\delta \neq 0$. Replace the line bundle J by

$$J_0 = J^{\otimes n_0}\,, \tag{4.28}$$

and keep the line bundle L unchanged. From Eq. (4.26) and Eq. (4.27),

$$\overline{\rho}_{J_0} \cup \overline{\rho}_L \equiv d \mod r\,, \tag{4.29}$$

where $\overline{\rho}_{J_0} \in H^1(X,\, \mathbb{Z}/r\mathbb{Z})$ is the cohomology class constructed as in Eq. (4.17) for the line bundle J_0 in Eq. (4.28).

We noted above that from Eq. (4.29) it follows that

$$\mathcal{M}_\xi(r)^{J_0} \bigcap \mathcal{M}_\xi(r)^L \neq \emptyset$$

(see Eq. (4.25)). Take any

$$V \in \mathcal{M}_\xi(r)^{J_0} \bigcap \mathcal{M}_\xi(r)^L\,. \tag{4.30}$$

Since V is isomorphic to $V \bigotimes J_0$, it follows that V is isomorphic to $V \bigotimes J_0^{\otimes n}$ for all n. Indeed, if

$$f : V \longrightarrow V \bigotimes J_0$$

is an isomorphism, then the composition

$$V \xrightarrow{f} V \bigotimes J_0 \xrightarrow{f\otimes \mathrm{Id}_{J_0}} V \bigotimes J_0^{\otimes 2} \xrightarrow{f\otimes \mathrm{Id}_{J_0^{\otimes 2}}} V \bigotimes J_0^{\otimes 3} \xrightarrow{f\otimes \mathrm{Id}_{J_0^{\otimes 3}}} \cdots \xrightarrow{f\otimes \mathrm{Id}_{J_0^{\otimes (n-1)}}} V \bigotimes J_0^{\otimes n}$$

is an isomorphism for all $n \geq 1$.

Take a positive integer m_0 such that $m_0 n_0 \equiv 1 \bmod r$, where n_0 is the integer in Eq. (4.27). Therefore, the line bundle $J_0^{\otimes m_0}$ is isomorphic to J (see Eq. (4.28)). Since V is isomorphic to $V \bigotimes J_0^{\otimes m_0} = V \bigotimes J$, we conclude that

$$V \in \mathcal{M}_\xi(r)^J\,.$$

Hence from Eq. (4.30),

$$V \in \mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L\,.$$

Therefore, we have proved that

$$\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L \neq \emptyset$$

if $\overline{\rho}_J \cup \overline{\rho}_L \neq 0$.

To prove the converse, assume that

$$\overline{\rho}_J \cup \overline{\rho}_L = 0\,. \tag{4.31}$$

If $E \longrightarrow X$ is a holomorphic vector bundle such that

$$End(E) = \bigoplus_{i,j=0}^{r-1} J^{\otimes i} \bigotimes L^{\otimes j}\,,$$

then from Eq. (4.31) it follows that $\text{degree}(E) \equiv 0 \mod r$. Since $d := \text{degree}(\xi)$ is coprime to r, this implies that

$$\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L = \emptyset .$$

Therefore,

$$\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L = \emptyset$$

if and only $\overline{\rho}_J \cup \overline{\rho}_L = 0$.

To prove the last statement in the lemma, assume that

$$\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L \neq \emptyset .$$

Fix a vector bundle $E \in \mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L$. Then from Corollary 4.2 and Remark 4.3 we know that $\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L$ is the orbit of E under the action of Γ on $\mathcal{M}_\xi(r)$. The isotropy subgroup $\Gamma_E \subset \Gamma$ for E is generated by J and L. Now, Eq. (4.23) holds because $\#\Gamma = r^{2g}$, and the order of the subgroup of Γ generated by J and L is r^2. This completes the proof of the lemma. □

5. The cohomologies

The cohomology groups $H^i(\mathcal{M}_\xi(r), \mathbb{Q})$, $i \geq 0$, are computed in [7], [2]. The cohomology algebra $\bigoplus_{i\geq 0} H^i(\mathcal{M}_\xi(r), \mathbb{Q})$ is computed by Kirwan in [8].

