# Semigroups of finite-dimensional random projections

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**Abstract**

In this paper we present a complete description of a stochastic semigroup of finite-dimensional projections in Hilbert space. The geometry of such semigroups is characterized by the asymptotic behavior of the widths of compact subsets with respect to the subspaces generated by the semigroup operators.

**Key words:** Stochastic semigroup, random operator, Kolmogorov widths, stochastic flow.

## 1 Introduction.

This paper is devoted to a very special type of stochastic operator-valued semigroups, namely to semigroups of finite-dimensional projections. The study of stochastic operator-valued semigroups was originated from the works of A.V. Skorokhod [1,2], where he gave a representation of such a semigroup as a solution to a stochastic differential equation with an operator-valued martingale or with a process with independent increments. Skorokhod treated stochastic operator-valued semigroup as the Dolean exponent for some operator-valued martingale. This approach requires that the mean operators have continuous inverse. Semigroups of finite-dimensional projections, which we study in this paper, do not satisfy this condition. Moreover, the nature of semigroups of finite-dimensional projections is different from those studied in [1,2]. They arose as operators describing shifts of functions or measures along a stochastic flow. In [2] A.V.Skorokhod briefly noted that in the case when a stochastic flow is generated by a stochastic differential equation, such semigroups must satisfy a stochastic differential equation with unbounded operators in coefficients, but detail investigation was not provided.

Stochastic flows with coalescence in general can not be represented as a solution to a stochastic differential equation. An example of such flows is the Arratia flow of Brownian particles [4]. It is a partial case of so-called Harris flows [5], which consist of Brownian particles with spatial correlation depending on the difference between the positions of the particles. In contrast to the flows generated by stochastic differential equations such flows can lose the gomeomorphic property [5]. Moreover, the Arratia flow maps every bounded interval into a finite number of points. Consequently, for the investigation of coaleshing stochastic flows we can not apply the tools from differential geometry used for smooth stochastic flows (about smooth stochastic flows see [6]).

One of the possible approaches to the understanding of the geometry of stochastic flows with coalescence is the investigation of random operators describing shifts of functions or measures along a flow. It will be shown in the next section that such random operators can have a finite-dimensional image almost surely and be unbounded. The corresponding semigroup can have a complicated structure. So in this article we propose to consider a stochastic semigroup consisting from finite-dimensional projections. It occurs that such a semigroup in Hilbert space can be described in terms of a certain random Poisson measure on the product of the positive half-line and some orthonormal basis (Theorem 3.1). The geometry of such semigroup can be characterized in terms of widths of compact sets with respect to it. The asymptotic behavior of such widths for two types of compacts we establish in Section 4. The study of the semigroup related to a stochastic flow with coalescence in the general case is the subject to the ongoing work.

## 2 Random projections in Hilbert space.

Let us recall the definition of the random operator. Let $H$ be a real separable Hilbert space with an inner product $(.,.)$ and $L_2(\Omega, P, H)$ be the space of square-integrable random elements in $H$.

**Definition 2.1.** Strong random operator in $H$ is a continuous linear map from $H$ to $L_2(\Omega, P, H)$ .

In this section we will consider a special case of the strong random operators. Namely we will consider finite-dimensional random projections. They are bounded random operators.

**Definition 2.2.** A strong random operator $A$ is a bounded random operator if there exists a family $\{A_\omega, \omega \in \Omega\}$ of deterministic bounded linear operators in $H$ such, that

$\forall x \in H: \ (Ax)_\omega = A_\omega x \ a.s.$

Note that boundedness of a strong random operator and finiteness of its image dimension are not connected, as it is demonstrated in the next example.

