Local well-posedness and blow up criterion for the Inviscid Boussinesq system in Hölder spaces

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Abstract: We prove the local in time existence and a blow up criterion of solution in the Hölder spaces for the inviscid Boussinesq system in $R^N, N \geq 2$, under the assumptions that the initial values $\theta_0, u_0 \in C^r$, with $r > 1$.

Key Words: inviscid Boussinesq system, local well-posedness, blow-up criterion

AMS(2000)Subject Classification: 35Q35, 76B03

1. Introduction

The Cauchy problem for the Boussinesq system in $R^N(N \geq 2)$ can be written as

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = g, \\ \partial_t u + u \cdot \nabla u + \nabla \Pi - \nu \Delta u = f, \\ \text{div} u = 0, \end{cases} \tag{1.1}$$

with initial data

$$\theta|_{t=0} = \theta_0, \quad u|_{t=0} = u_0. \tag{1.2}$$

Here $u(x,t), (x,t) \in R^N \times (0, \infty), N \geq 2$, is the velocity vector field, $\theta(x,t)$ is the scalar temperature, $\Pi(x,t)$ is the scalar pressure, $f(x,t)$ is the external forces, which is a vector function, and g is a known scalar function. $\nu \geq 0$ is the kinematic viscosity, and $\kappa \geq 0$ is the thermal diffusivity.

The Boussinesq system is extensively used in the atmospheric sciences and oceanographic turbulence in which rotation and stratification are important (see [18, 19] and references therein). When $\kappa > 0$ and $\nu > 0$, the 2D Boussinesq system with $g = 0, f = \theta e_2 (e_2 = (0,1))$ has been well-understood (see [3], [14],

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[1]). When $\kappa > 0$ and $\nu = 0$ or $\kappa = 0$ and $\nu > 0$, the Boussinesq system is usually called the partial viscosity one. In these cases, the Boussinesq system has also been extensively and successfully studied. In particular, in the case $\kappa > 0$ and $\nu = 0$, D. Chae ([4]) proved that the Boussinesq system is globally well-posedness in R^m for any $m \geq 3$ and this result was extended by T. Hmidi and S. Keraani ([15]), R. Danchin and M. Paicu ([11]) to rough initial data in Besov space framework. In the case $\kappa = 0$ and $\nu > 0$, the global well-posedness was proved by D. Chae ([4]), T. Y. Hou and C. Li ([16]) in $H^m(R^2))$ space with $m \geq 3$.

When $\kappa = 0$ and $\nu = 0$, the Boussinesq system (1.1) becomes the inviscid one. In this case, it is clear that if $\theta \equiv 0$, the inviscid Boussinesq system reduces to the classical Euler equations. And the two-dimensional Boussinesq system can be used a model for the three-dimensional axisymmetric Euler equations with swirl (see [12]). However, the global well-posedness problem of the inviscid Boussinesq system is still completely open in general (an exceptional case is the two-dimensional Euler equations which correspond to $\theta \equiv 0$, see [8] and references therein). Local existence and blow-up criteria have been established for the inviscid Boussinesq system (see [6, 7, 13, 20] and references therein). In particular, D. Chae, S. K. Kim and H. S. Nam considered in [6] the inviscid Boussinesq system with $g = 0$ and $f = \theta f_1$, where f_1 satisfies that $curl f = 0$ and $f \in L^\infty_{loc}([0, \infty); W^{1,\infty}(R^2))$. They proved that there exists a unique and local $C^{1+\gamma}$ solution of the inviscid Boussinesq system with initial data $u_0, \theta_0 \in C^{1+\gamma}, \omega_0, \Delta\theta_0 \in L^q$ for $0 < \gamma < 1$ and $1 < q < 2$, where ω_0 is the initial vorticity of the initial velocity u_0. They also proved a blow-up criterion for the local solution, which says that the gradient of the passive scalar θ controls the breakdown of $C^{1+\gamma}$ solutions of the Boussinesq system.

In this paper, we devote to the following inviscid Boussinesq system:

$$\begin{cases} \partial_t\theta + u \cdot \nabla\theta = 0, \\ \partial_t u + u \cdot \nabla u + \nabla\Pi = \theta e_N, \\ \mathrm{div} u = 0, \end{cases} \tag{1.3}$$

with the initial data

$$\theta|_{t=0} = \theta_0, \quad u|_{t=0} = u_0. \tag{1.4}$$

In [8], the local in time existence and blow up criterion for the Euler equations with the initial data $u_0 \in C^r(r > 1)$ was proved. Here we will extend the approaches and results in [8] to the inviscid Boussinesq system (1.3). It should be noted that our results will relax the initial conditions in [6]. More precisely, we prove that the inviscid Boussinesq system has a local and unique $C^r(r > 1)$-solution under the assumptions that the initial data $u_0, \theta_0 \in C^r(r > 1)$. It does not require that $\omega_0, \Delta\theta_0 \in L^q$ for some $1 < q < 2$ which are needed in [6].

The plan of the paper is as follows. In Section 2, we give some preliminaries and our main results. To prove our main results, we first present a priori estimates in Section 3 and then in Section 4 we construct approximate solutions and furthermore prove that they are Cauchy sequences in appropriate Hölder spaces. Lastly, in Section 5, we give the proof of the blow-up criterion.

2. Preliminaries and Main Results

Let us start with the definition of the dyadic decomposition of the full space R^N (see[8]).

