ON HADAMARD-TYPE INEQUALITIES FOR CO–ORDINATED $r-$CONVEX FUNCTIONS

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Abstract. In this paper we defined $r-$convexity on the coordinates and we established some Hadamard-Type Inequalities.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

This inequality is well known in the literature as Hadamard's inequality.

In [1], C.E.M. Pearce, J. Pecaric and V. Simic generalized this inequality to $r-$convex positive function f which is defined on an interval $[a, b]$, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$;

$$f(\lambda x + (1-\lambda)y) \leq \begin{cases} \left(\lambda \left[f(x)\right]^r + (1-\lambda)\left[f(y)\right]^r\right)^{\frac{1}{r}}, & if\ r \neq 0 \\ \left[f(x)\right]^{\lambda} \left[f(y)\right]^{1-\lambda}, & if\ r = 0 \end{cases}$$

We have that $0-$convex functions are simply log-convex functions and $1-$convex functions are ordinary convex functions.

In [3], N.P.G. Ngoc, N.V. Vinh and P.T.T. Hien established following theorems for $r-$convex functions:

Theorem 1. *Let $f : [a, b] \to (0, \infty)$ be $r-$convex function on $[a, b]$ with $a < b$. Then the following inequality holds for $0 < r \leq 1$:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{r}{r+1}\right)^{\frac{1}{r}} \left(\left[f(a)\right]^r + \left[f(b)\right]^r\right)^{\frac{1}{r}} \tag{1.1}$$

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Theorem 2. *Let $f,g:[a,b]\to(0,\infty)$ be $r-$convex and $s-$convex functions respectively on $[a,b]$ with $a<b$. Then the following inequality holds for $0<r,s\leq 2:$*

$$\begin{aligned}(1.2)\qquad \frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \frac{1}{2}\left(\frac{r}{r+2}\right)^{\frac{2}{r}}\left([f(a)]^r+[f(b)]^r\right)^{\frac{2}{r}}\\ &+\frac{1}{2}\left(\frac{s}{s+2}\right)^{\frac{2}{s}}\left([g(a)]^s+[g(b)]^s\right)^{\frac{2}{s}}\end{aligned}$$

Theorem 3. *Let $f,g:[a,b]\to(0,\infty)$ be $r-$convex and $s-$convex functions respectively on $[a,b]$ with $a<b$.Then the following inequality holds if $r>1$, and $\frac{1}{r}+\frac{1}{s}=1:$*

$$(1.3)\qquad \frac{1}{b-a}\int_a^b f(x)g(x)dx\leq\left(\frac{[f(a)]^r+[f(b)]^r}{2}\right)^{\frac{1}{r}}\left(\frac{[g(a)]^s+[g(b)]^s}{2}\right)^{\frac{1}{s}}$$

Similar results can be found for several kind of convexity, in [8], [9], [10] and [12].
In [5], a convex function on the co-ordinates defined by S.S. Dragomir as follow:

Definition 1. *A function $f:\Delta\to\mathbb{R}$ which is convex on Δ is called co-ordinated convex on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex for all $y\in[c,d]$ and $x\in[a,b]$.*

Again in [5], Dragomir gave the following inequalities related to definition given above.

Theorem 4. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}(1.4)\qquad & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ \leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\ \leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ & \left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\end{aligned}$$

The above inequalities are sharp.

In [6], M. Alomari and M. Darus proved some inequalities of the Hadamard and Jensen types for co-ordinated log-convex functions. In [7], M.K. Bakula and J.

Pecaric improved several inequalities of Jensen's type for convex functions on the co-ordinates. In [11], M.E. Özdemir, E. Set and M.Z. Sarıkaya established Hadamard's type inequalities for co-ordinated $m-$convex and $(\alpha, m)-$convex functions. Similar results can be found in [8], [9], [10] and [12].

The main purpose of this present note is to give definition of $r-$convexity on the coordinates and to prove some Hadamard-type inequalities for co-ordinated $r-$convex functions.