**Example 2.1.** Suppose that $H = L_2([0;1])$, $\theta$ is a random variable uniformly distributed on $[0;1]$. Define a random linear functional on $H$ as follows

$$H \ni f \longmapsto \Phi(f) = f(\theta).$$

Then

$$E\Phi(f)^2 = \int_0^1 f(s)^2 ds.$$

Consequently, $\Phi$ is continuous in the square-mean. Let us check that $\Phi$ is not a random bounded functional. Suppose, that it is and denote by $\Phi_\omega$, $\omega \in \Omega$ the corresponding family of continuous linear functionals on $H$. Let us identify the functional $\Phi_\omega$ with the function from $H$ for every $\omega \in \Omega$ . Let $\{f_n \in H,\ n \geq 1\}$ be a sequence of i.i.d. random variables with a finite second moment when considered on the standard probability space $[0;1]$ with the Lebesgue measure. Suppose also that

$$\int_0^1 f_1(s)^2 ds = 1,\ \int_0^1 f_1(s) ds = 0.$$

Then the sequence $\{\Phi(f_n),\ n \geq 1\}$ is the sequence of i.i.d. random variables. From the other side

$$\Phi(f_n)_\omega = (f_n, \Phi_\omega) \to 0,\ n \to \infty.$$

This contradiction proves our statement.

In what follows we say that $A$ is a random Hilbert-Schmidt or nuclear or finite-dimensional operator if the corresponding family $\{A_\omega, \omega \in \Omega\}$ consists of Hilbert-Schmidt, nuclear or finite-dimensional operators. Let us start with a characterization of random Hilbert-Schmidt operators.

**Theorem 2.1.** A strong random operator $A$ is a random Hilbert-Schmidt operator if and only if for some orthonormal basis $\{e_n; n \geq 1\}$ in $H$

$$\sum_{n=1}^{\infty} ||Ae_n||^2 < \infty. \tag{1}$$

**Proof.** If $A$ is a random Hilbert-Schmidt operator, then the condition of the theorem holds obviously. Suppose that strong random operator $A$ satisfies (1). Define a subset $\Omega_0$ of the probability space as follows

$\forall \omega \in \Omega_0, \alpha_1, ..., \alpha_n \in R:$

$$\sum_{k=1}^{n} \alpha_k A e_k(\omega) = A(\sum_{k=1}^{n} \alpha_k e_k)(\omega),\ \sum_{n=1}^{\infty} ||Ae_n||^2(\omega) < \infty.$$

Now for every $\omega \in \Omega_0$ define a Hilbert-Schmidt operator $A_\omega$ as follows

$$A_\omega x = \sum_{n=1}^{\infty} (x, e_n)(Ae_n)(\omega).$$

For $\omega \notin \Omega_0$ put $A_\omega = 0$. Then $A$ satisfies Definition 2.2 with the family $\{A_\omega, \omega \in \Omega\}$. $\square$

The previous statement allows us to characterize random finite-dimensional projections.

**Theorem 2.2.** Suppose that a strong random operator $A$ satisfies the condition of Theorem 2.1 and the following conditions hold:

- for all $x, y \in H \quad (Ax, y) = (Ay, x),\ (Ax, x) \geq 0,$

- $A = A^2$.

Then $A$ is a random finite-dimensional projection.

**Proof.** The proof follows immediately from the well-known characterization of projections in a Hilbert space and the fact that any Hilbert-Schmidt projection is finite-dimensional. □

Let us consider some examples of random finite-dimensional projections.

**Example 2.2.** Let $n(t),\ t \in [0;1]$ be a Poisson process. Denote by $\tau_1, ..., \tau_\nu$ the subsequent jumps of $n$. The intervals $[0;\tau_1), \ldots, [\tau_\nu;1]$ generate a finite $\sigma-$field $\mathcal{A}$. Define a random operator $A$ in $L_2([0;1])$ as a conditional expectation with respect to the $\sigma-$field $\mathcal{A}$. Then $A$ is a random finite-dimensional projection.

**Example 2.3.** Let $(X, \rho)$ be a Polish space and $\mu$ be a probability measure on the Borel $\sigma-$field in $X$. Consider a measurable map $\phi : X \times \Omega \mapsto X$ such, that for every $\omega \in \Omega$ the image $\phi(X, \omega)$ contains a finite number of elements. Define a random operator $A$ in $L_2(X, \mu)$ as a conditional expectation with respect to the $\sigma-$field generated by $\phi$. Then $A$ is a finite-dimensional random projection related to $\phi$.

**Remark.** If the map $\phi$ has not the finite image the operator $A$ still will be well-defined random projection but not finite-dimensional.

## 3 Semigroups of projections.

In this section we introduce the notion of semigroups of random projections which is the main object of investigation in the article.