Consider the action of Γ on $\bigoplus_{i\geq 0} H^i(\mathcal{M}_\xi(r), \mathbb{Q})$ given by the action of Γ on $\mathcal{M}_\xi(r)$. It is known that this action is trivial [7, p. 220, Theorem 1]. Therefore, the cohomology algebra $\bigoplus_{i\geq 0} H^i(\mathcal{M}_\xi(r), \mathbb{Q})$ is identified with the cohomology algebra $\bigoplus_{i\geq 0} H^i(\mathcal{M}_\xi(r)/\Gamma, \mathbb{Q})$.

Take any nontrivial line bundle $L \in \Gamma \setminus \{\mathcal{O}_X\}$. Since r is a prime number, the order of L is r. Let

$$\gamma_L : Y_L \longrightarrow X$$

be the Galois covering of degree r constructed as in Eq. (2.7). Let

$$\text{Prym}_\xi(\gamma_L) \subset \text{Pic}^d(Y_L) \tag{5.1}$$

be the Prym variety parametrizing all line bundles $\eta \longrightarrow Y_L$ such that

$$\det \gamma_{L*}\eta := \bigwedge^r (\gamma_{L*}\eta) = \xi .$$

Note that the Galois group $\text{Gal}(\gamma_L)$ acts on $\text{Prym}_\xi(\gamma_L)$. The action of $\tau \in \text{Gal}(\gamma_L)$ on $\text{Prym}_\xi(\gamma_L)$ sends any line bundle η to $\tau^*\eta$. From Lemma 2.1 we know that

$$\text{Prym}_\xi(\gamma_L)/\text{Gal}(\gamma_L) = \mathcal{M}_\xi(r)^L . \tag{5.2}$$

We also know that the action of $\text{Gal}(\gamma_L)$ on $\text{Prym}_\xi(\gamma_L)$ is free (see Lemma 3.5).

The group Γ acts on $\text{Prym}_\xi(\gamma_L)$. The action of any $\zeta \in \Gamma$ is given by the map $\eta \longmapsto \eta \bigotimes \gamma_L^*\zeta$; note that by the projection formula,

$$\bigwedge^r \gamma_{L*}(\eta \bigotimes \gamma_L^*\zeta) = (\bigwedge^r \gamma_{L*}\eta) \bigotimes \zeta^{\otimes r} = \bigwedge^r \gamma_{L*}\eta ,$$

hence $\eta \bigotimes \gamma_L^*\zeta \in \text{Prym}_\xi(\gamma_L)$ if $\eta \in \text{Prym}_\xi(\gamma_L)$.

It is straight–forward to check that the actions of Γ and $\text{Gal}(\gamma_L)$ on $\text{Prym}_\xi(\gamma_L)$ commute.

For any $i \geq 0$, consider the action of Γ on $H^i(\mathrm{Prym}_\xi(\gamma_L), \mathbb{Q})$ given by the above action of Γ on $\mathrm{Prym}_\xi(\gamma_L)$. Since the action of Γ on $\mathrm{Prym}_\xi(\gamma_L)$ is through translations, it follows immediately that this action of Γ on $H^i(\mathrm{Prym}_\xi(\gamma_L), \mathbb{Q})$ is the trivial one.

We will now recall the topological model of a cyclic covering of X of degree r.

The isomorphism classes of unramified cyclic coverings

$$Y \longrightarrow X$$

of degree r with Y connected are parametrized by the complement

$$H^1(X, \mathbb{Z}/r\mathbb{Z})_0 := H^1(X, \mathbb{Z}/r\mathbb{Z}) \setminus \{0\},$$