2. MAIN RESULTS

We can define $r-$convex functions on the coordinates as follow:

Definition 2. *A function $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ will be called $r-$convex on Δ, for all $t, \lambda \in [0, 1]$ and $(x, y), (u, v) \in \Delta$, if the following inequalities hold:*

$$
\begin{aligned}
& f\left(tx + (1-t)y, \lambda u + (1-\lambda)v\right) \\
\leq & \begin{cases} \left(t\left[f(x,u\right]^r + (1-t)\left[f(y,v)\right]^r\right)^{\frac{1}{r}} & , \; if \;\; r \neq 0 \\ \\ f^{t\lambda}(x,u) f^{t(1-\lambda)}(x,v) f^{(1-t)\lambda}(y,u) f^{(1-t)(1-\lambda)}(y,v), & if \;\; r = 0 \end{cases}
\end{aligned}
$$

It is simply to see that if we choose $r = 0$, we have co-ordinated log-convex functions and if we choose $r = 1$, we have co-ordinated convex functions. A function $f : \Delta \to \mathbb{R}_+$ is $r-$convex on Δ is called co-ordinated $r-$convex on Δ if the partial mappings

$$f_y : [a, b] \to \mathbb{R}_+, \;\; f_y(u) = f(u, y)$$

and

$$f_x : [c, d] \to \mathbb{R}_+, \;\; f_x(v) = f(x, v)$$

are $r-$convex for all $y \in [c, d]$ and $x \in [a, b]$.

We need the foolowing lemma for our main results.

Lemma 1. *Every $r-$convex mapping $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ is $r-$convex on the co-ordinates, where $t \in [0, 1]$.*

Proof. Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ is $r-$convex on Δ. Consider the mapping

$$f_y : [a, b] \to \mathbb{R}_+, \;\; f_y(u) = f(u, y)$$

Case 1: For $r = 0$ and $u_1, u_2 \in [a, b]$, then we have:

$$
\begin{aligned}
f_y(tu_1 + (1-t)u_2) &= f(tu_1 + (1-t)u_2, y) \\
&= f(tu_1 + (1-t)u_2, ty + (1-t)y) \\
&\leq tf(u_1, y) + (1-t)f(u_2, y) \\
&= tf_y(u_1) + (1-t)f_y(u_2)
\end{aligned}
$$

Case 2: For $r \neq 0$ and $u_1, u_2 \in [a, b]$, then we have:

$$
\begin{aligned}
f_y(tu_1 + (1-t)u_2) &= f(tu_1 + (1-t)u_2, y) \\
&= f(tu_1 + (1-t)u_2, ty + (1-t)y) \\
&\leq \left(t\left[f(u_1, y)\right]^r + (1-t)\left[f(u_2, y)\right]^r\right)^{\frac{1}{r}} \\
&= \left(t\left[f_y(u_1)\right]^r + (1-t)\left[f_y(u_2)\right]^r\right)^{\frac{1}{r}}
\end{aligned}
$$

Therefore $f_y(u) = f(u, y)$ is $r-$convex on $[a, b]$. By a similar argument one can see $f_x(v) = f(x, v)$ is $r-$convex on $[c, d]$. □

Theorem 5. *Suppose that $f : \Delta \to \mathbb{R}_+$ be a positive co-ordinated $r-$convex function on Δ. If $t, \lambda \in [0, 1]$ and $(x, y), (u, v) \in \Delta$, then one has the inequality:*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \tag{2.1}$$
$$\leq \frac{1}{2}\left(\frac{r}{r+1}\right)^{\frac{1}{r}}\left[\left(\frac{1}{b-a}\int_a^b [f(x,c)]^r dx + \frac{1}{b-a}\int_a^b [f(x,d)]^r dx\right)^{\frac{1}{r}}\right.$$
$$\left. + \left(\frac{1}{d-c}\int_c^d [f(a,y)]^r dy + \frac{1}{d-c}\int_c^d [f(b,y)]^r dy\right)^{\frac{1}{r}}\right]$$

where $0 < r \leq 1$.

Proof. Since $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ is co-ordinated $r-$convex on Δ, then the partial mappings

$$f_x : [c, d] \to \mathbb{R}_+,\ f_x(v) = f(x, v)$$

and

$$f_y : [a, b] \to \mathbb{R}_+,\ f_y(u) = f(u, y)$$

are $r-$convex, by inequality (1.1), we can write:

$$\frac{1}{d-c}\int_c^d f_x(y)dy \leq \left(\frac{r}{r+1}\right)^{\frac{1}{r}}\left([f_x(c)]^r + [f_x(d)]^r\right)^{\frac{1}{r}}$$

or

$$\frac{1}{d-c}\int_c^d f(x,y)dy \leq \left(\frac{r}{r+1}\right)^{\frac{1}{r}}\left([f(x,c)]^r + [f(x,d)]^r\right)^{\frac{1}{r}}$$

Dividing both side of inequality $(b - a)$ and integrating respect to x on $[a, b]$, we get

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \tag{2.2}$$
$$\leq \left(\frac{r}{r+1}\right)^{\frac{1}{r}}\left[\frac{1}{b-a}\int_a^b [f(x,c)]^r dx + \frac{1}{b-a}\int_a^b [f(x,d)]^r dx\right]^{\frac{1}{r}}$$