**Definition 3.1.** A family of random bounded operators $\{G_{s,t},\ 0 \le s \le t < \infty\}$ is referred to as a semigroup if the following conditions hold:

1. For any $s, t, r \ge 0 :\ G_{s,t}$ and $G_{s+r,t+r}$ are equidistributed.
2. For any $x \in H :\ E||G_{0,t}x - x||^2 \mapsto 0,\ n \mapsto \infty$.
3. For any $0 \le s_1 \le \cdots \le s_n < \infty :\ G_{s_1,s_2}, \ldots, G_{s_{n-1},s_n}$ are independent.
4. For any $r \le s \le t :\ G_{r,s}G_{s,t} = G_{r,t},\ G_{r,r} = I$, where $I$ is an identity operator.

To illustrate a connection of the stochastic semigroup with stochastic flows consider the following example. Suppose, that $X$ is a Polish space. Denote by $\mathbf{M}$ the space of all finite signed measures on the Borel $\sigma-$field in $X$ equipped with the topology of weak convergence. $\mathbf{M}$ is a linear topological space.

**Definition 3.2.** A family of measurable mappings $\phi_{s,t} : X \times \Omega \mapsto X$, $0 \le s \le t < \infty$ is referred to as a random flow on $X$ if the following conditions hold:

- For any $0 \le s_1 \le s_2 \le \ldots s_n < \infty :\ \phi_{s_1,s_2}, \ldots, \phi_{s_{n-1},s_n}$ are independent.
- For any $s, t, r \ge 0 : \phi_{s,t}$ and $\phi_{s+r,t+r}$ are equidistributed.

- For any $r \le s \le t$ and $u \in X$ : $\phi_{r,s}\phi_{s,t}(u) = \phi_{r,t}(u)$, $\phi_{r,r}$ is an identity map.
- For any $u \in X : \phi_{0,t}(u) \mapsto u$ in probability when $t \mapsto 0$.

**Example 3.1.** Suppose that the random flow $\{\phi_{s,t},\ 0 \le s \le t < \infty\}$ on $X$ has the following additional property. For any $t$ the function $\phi_{0,t}$ is continuous with probability one. Define the operator $G_{s,t}$ in $\mathbf{M}$ by the formula

$$G_{s,t}(\mu) = \mu\phi_{s,t}^{-1}.$$

It can be easily checked that the family $\{G_{s,t},\ 0 \le s \le t < \infty\}$ satisfies the analog of Definition 3.1 for a linear topological space.

In the next example we consider a semigroup of random finite-dimensional projections in Hilbert space.

**Example 3.2.** Let $H$ be Hilbert space with an orthonormal basis $\{e_k, k \ge 1\}$. Consider the sequence $\{n_k,\ k \ge 1\}$ of independent Poisson processes with intensities $\{\lambda_k,\ k \ge 1\}$. Suppose that

$$\forall \rho > 0 : \quad \sum_{k=1}^{\infty} \exp(-\rho\lambda_k) < +\infty \tag{2}$$

Define for every $k \ge 1$ and $0 \le s \le t$

$$\nu_{s,t}^k = \begin{cases} 0,\ n_k(t) - n_k(s) > 0, \\ 1,\ n_k(t) - n_k(s) = 0. \end{cases}$$

Finally define the projection $G_{s,t}$ as follows

$$G_{s,t}(u) = \sum_{k=1}^{\infty} (u, e_k)\nu_{s,t}^k e_k.$$

Condition (2) implies that $G_{s,t}$ is a finite-dimensional projection with probability one. The conditions of Definition 3.1 trivially hold.

The next lemma shows that deterministic semigroup of finite-dimensional projections does not exist.

**Lemma 3.1.** Suppose that $\{G_t,\ 0 \le t < \infty\}$ is a strongly continuous semigroup of bounded operators in separable Banach space $\mathcal{B}$. Assume that $\dim G_t(\mathcal{B}) < \infty$ for every $t > 0$. Then $\dim \mathcal{B} < \infty$.