because the space of all the surjective homomorphisms

$$\pi_1(X, x_0) \longrightarrow \mathbb{Z}/r\mathbb{Z}$$

is parametrized by $H^1(X, \mathbb{Z}/r\mathbb{Z})_0$. Let $\mathrm{Diff}^+(X)$ denote the group of all orientation preserving diffeomorphisms of X. This group $\mathrm{Diff}^+(X)$ has a natural action on $H^1(X, \mathbb{Z}/r\mathbb{Z})$. The action of $\mathrm{Diff}^+(X)$ on $H^1(X, \mathbb{Z}/r\mathbb{Z})_0$ can be shown to be transitive. To prove this, let $\mathrm{Aut}(H^1(X, \mathbb{Z}))$ denote the group of automorphisms of $H^1(X, \mathbb{Z})$ preserving the cup product; so $\mathrm{Aut}(H^1(X, \mathbb{Z}))$ can be identified with the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ after fixing a symplectic basis of $H^1(X, \mathbb{Z})$. It is known that the natural homomorphism

$$\mathrm{Diff}^+(X) \longrightarrow \mathrm{Aut}(H^1(X, \mathbb{Z}))$$

is surjective [6, p. 114]. On the other hand, it is easy to check that the natural action of $\mathrm{Aut}(H^1(X, \mathbb{Z}))$ on $H^1(X, \mathbb{Z}/r\mathbb{Z})_0$ is transitive. Hence, we conclude that the action of $\mathrm{Diff}^+(X)$ on $H^1(X, \mathbb{Z}/r\mathbb{Z})_0$ is transitive.

Therefore, given two unramified cyclic coverings

$$\pi_1 : Y_1 \longrightarrow X \quad \text{and} \quad \pi_2 : Y_2 \longrightarrow X$$

of degree r with both Y_1 and Y_2 connected, there is a diffeomorphism

$$\varphi : X \longrightarrow X$$

such that φ pulls back the covering π_2 to π_1.

One example of a cyclic covering of degree r is the following:

Let X_0 be a compact surface of genus one, and let X_1 be a compact surface of genus $g-1$. Take an orientation preserving free action of $\mathbb{Z}/r\mathbb{Z}$ on X_0. Let X_0' be the complement of r open disks in X_0 such that X_0' is preserved by the action of $\mathbb{Z}/r\mathbb{Z}$ on X_0. Let X_1' be the complement of a closed disk in X_1. Now attach r copies of X_1' to X_0' along the r boundary circles of X_0'. The resulting compact connected surface of genus $r(g-1)+1$ will be denoted by Y. The action of $\mathbb{Z}/r\mathbb{Z}$ on X_0' and the permutation action of $\mathbb{Z}/r\mathbb{Z}$ on the r copies of X_1' together define an action of $\mathbb{Z}/r\mathbb{Z}$ on Y. This action is clearly free, and the quotient is of genus g. Up to diffeomorphisms of $Y/(\mathbb{Z}/r\mathbb{Z})$, all connected unramified cyclic coverings of $Y/(\mathbb{Z}/r\mathbb{Z})$ of degree r coincide with the covering

$$Y \longrightarrow Y/(\mathbb{Z}/r\mathbb{Z}). \tag{5.3}$$

In particular, the topological model of the covering γ_L in Eq. (2.7) is the covering in Eq. (5.3).

Using this model of γ_L it follows that $\mathrm{Prym}_\xi(\gamma_L)$ (defined in Eq. (5.1)) is topologically isomorphic to a real torus of dimension $2(r-1)(g-1)$. The action of the Galois group

$\text{Gal}(\gamma_L)$ on $H^1(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$ can also be calculated using the above topological model of the covering γ_L.

To calculate the action of $\text{Gal}(\gamma_L)$ on $H^1(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$, consider the group μ_r defined in Eq. (2.8), which is identified with $\text{Gal}(\gamma_L)$. Let

$$\widehat{\mu}_r \,:=\, \text{Hom}(\mu_r\,, \mathbb{C}^*)$$

be the group of characters of μ_r. It is a cyclic group of order r generated by the tautological character of μ_r defined by the inclusion of μ_r in $\mathbb{C}^*$.

Each nontrivial element of $\widehat{\mu}_r$ is an eigen–character of μ_r for the action of $\text{Gal}(\gamma_L) \,=\, \mu_r$ on $H^1(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$, and furthermore, the multiplicity of each eigen–character is $2(g-1)$.