By a similar argument for the mapping, we have

$$f_y : [a, b] \to \mathbb{R}_+,\ f_y(u) = f(u, y)$$

$$(2.3)\qquad \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$

$$\leq \left(\frac{r}{r+1}\right)^{\frac{1}{r}}\left[\frac{1}{d-c}\int_c^d [f(a,y)]^r\,dy+\frac{1}{d-c}\int_c^d [f(b,y)]^r\,dy\right]^{\frac{1}{r}}$$

By addition (2.2) and (2.3), (2.1) is proved. □

Corollary 1. *In (2.1), if we choose $r=1$ we have the mid inequality of (1.4).*

Theorem 6. *Suppose that $f,g:\Delta\to\mathbb{R}_+$ be co-ordinated r_1-convex function and co-ordinated r_2-convex function on Δ. Then one has the inequality:*

$$(2.4)\qquad \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$\leq \frac{1}{4}\left(\frac{r_1}{r_1+2}\right)^{\frac{2}{r_1}}\left(\frac{1}{(b-a)}\int_a^b [f(x,c)]^{r_1}\,dx+\frac{1}{(b-a)}\int_a^b [f(x,d)]^{r_1}\,dx\right)^{\frac{2}{r_1}}$$

$$+\frac{1}{4}\left(\frac{r_2}{r_2+2}\right)^{\frac{2}{r_2}}\left(\frac{1}{(b-a)}\int_a^b [g(x,c)]^{r_2}\,dx+\frac{1}{(b-a)}\int_a^b [g(x,d)]^{r_2}\,dx\right)^{\frac{2}{r_2}}$$

$$+\frac{1}{4}\left(\frac{r_1}{r_1+2}\right)^{\frac{2}{r_1}}\left(\frac{1}{(d-c)}\int_c^d [f(a,y)]^{r_1}\,dy+\frac{1}{(d-c)}\int_c^d [f(b,y)]^{r_1}\,dy\right)^{\frac{2}{r_1}}$$

$$+\frac{1}{4}\left(\frac{r_2}{r_2+2}\right)^{\frac{2}{r_2}}\left(\frac{1}{(d-c)}\int_c^d [g(a,y)]^{r_2}\,dy+\frac{1}{(d-c)}\int_c^d [g(b,y)]^{r_2}\,dy\right)^{\frac{2}{r_2}}$$

where $r_1>0$, $r_2\leq 2$.

Proof. Since $f,g:\Delta=[a,b]\times[c,d]\to\mathbb{R}_+$ is co-ordinated r_1-convex and r_2-convex on Δ. Then the partial mappings

$$f_x:[c,d]\to\mathbb{R}_+,\ \ f_x(v)=f(x,v)$$

and

$$f_y:[a,b]\to\mathbb{R}_+,\ \ f_y(u)=f(u,y)$$

are r_1-convex on Δ. On the other hand the partial mappings

$$g_x:[c,d]\to\mathbb{R}_+,\ \ g_x(v)=g(x,v)$$

and

$$g_y:[a,b]\to\mathbb{R}_+,\ \ g_y(u)=g(u,y)$$

are r_2-convex on Δ. From (1.2), we get

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy \leq \frac{1}{2}\left(\frac{r_1}{r_1+2}\right)^{\frac{2}{r_1}}\left([f_x(c)]^{r_1}+[f_x(d)]^{r_1}\right)^{\frac{2}{r_1}} +\frac{1}{2}\left(\frac{r_2}{r_2+2}\right)^{\frac{2}{r_2}}\left([g_x(c)]^{r_2}+[g_x(d)]^{r_2}\right)^{\frac{2}{r_2}}$$

or

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \leq \frac{1}{2}\left(\frac{r_1}{r_1+2}\right)^{\frac{2}{r_1}}\left([f(x,c)]^{r_1}+[f(x,d)]^{r_1}\right)^{\frac{2}{r_1}} +\frac{1}{2}\left(\frac{r_2}{r_2+2}\right)^{\frac{2}{r_2}}\left([g(x,c)]^{r_2}+[g(x,d)]^{r_2}\right)^{\frac{2}{r_2}}$$

Dividing both side of inequality $(b-a)$ and integrating respect to x on $[a,b]$, we have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$
$$\leq \frac{1}{2}\left(\frac{r_1}{r_1+2}\right)^{\frac{2}{r_1}}\left(\frac{1}{(b-a)}\int_a^b [f(x,c)]^{r_1}dx+\frac{1}{(b-a)}\int_a^b [f(x,d)]^{r_1}dx\right)^{\frac{2}{r_1}}$$
$$+\frac{1}{2}\left(\frac{r_2}{r_2+2}\right)^{\frac{2}{r_2}}\left(\frac{1}{(b-a)}\int_a^b [g(x,c)]^{r_2}dx+\frac{1}{(b-a)}\int_a^b [g(x,d)]^{r_2}dx\right)^{\frac{2}{r_2}} \tag{2.5}$$