**Proof.** Define the function $\nu(t) = \dim G_t(\mathcal{B})$, $t > 0$. It is clear that this function is decreasing, takes integer values and

$$\lim_{t \to 0} \nu(t) = +\infty.$$

Let $t_0$ be a positive point of jump for the function $\nu$. Then there exists a nonzero element $x \in G_{t_0}(\mathcal{B})$ such that $x \notin G_t(\mathcal{B})$ for all $t > t_0$. Since $\{G_t,\ 0 \le t < \infty\}$ is a semigroup, then for arbitrary $s > 0$ $G_s(x) = 0$. This contradicts to the strong continuity of $G$. □

Actually Example 3.2 describes the unique possibility of the construction of the semigroup of random finite-dimensional projections in Hilbert space.

**Theorem 3.1.** Let $\{G_{s,t},\ 0 \le s \le t < \infty\}$ be a semigroup of random finite-dimensional projections in separable Hilbert space H. Then there exists an orthonormal basis $\{e_k,\ k \ge 1\}$ in $H$ and Poisson processes $\{n_k,\ k \ge 1\}$ which have jointly independent increments, such that

$$G_{s,t}(u) = \sum_{k=1}^{\infty} (u, e_k) \nu^k_{s,t} e_k,$$

where for every $k$ $\nu^k_{s,t}$ is built from $n_k$ exactly in the same way as in Example 3.2.

**Proof.** Consider two projections $R_1,\ R_2$ in $H$ such that their product $Q = R_1 R_2$ is a projection. Then $R_2 R_1 = R_1 R_2$. To check this relation introduce the notations $R_i(H) = L_i,\ i = 1, 2$. Suppose, that $u \in H$ is such that $||u|| = ||R_1 R_2 u||$. Then $||u|| = ||R_2 u||$, $||u|| = ||R_1 u||$ i.e. $u \in L_1 \cap L_2$. This means that $Q$ is a projection on $L_1 \cap L_2$. Since the subspace $L_2$ can be represented as $L_2 = L_2^{'} \oplus L_1 \cap L_2$, then $R_1(L_2^{'}) = \{0\}$. This imply that $R_2 R_1 = R_1 R_2$. Now consider the semigroup $\{G_{s,t},\ 0 \le s \le t < \infty\}$. From the definition of the semigroup and the previous considerations one can conclude that for every $s_1 \le s_2 \le s_3$ the operators $G_{s_1,s_2}$ and $G_{s_2,s_3}$ commute with probability one. Since the space $H$ is separable, then there exists a sequence $\{\triangle_{kn};\ k,\ n \ge 1\}$ of increasing partitions of the probability space which generates $\sigma(G_{s_1,s_2})$ . The following integrals define bounded operators in $H$

$$Q^{kn}_{s_1,s_2} u = \frac{1}{P(\triangle_{kn})} \int_{\triangle_{kn}} G_{s_1,s_2} u P(d\omega),\ u \in H.$$

The norm of $Q^{kn}_{s_1,s_2}$ is less or equal to one. Consider the following random operators

$$G^n_{s_1,s_2} = \sum_{k=1}^{\infty} Q^{kn}_{s_1,s_2} 1_{\triangle_{kn}},\ n \ge 1.$$

It can be easily checked that for every $u \in H$ with probability one

$$G^n_{s_1,s_2} u \to G_{s_1,s_2} u,\ n \to \infty.$$

The random operators $G^n_{s_1,s_2}$ take commuting values. Namely, for the arbitrary $k,\ n,\ s_1,\ s_2$ and $l,\ m,\ t_1,\ t_2$ we have

$$Q^{kn}_{s_1,s_2} Q^{lm}_{t_1,t_2} = Q^{lm}_{t_1,t_2} Q^{kn}_{s_1,s_2}.$$

To prove this relation note that the values of $Q^{kn}_{s_1,s_2}$ depend only from the distribution of $G_{s_1,s_2}$. Consequently, it is enough to consider the case, when $0 = s_1 < s_2 = t_1 < t_2 = s_2 - s_1 + t_2 - t_1$. Now the subsets which we use in the definition of $Q^{kn}_{s_1,s_2}$ and $Q^{lm}_{t_1,t_2}$ are independent. Hence

$$\int_{\triangle_{kn}} G_{s_1,s_2} P(d\omega) \int_{\triangle_{lm}} G_{t_1,t_2} P(d\omega) =$$

$$= \int_{\Omega} G_{s_1,s_2} G_{t_1,t_2} 1_{\triangle_{kn}} 1_{\triangle_{lm}} P(d\omega) G_{s_1,s_2} P(d\omega) =$$

$$= \int_{\Omega} G_{t_1,t_2} G_{s_1,s_2} 1_{\triangle_{lm}} 1_{\triangle_{kn}} P(d\omega) = \int_{\triangle_{lm}} G_{t_1,t_2} P(d\omega) \int_{\triangle_{kn}} G_{s_1,s_2} P(d\omega).$$