To prove the above assertion, consider the covering in Eq. (5.3). We noted earlier that it is the topological model of the covering γ_L. Let

$$A \,:\, H^1(X_1,\, \mathbb{C})^{\oplus r} \,\longrightarrow\, H^1(X_1,\, \mathbb{C}) \tag{5.4}$$

be the homomorphism defined by

$$(c_1\,, \cdots, c_r) \,\longmapsto\, \sum_{j=1}^{r} c_j\,;$$

the surface X_1 is the one used in the construction of the covering in Eq. (5.3). The complex vector space $H^1(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$ is identified with the kernel of the homomorphism

$$H^1(\text{Pic}^d(Y_L),\, \mathbb{C}) \,=\, H^1(Y_L,\, \mathbb{C}) \,\longrightarrow\, H^1(X,\, \mathbb{C})\mathbb{C})$$

that sends any $c \,\in\, H^1(Y_L,\, \mathbb{C})$ to the class in

$$H^1(X,\, \mathbb{C})\mathbb{C}) \,=\, H^1(Y_L,\, \mathbb{C})^{\text{Gal}(\gamma_L)}$$

defined by $\sum_{\tau\in\text{Gal}(\gamma_L)} \tau^* c$. We have

$$H^1(\text{Pic}^d(Y_L),\, \mathbb{C}) \,=\, H^1(X_1,\, \mathbb{C})^{\oplus r} \bigoplus H^1(X_0,\, \mathbb{C})$$

(X_0 is the surface of genus one in the construction of the covering in Eq. (5.3)), and

$$H^1(\text{Prym}_\xi(\gamma_L),\, \mathbb{C}) \,=\, \text{kernel}(A) \,\subset\, H^1(X_1,\, \mathbb{C})^{\oplus r} \,\subset\, H^1(\text{Pic}^d(Y_L),\, \mathbb{C})\,,$$

where A is the homomorphism in Eq. (5.4). The action of

$$\exp(2\pi\sqrt{-1}/r) \,\in\, \mu_r \,=\, \text{Gal}(\gamma_L)$$

on $H^1(\text{Pic}^d(Y_L),\, \mathbb{C}) \,=\, H^1(X_1,\, \mathbb{C})^{\oplus r} \bigoplus H^1(X_0,\, \mathbb{C})$ is given by the automorphism defined by

$$(c_1\,, \cdots, c_r\,; d) \,\longmapsto\, (c_2\,, \cdots, c_r\,, c_1\,; d) \,\in\, H^1(X_1,\, \mathbb{C})^{\oplus r} \bigoplus H^1(X_0,\, \mathbb{C})\,.$$

Also, $\dim H^1(X_1,\, \mathbb{C}) \,=\, 2(g-1)$. It is now easy to see that each nontrivial character of μ_r is an eigen–character of multiplicity $2(g-1)$ for the action of $\text{Gal}(\gamma_L) \,=\, \mu_r$ on $H^1(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$.

The cohomology algebra $\bigoplus_{i\geq 0} H^i(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$ is identified with the exterior algebra $\bigoplus_{i\geq 0} \bigwedge^i H^1(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$. Therefore, from the above description of the action of $\text{Gal}(\gamma_L)$ on $H^1(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$ we obtain a description of the action of $\text{Gal}(\gamma_L)$ on the cohomology algebra $\bigoplus_{i\geq 0} H^i(\text{Prym}_\xi(\gamma_L),\, \mathbb{C})$.

5.1. **The Chen–Ruan cohomology.** The case of $r = 2$ was already considered in [3]. Here we will assume that $r \geq 3$.