By a similar argument, we have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$
$$\leq \frac{1}{2}\left(\frac{r_1}{r_1+2}\right)^{\frac{2}{r_1}}\left(\frac{1}{(d-c)}\int_c^d [f(a,y)]^{r_1}dy+\frac{1}{(d-c)}\int_c^d [f(b,y)]^{r_1}dy\right)^{\frac{2}{r_1}}$$
$$+\frac{1}{2}\left(\frac{r_2}{r_2+2}\right)^{\frac{2}{r_2}}\left(\frac{1}{(d-c)}\int_c^d [g(a,y)]^{r_2}dy+\frac{1}{(d-c)}\int_c^d [g(b,y)]^{r_2}dy\right)^{\frac{2}{r_2}} \tag{2.6}$$

Addition (2.5) and (2.6), we can write

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\
\leq\;& \frac{1}{4}\left(\frac{r_1}{r_1+2}\right)^{\frac{2}{r_1}}\left(\frac{1}{(b-a)}\int_a^b [f(x,c)]^{r_1}\,dx+\frac{1}{(b-a)}\int_a^b [f(x,d)]^{r_1}\,dx\right)^{\frac{2}{r_1}}\\
&+\frac{1}{4}\left(\frac{r_2}{r_2+2}\right)^{\frac{2}{r_2}}\left(\frac{1}{(b-a)}\int_a^b [g(x,c)]^{r_2}\,dx+\frac{1}{(b-a)}\int_a^b [g(x,d)]^{r_2}\,dx\right)^{\frac{2}{r_2}}\\
&+\frac{1}{4}\left(\frac{r_1}{r_1+2}\right)^{\frac{2}{r_1}}\left(\frac{1}{(d-c)}\int_c^d [f(a,y)]^{r_1}\,dy+\frac{1}{(d-c)}\int_c^d [f(b,y)]^{r_1}\,dy\right)^{\frac{2}{r_1}}\\
&+\frac{1}{4}\left(\frac{r_2}{r_2+2}\right)^{\frac{2}{r_2}}\left(\frac{1}{(d-c)}\int_c^d [g(a,y)]^{r_2}\,dy+\frac{1}{(d-c)}\int_c^d [g(b,y)]^{r_2}\,dy\right)^{\frac{2}{r_2}}
\end{aligned}
$$

which completes the proof. □

Corollary 2. *In (2.4), if we choose $r_1 = r_2 = 2$, we have*

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\
\leq\;& \frac{1}{8(b-a)}\left(\int_a^b [f(x,c)]^2\,dx+\int_a^b [f(x,d)]^2\,dx+\int_a^b [g(x,c)]^2\,dx+\int_a^b [g(x,d)]^2\,dx\right)\\
&+\frac{1}{8(d-c)}\left(\int_c^d [f(a,y)]^2\,dy+\int_c^d [f(b,y)]^2\,dy+\int_c^d [g(a,y)]^2\,dy+\int_c^d [g(b,y)]^2\,dy\right)
\end{aligned}
$$

Corollary 3. *In (2.4), if we choose $r_1 = r_2 = 2$, and $f(x,y) = g(x,y)$, we have*

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)^2dydx\\
\leq\;& \frac{1}{4(b-a)}\left(\int_a^b [f(x,c)]^2\,dx+\int_a^b [f(x,d)]^2\,dx\right)+\frac{1}{4(d-c)}\left(\int_c^d [f(a,y)]^2\,dy+\int_c^d [f(b,y)]^2\,dy\right)
\end{aligned}
$$

Theorem 7. *Suppose that $f,g:\Delta\to\mathbb{R}_+$ be co-ordinated r_1-convex function and co-ordinated r_2-convex function on Δ. Then one has the inequality:*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \\
&\leq \frac{1}{2}\left(\frac{1}{2(b-a)}\int_a^b[f(x,c)]^{r_1}dx+\frac{1}{2(b-a)}\int_a^b[f(x,d)]^{r_1}dx\right)^{\frac{1}{r_1}} \\
&\times\left(\frac{1}{2(b-a)}\int_a^b[g(x,c)]^{r_2}dx+\frac{1}{2(b-a)}\int_a^b[g(x,d)]^{r_2}dx\right)^{\frac{1}{r_2}} \\
&+\frac{1}{2}\left(\frac{1}{2(d-c)}\int_c^d[f(a,y)]^{r_1}dy+\frac{1}{2(d-c)}\int_c^d[f(b,y)]^{r_1}dy\right)^{\frac{1}{r_1}} \\
&\times\left(\frac{1}{2(d-c)}\int_c^d[g(a,y)]^{r_2}dy+\frac{1}{2(d-c)}\int_c^d[g(b,y)]^{r_2}dy\right)^{\frac{1}{r_2}}
\end{aligned}\tag{2.7}$$

where $r_1>1$ and $\frac{1}{r_1}+\frac{1}{r_2}=1$.