Also note that the operators $Q^{kn}_{s_1,s_2}$ are self-adjoint and non-negative. Consequently, one can build a countable family $\Gamma$ of commuting self-adjoint non-negative operators such that random projections $G_{s_1,s_2}$ can be approximated by random operators with the values from $\Gamma$. Let us verify that we can choose the family $\Gamma$ in such a way that it consists of the nuclear operators. Since our projections are finite-dimensional, then for arbitrary $s_1 \leq s_2$

$$trG_{s_1,s_2} < +\infty.$$

Then truncating the sets $\triangle_{kn}$ to their intersections with the sets $\{trG_{s_1,s_2} < R\}$ one can achieve that the family $\Gamma$ will consist of the nuclear operators. Finally denote by $\{e_n;\ n \geq 1\}$ the orthonormal basis in $H$ which is a common eighenbasis for all operators from $\Gamma$ . Denote by $\check{\Gamma}$ the family of all projections onto subspaces generated by a finite number of the vectors from $\{e_n;\ n \geq 1\}$. Then every operator $G_{s_1,s_2}$takes values in $\check{\Gamma}$.

Now describe a random structure of $G_{s_1,s_2}$. For the fixed $n$ consider a random process in $H$

$$\xi_n(t) = G_{0,t} e_n,\ t \geq 0.$$

By the definition of a random semigroup $\xi_n$ is a homogeneous Markov process. From the other side there exists a random moment $\tau_n$ such that

$$\xi_n(t) = e_n,\ t < \tau_n,\ \xi_n(t) = 0,\ t \geq \tau_n.$$

The random moment $\tau_n$ has an exponential distribution with parameter $\lambda_n$. Define the random measure $\nu$ on the product $[0;+\infty) \times N$ as follows

$$\nu((s;t] \times A) = \sum_{n \in A} 1_{\{G_{s,t} e_n = 0\}}.$$

Note that the measure $\nu$ has independent values on the sets which have disjoint projections on $[0;+\infty)$. For arbitrary $n \geq 1$ the process $\{\nu((0;t] \times n);\ t \geq 0\}$ is Poissonian with the parameter $\lambda_n$. Also note that for arbitrary $t > 0$

$$P\{\exists n_0\, \forall n \geq n_0 :\ \xi_n(t) = 0\} = 1.$$

The theorem is proved. □

# 4 Widths of compact sets defined with respect to the semigroups of projections.

The last statement of the previous section gives us a description of the semigroups of random finite-dimensional projections in terms of integer-valued random measure. To understand the relationships between this measure and geometrical properties of the semigroup consider the asymptotic of widths of compact sets with respect to the images of the semigroup projections. Let $\{G_{s,t},\ 0 \le s \le t < \infty\}$ be a random semigroup of finite-dimensional projections and $K$ be a compact subset of $H$. We will investigate the behavior of the value

$$\varsigma_K(t) = \max_{x\in K} ||x - G_{0,t}x||$$

as $t \to 0$. The value $\varsigma_K(t)$ is exactly the width of $K$ with respect to the linear subspace $G_{0,t}(H)$ [3]. Theorem 3.1 implies that with probability one $G_{0,t}$ strongly converges to identity when $t \to 0$. Consequently, with probability one $\varsigma_K(t) \to 0,\ t \to 0$. We will investigate the rate of the convergence. Let us consider the case, when the processes $\{\xi_n;\ n \ge 1\}$ which arose in the description of the structure of the semigroup are independent and the compact $K$ has a simple description in the basis $\{e_n;\ n \ge 1\}$.