The i–th Chen–Ruan cohomology group is the degree shifted direct sum

$$H^i_{CR}(\mathcal{M}_\xi(r)/\Gamma, \mathbb{Q}) = \bigoplus_{L\in\Gamma} H^{i-2\iota(L)}(\mathcal{M}_\xi(r)^L/\Gamma, \mathbb{Q}). \tag{5.5}$$

The degree shifting number $\iota(L)$ is obtained from Proposition 3.4:

$$\iota(L) = \begin{cases} 0 & \text{if } L = \mathcal{O}_X \\ \frac{1}{2}(r^2-r)(g-1) & \text{otherwise.} \end{cases} \tag{5.6}$$

As in [3], we will denote $H^{*+2\iota(L)}(\mathcal{M}_\xi(r)^L/\Gamma, \mathbb{Q})$ by $A^*(L)$. Then for $\alpha_1 \in A^p(L_1)$ and $\alpha_2 \in A^q(L_2)$, the Chen–Ruan product

$$\alpha_1 \cup \alpha_2 \in A^{p+q}(L_1 \bigotimes L_2)$$

is defined via the relation

$$\langle \alpha_1 \cup \alpha_2 \,, \alpha_3\rangle = \int_{\mathbf{S}/\Gamma} e_1^*\alpha_1 \bigwedge e_2^*\alpha_2 \bigwedge e_3^*\alpha_3 \bigwedge c_{\text{top}}\mathcal{F} \tag{5.7}$$

for all $\alpha_3 \in A^*(L_3)$ such that $L_1 \bigotimes L_2 \bigotimes L_3 = \mathcal{O}_X$, where $\langle\, ,\, \rangle$ is the nondegenerate bilinear Poincaré pairing for Chen–Ruan cohomology (see [4], [3, (6.20)]),

$$\mathbf{S} := \mathcal{M}_\xi(r)^{L_1} \bigcap \mathcal{M}_\xi(r)^{L_2}\,,$$

and $e_i : \mathbf{S}/\Gamma \longrightarrow \mathcal{M}_\xi(r)^{L_i}/\Gamma$ are the canonical inclusions, and $\mathcal{F}$ is a complex Γ–bundle over $\mathbf{S}$, or equivalently, an orbifold vector bundle over $\mathbf{S}/\Gamma$, of rank

$$\text{rank}(\mathcal{F}) = \dim_{\mathbb{C}} \mathbf{S} - \dim_{\mathbb{C}} \mathcal{M}_\xi(r) + \sum_{j=1}^{3} \iota(L_j)\,. \tag{5.8}$$

From Lemma 4.6 it follows that $\mathbf{S}$ is empty or zero dimensional if L_1 and L_2 are linearly independent. If $\mathbf{S}$ is empty, then the corresponding Chen–Ruan products are automatically zero. Even otherwise, since L_3 is also nontrivial if L_1 and L_2 are linearly independent, using Eq. (5.6) we get that

$$\text{rank}(\mathcal{F}) = \frac{1}{2}(r-1)(r-2)(g-1) > 0 = \dim_{\mathbb{C}} \mathbf{S}$$

(recall that $r \geq 3$). Hence, $c_{\text{top}}\mathcal{F} = 0$, and again all the corresponding Chen–Ruan products are zero.

If L_1 and L_2 are linearly dependent then we have the following three cases:

(1) If L_1 and L_2 are both trivial then the Chen–Ruan products are just the usual products in the singular cohomology of $\mathcal{M}_\xi(r)/\Gamma$.
(2) If all the three line bundles are nontrivial, meaning $L_i = L^{\otimes k_i}$ for some $L \neq \mathcal{O}_X$ and $1 \leq k_i \leq (r-1)$, $i \in \{1\,,2\,,3\}$, then we have $\mathbf{S} = \mathcal{M}_\xi(r)^L$, and

$$\text{rank}(\mathcal{F}) = \frac{1}{2}(r^2-r)(g-1) > (r-1)(g-1) = \dim_{\mathbb{C}} \mathbf{S}$$

(recall that $r \geq 3$). Hence $c_{\text{top}}\mathcal{F} = 0$, and all the corresponding Chen–Ruan products are zero.

(3) If exactly two of the three L_i's are nontrivial, then we must have $L_{i_1} = L$, $L_{i_2} = L^{\otimes(r-1)}$ and $L_{i_3} = \mathcal{O}_X$ for some $L \neq \mathcal{O}_X$ and some permutation $\{i_1, i_2, i_3\}$ of $\{1, 2, 3\}$. The calculations of the Chen–Ruan products are analogous to the cases a), b) and c) in Section 6.1 of [3].

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