Proof. Since $f,g:\Delta=[a,b]\times[c,d]\to\mathbb{R}_+$ is co-ordinated r_1-convex and r_2-convex on Δ. Then the partial mappings

$$f_x:[c,d]\to\mathbb{R}_+,\ \ f_x(v)=f(x,v)$$

and

$$f_y:[a,b]\to\mathbb{R}_+,\ \ f_y(u)=f(u,y)$$

are r_1-convex on Δ. On the other hand the partial mappings

$$g_x:[c,d]\to\mathbb{R}_+,\ \ g_x(v)=g(x,v)$$

and

$$g_y:[a,b]\to\mathbb{R}_+,\ \ g_y(u)=g(u,y)$$

are r_2-convex on Δ. From (1.3), we can write

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \\
&\leq \left(\frac{[f(x,a)]^{r_1}+[f(x,b)]^{r_1}}{2}\right)^{\frac{1}{r_1}}\left(\frac{[g(x,a)]^{r_2}+[g(x,b)]^{r_2}}{2}\right)^{\frac{1}{r_2}}
\end{aligned}$$

Integrating this inequality respect to x on $[a,b]$, we get

$$\begin{aligned}
(2.8) \qquad & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \\
&\leq \left(\frac{1}{2(b-a)}\int_a^b [f(x,c)]^{r_1}\,dx+\frac{1}{2(b-a)}\int_a^b [f(x,d)]^{r_1}\,dx\right)^{\frac{1}{r_1}} \\
&\times\left(\frac{1}{2(b-a)}\int_a^b [g(x,c)]^{r_2}\,dx+\frac{1}{2(b-a)}\int_a^b [g(x,d)]^{r_2}\,dx\right)^{\frac{1}{r_2}}
\end{aligned}$$

Similarly, we can write

$$\begin{aligned}
(2.9) \qquad & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \\
&\leq \left(\frac{1}{2(d-c)}\int_c^d [f(a,y)]^{r_1}\,dy+\frac{1}{2(d-c)}\int_c^d [f(b,y)]^{r_1}\,dy\right)^{\frac{1}{r_1}} \\
&\times\left(\frac{1}{2(d-c)}\int_c^d [g(a,y)]^{r_2}\,dy+\frac{1}{2(d-c)}\int_c^d [g(b,y)]^{r_2}\,dy\right)^{\frac{1}{r_2}}
\end{aligned}$$

Adding (2.8) and (2.9), (2.7) is proved. □

Corollary 4. *In (2.7), if we choose* $r_1=r_2=2$, *we have*

$$\begin{aligned}
(2.10) \qquad & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \\
&\leq \frac{1}{2}\sqrt{\frac{1}{2(b-a)}\int_a^b [f(x,c)]^2\,dx+\frac{1}{2(b-a)}\int_a^b [f(x,d)]^2\,dx} \\
&\times\sqrt{\frac{1}{2(b-a)}\int_a^b [g(x,c)]^2\,dx+\frac{1}{2(b-a)}\int_a^b [g(x,d)]^2\,dx} \\
&+\frac{1}{2}\sqrt{\frac{1}{2(d-c)}\int_c^d [f(a,y)]^2\,dy+\frac{1}{2(d-c)}\int_c^d [f(b,y)]\,dy} \\
&\times\sqrt{\frac{1}{2(d-c)}\int_c^d [g(a,y)]^2\,dy+\frac{1}{2(d-c)}\int_c^d [g(b,y)]^2\,dy}
\end{aligned}$$

Corollary 5. *In (2.7), if we choose $r_1 = r_2 = 2$, and $f(x,y) = g(x,y)$, we have*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)^2 dydx \tag{2.11}$$

$$\leq \frac{1}{4(b-a)}\left[\int_a^b [f(x,c)]^2 dx + \int_a^b [f(x,d)]^2 dx\right]$$

$$+\frac{1}{4(d-c)}\left[\int_c^d [f(a,y)]^2 dy + \int_c^d [f(b,y)]^2 dy\right]$$

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