**Example 4.1.** Suppose that $\lambda_n = n,\ n \ge 1$ and

$$K = \{x:\ (x,\ e_n)^2 \le \frac{1}{n^2},\ n \ge 1\}.$$

Now

$$\varsigma_K(t)^2 = \sum_{n=1}^{\infty} \frac{\xi_n(t)}{n^2}.$$

One can check that

$$E\varsigma_K(t)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2}(1 - \exp(-nt)) = \sum_{n=1}^{\infty} \int_0^t \int_s^{+\infty} \exp(-nr)drds =$$

$$\int_0^t \int_s^{+\infty} \sum_{n=1}^{\infty} \exp(-nr)drds = \int_0^t \int_s^{+\infty} \frac{\exp(-r)}{1-\exp(-r)}drds.$$

Consequently,

$$E\varsigma_K(t)^2 = \int_0^t ln(1 - \exp(-s))ds \sim tlnt,\ t \to 0.$$

In the similar way the forth moment can be estimated

$$E\varsigma_K(t)^4 = E\sum_{i,j=1}^{\infty} \frac{\xi_i(t)\xi_j(t)}{i^2j^2} = \sum_{i=1}^{\infty} \frac{1}{i^4}(1 - \exp(-it)) +$$

$$+\sum_{i\ne j}^{\infty} \frac{1}{i^2j^2}(1-\exp(-it))(1-\exp(-jt)) = (E\varsigma_K(t)^2)^2 + \sum_{i=1}^{\infty} \frac{1}{i^4}(1-\exp(-it))\exp(-it).$$

Note that

$$\sum_{i=1}^{\infty} \frac{1}{i^4}(1 - \exp(-it))\exp(-it) \le \sum_{i=1}^{\infty} \frac{1}{i^4}(1 - \exp(-it)) \le \sum_{i=1}^{\infty} \frac{t}{i^3} = ct.$$

Hence

$$V\varsigma_K(t)^2 = o(tlnt),\ t \to 0.$$

Finally, one can conclude that

$$P - \lim_{t\to 0} \frac{\varsigma_K(t)}{\sqrt{tlnt}} = 1.$$

In the next example we consider another type of compact and the same semigroup.

**Example 4.2.** Consider the following compact set

$$K = \{x : \sum_{n=1}^{\infty} n^2 (x,\ e_n)^2 \le 1\}.$$

For the same semigroup $\{G_{s,t},\ 0 \le s \le t < \infty\}$ as in the previous example let us study the behavior of $\varsigma_K(t)$. Now

$$\varsigma_K(t)^2 = \max_{n:\, \xi_n(t)=0} \frac{1}{n^2}.$$

Let us find the expectation

$$E\varsigma_K(t)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \prod_{i=1}^{n-1} \exp(-it)(1-\exp(-nt)) = \sum_{n=2}^{\infty} \frac{1}{n^2} \exp(-\frac{n(n-1)}{2}t)\times$$

$$\times(1-\exp(-nt)) + 1 - \exp(-t).$$

The asymptotic behavior when $t \to 0$ of the last summand is trivial. To obtain the estimation from belaw let us rewrite the sum as follows

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \exp(-\frac{n(n-1)}{2}t)(1-\exp(-nt)) = \sum_{n=2}^{\infty} \frac{1}{n} \exp(-\frac{n(n-1)}{2}t) \int_0^t \exp(-ns)ds \ge$$

$$\ge \sum_{n=2}^{\infty} \frac{1}{n} \exp(-\frac{n(n-1)}{2}t)t\exp(-nt) \ge t \int_2^{+\infty} \frac{1}{x} \exp(-\frac{x(x-1)}{2}t - xt)dx.$$

For arbitrary sufficiently small positive $\alpha$ there exists such positive $c$ that

$$\int_c^{+\infty} \frac{1}{x} \exp(-\frac{x(x-1)}{2}t - xt)dx \ge \int_c^{+\infty} \frac{1}{x} \exp(-(\frac{1}{2}+\alpha)x^2 t)dx =$$

$$= \int_{c\sqrt{(\frac{1}{2}+\alpha)t}}^{+\infty} \frac{1}{x} \exp(-x^2)dx \sim \frac{1}{2} \ln t,\ t \to 0.$$

For the upper estimate let us proceed in the same way

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \exp(-\frac{n(n-1)}{2}t)(1-\exp(-nt)) = \sum_{n=2}^{\infty} \frac{1}{n} \exp(-\frac{n(n-1)}{2}t) \int_0^t \exp(-ns)ds \le$$

$$\leq \sum_{n=2}^{\infty} \frac{1}{n} \exp(-\frac{n(n-1)}{2}t)t \leq t \int_1^{+\infty} \frac{1}{x} \exp(-\frac{x(x-1)}{2}t)dx \leq$$

$$\leq t \int_1^c \frac{1}{x} \exp(-\frac{x(x-1)}{2}t)dx + t \int_c^{+\infty} \frac{1}{x} \exp(-(\frac{1}{2}-\alpha)x^2 t)dx =$$

$$= t \int_1^c \frac{1}{x} \exp(-\frac{x(x-1)}{2}t)dx + t \int_{c\sqrt{(\frac{1}{2}-\alpha)t}}^{+\infty} \frac{1}{x} \exp(-x^2)dx \sim \frac{1}{2} t \ln t, \ t \to 0.$$

To understand a piecewise behavior of $\varsigma_K(t)$ when $t \to 0$ let us introduce a family $\{\tau_n;\ n \geq 1\}$ of independent exponentially distributed random variables with intensities $n$. Then the sequence of random processes $\{\xi_n(t);\ n \geq 1\}$ is equidistributed with the sequence $\{1_{[0;\,t]}(\tau_n);\ n \geq 1\}$. Consequently, for a continuous strictly decreasing positive function $a$ and a constant $c > 0$

$$P\{\liminf_{t\to 0} \frac{1}{\varsigma_K(t)a(t)} > c\} = P\{\exists N\ \forall n \geq N:\ \min_{j=1,\dots,n} \tau_j > a^{-1}(\frac{n}{c})\} =$$

$$P\{\exists N\ \forall n \geq N:\ \tau_n > a^{-1}(\frac{n}{c})\}.$$

The last probability equals to one if and only if the infinite product

$$\prod_{n=1}^{\infty} P\{\tau_n > a^{-1}(\frac{n}{c})\}$$

converges. This condition is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} P\{\tau_n < a^{-1}(\frac{n}{c})\} = \sum_{n=1}^{\infty} (1 - \exp(-n) a^{-1}(\frac{n}{c})).$$

This series converges if and only if

$$\sum_{n=1}^{\infty} n a^{-1}(\frac{n}{c}) < +\infty.$$

This inequality holds simultaneously for all positive $c$, which means that

$$\lim_{t\to 0} \varsigma_K(t)a(t) = 0.$$

For example, this condition is true for the function

$$a^{-1}(n) = \frac{1}{n^2 \ln^2 n}.$$

The upper bound can be obtained using the equality

$$P\{\frac{1}{\varsigma_K(t)} \geq n\} = \exp(-\frac{n(n-1)}{2}t),\ n \geq 2.$$

Define a function $\varphi$ by the formula

$$\varphi(t)=\sqrt{\frac{2}{t}llnt}$$

for sufficiently small positive $t$ with the usual agreement for $lln$. Then, taking $t_n=q^n$ for some $0<q<1$ one can get

$$\sum_{n=1}^{\infty}P\{\frac{1}{\varsigma_K(t_n)}\geq(1+\varepsilon)\varphi(t_n)\}<+\infty$$

for any positive $\varepsilon$. Consequently,

$$\limsup_{n\to\infty}\frac{1}{\varsigma_K(t_n)\varphi(t_n)}\leq 1.$$

Since the function $\varsigma_K$ is increasing, then for $t_{n+1}<t\leq t_n$

$$\frac{1}{\varsigma_K(t)\varphi(t)}\leq\frac{1}{\varsigma_K(t_{n+1})\varphi(t_{n+1})}\frac{\varphi(t_{n+1})}{\varphi(t_n)}.$$

Hence

$$\limsup_{t\to 0}\frac{1}{\varsigma_K(t)\varphi(t)}\leq\frac{1}{q}.$$

Finally

$$\liminf_{t\to 0}\varsigma_K(t)\varphi(t)\geq 1.$$

In general the structure of $K$ can be more complicated and does not allow an explicit form for $\varsigma_K(t)$. In some cases one can have only the estimation for the Kolmogorov width for $K$

$$d_n(K)=\inf_{dimL=n}\max_{x\in K}\rho(x,\ K),$$

where inf is taken over all subspaces $L$ of $H$, which have the dimension $n$. From this reason it is useful to estimate the growth of $\dim G_{0,t}(H)$ when $t\to 0$. For the semigroup from the previous examples such an estimation can be obtained as follows.

**Example 4.3.** Define $\alpha(t)=\dim G_{0,t}(H)$. Using the random variables $\{\tau_n;\ n\geq 1\}$ which were defined in Example 4.2 one can check, that

$$P\{\alpha(t)\geq n\}=P\{\exists\, k_1<k_2<\ldots<k_n:\ \tau_{k_1}\geq t,\ \tau_{k_2}\geq t,\ldots,\ \tau_{k_n}\geq t\}.$$

Consequently,

$$P\{\alpha(t)\geq n\}\leq P\{\tau_1\geq t,\ \tau_2\geq t,\ldots,\ \tau_n\geq t\}+1-P\{\forall\, k>n:\ \tau_k<t\}\leq$$

$$\leq\exp(-nt)+1-\prod_{k=n+1}^{\infty}(1-\exp(-kt))\leq\exp(-nt)+\sum_{k=n+1}^{\infty}\exp(-kt)\leq$$

$$\leq \exp(-nt) + \frac{\exp(-nt)}{1-\exp(-t)} \leq \exp(-nt)(1+(1-e^{-1})\frac{1}{t})$$

for $t \in (0;\, t)$. Taking $t_k = \frac{1}{k}$, $n_k = [(2+\delta)k\ln k]$ for positive $\delta$ one can get that

$$\limsup_{k\to\infty} \frac{\alpha(\frac{1}{k})}{k\ln k} \leq 2.$$

Here $[x]$ means an integer part of $x$. Using the monotonicity of $\alpha$ one can conclude that with probability one

$$\limsup_{t\to 0} \frac{t\alpha(t)}{2|\ln t|} \leq 1.$$

To obtain an estimation from below let us denote

$$c(t) = \prod_{j=1}^{\infty}(1-\exp(-jt)) = P\{\alpha(t)=0\}.$$

Note, that

$$\ln c(t) = \sum_{j=1}^{\infty}\ln(1-\exp(-jt)) \leq -\sum_{j=1}^{\infty}\exp(-jt) = -\frac{\exp(-t)}{1-\exp(-t)} \sim -\frac{1}{t},\ t\to 0.$$

Consequently,

$$\lim_{t\to 0} t\ln c(t) = -1.$$

Now

$$P\{\alpha(t)<n\} = \sum_{k=0}^{n-1}P\{\alpha(t)=k\} = \prod_{j=1}^{\infty}(1-\exp(-jt))(1+$$

$$+\sum_{k=1}^{n-1}\sum_{1\leq j_1<j_2<\ldots<j_k}\prod_{s=1}^{k}\exp(-j_s t)(1-\exp(-j_s t))^{-1}).$$

Consider the series

$$\sum_{j=2}^{\infty}\exp(-jt)(1-\exp(-jt))^{-1} \leq \int_1^{\infty}\exp(-xt)(1-\exp(-xt))^{-1}dx =$$

$$= -\frac{1}{t}\ln(1-\exp(-t)).$$

Hence

$$P\{\alpha(t)<n\} \leq \prod_{j=1}^{\infty}(1-\exp(-jt))\sum_{k=0}^{n-1}\frac{1}{k!}(\exp(-t)(1-\exp(-t))^{-1}-\frac{1}{t}\ln(1-\exp(-t)))^k.$$

Consequently, for arbitrary $\delta > 1$, $c > 1$ for sufficiently small $t$

$$P\{\alpha(t) < n\} \leq c(t)c^n \frac{1}{t^n} |\ln t|^n \leq$$

$$\leq \exp(-\frac{\delta}{t} + n \ln c + n|\ln t| + n \ln |\ln t|).$$

For $\beta > 2$ consider the sequences $\{t_k = \frac{1}{k};\ k \geq 1\}$ and $\{n_k = \frac{k}{\beta \ln k};\ k \geq 2\}$. Then

$$\sum_{k=2}^{\infty} P\{\alpha(t_k) < n_k\} < +\infty.$$

Using the monotonicity of $\alpha$ as above one can get that with probability one

$$\liminf_{t \to 0} \alpha(t) t |\ln t| \geq \frac{1}{2}.$$

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