Proposition 2.1 Denote by $\mathcal{C}$ the annulus of centre 0, short radius 3/4 and long radius 8/3. Then there exist two positive radial functions $\chi$and ϕ belonging respectively to $C_0^\infty(B(0,4/3))$ and $C_0^\infty(\mathcal{C})$ such that

$$\chi(\xi)+\sum_{q\geq -1}\varphi(2^{-q}\xi)=1,$$

$$|p-q|\geq 2\Rightarrow \mathrm{supp}\varphi(2^{-q}\cdot)\cap\mathrm{supp}\varphi(2^{-p}\cdot)=\emptyset,$$

$$q\geq 1\Rightarrow \mathrm{supp}\chi\cap\mathrm{supp}\varphi(2^{-q}\cdot)=\emptyset.$$

If $\widetilde{\mathcal{C}}=B(0,2/3)+\mathcal{C}$, then $\widetilde{\mathcal{C}}$ is an annulus and we have

$$|p-q|\geq 5\Rightarrow 2^p\widetilde{\mathcal{C}}\cap 2^q\mathcal{C}=\emptyset,$$

$$\frac{1}{3}\leq\chi^2(\xi)+\sum_{q\geq 0}\varphi^2(2^{-q}\xi)\leq 1.$$

Notation 2.1 For the inhomogeneous Besov spaces, we have the notations

$$h=\mathcal{F}^{-1}\varphi \quad\text{and}\quad \widetilde{h}=\mathcal{F}^{-1}\varphi,$$

$$\Delta_{-1}u=\chi(D)u=\mathcal{F}^{-1}(\chi(\xi)\widehat{u}(\xi)),$$

$$\text{if } q\geq 0,\ \Delta_q u=\varphi(2^{-q}D)u=2^{qN}\int h(2^q y)u(x-y)dy,$$

$$\text{if } q\leq -2,\ \Delta_q u=0,$$

$$S_q u=\sum_{p\leq q-1}\Delta_p u=\chi(2^{-q}D)u=2^{qN}\int\widetilde{h}(2^q y)u(x-y)dy.$$

The product uv can be formally divided into three parts as follows (see [2]) :

$$uv=T_u v+T_v u+R(u,v),$$

where

$$T_u v=\sum_q S_{q-1}\triangle_q v,\qquad R(u,v)=\sum_q\triangle_q u(\sum_{j=-1}^{1}\triangle_{q+j}v),$$

$T_u v$ is called paraproduct of v by u and $R(u,v)$ the remainder term.

Proposition 2.2 (Bernstein's Inequality) Let (r_1,r_2) be a pair of strictly positive numbers such that $r_1<r_2$. There exists a constant C such that for every

nonnegative integer k, and for every $1 \leq a \leq b$ and for all function $u \in L^a(R^N)$, we have

$$\mathrm{supp}\hat{u} \in B(0, \lambda r_1) \quad \Rightarrow \quad \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+N(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a},$$

$$\mathrm{supp}\hat{u} \in \mathcal{C}(0, \lambda r_1, \lambda r_2) \quad \Rightarrow \quad C^{-k}\lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a},$$

where $\hat{u}$ denotes the Fourier transform of u; $B(0,r)$ refers a ball with the center 0 and radius r; and $\mathcal{C}(0, r_1, r_2)$ denotes analogous as before a ring of center 0, short radius r_1 and long radius r_2.

Definition 2.1 Let $s \in R,\ p,\ q \in [1, \infty]$. The inhomogeneous Besov space $B^s_{p,q}$ and the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined as a space of $f \in S'$ (tempered distributions) such that

$$\|f\|_{B^s_{p,q}} = (\sum_{j=-1}^{\infty} 2^{jqs} \|\Delta_q f\|^q_{L^p})^{\frac{1}{q}} < \infty, \quad \|f\|_{\dot{B}^s_{p,q}} = (\sum_{j=-\infty}^{\infty} 2^{jqs} \|\dot{\Delta}_q f\|^q_{L^p})^{\frac{1}{q}} < \infty.$$

In case $q = \infty$, the expressions are understood as

$$\|f\|_{B^s_{p,\infty}} = \sup_{j \geq -1} 2^{js} \|\Delta_j f\|_{L^p}, \quad \|f\|_{\dot{B}^s_{p,\infty}} = \sup_{j \in Z} 2^{js} \|\dot{\Delta}_j f\|_{L^p}.$$

And, in the case $p = q = \infty$, we write $B^s_{\infty,\infty} \triangleq C^r$, and $\dot{B}^s_{\infty,\infty} \triangleq \dot{C}^r$. So the norms are written as $\|u\|_{\dot{B}^r_{\infty,\infty}} \triangleq \|u\|_{\dot{C}^r}$ and $\|u\|_{C^r} \triangleq \|u\|_r$.

Lemma 2.1 Let v be a smooth free-divergence vector field and f a smooth function. Then we have

$$\|[v \cdot \nabla, \Delta_q] f\|_{L^\infty} \leq C(r) 2^{-qr} \|f(t)\|_r \|\nabla v\|_{L^\infty} (q \geq -1, r > 1).$$

The detailed proof of the lemma is referred to [8].

The following embeddings will be used later.

Lemma 2.2

(1) Let $r > \frac{N}{p} + 1$, then $B^{r-1}_{p,q} \hookrightarrow L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$. When $p = q = \infty$, we have the special case $B^r_{\infty,\infty} \hookrightarrow L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$, with $r > 0$.

(2) Let $s > 0,\ 1 \leq p, q < \infty$, and $s_1 - \frac{N}{p_1} = s_2 - \frac{N}{p_2},\ s_1, s_2 \in R,\ s_1 < s_2$, then we get $B^s_{p,q} \hookrightarrow \dot{B}^s_{p,q}$ and $\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q}$.

(3) $B^r_{\infty,\infty} \hookrightarrow \dot{B}^1_{\infty,1}$ with $r > 1$, and $B^r_{\infty,\infty} \hookrightarrow \dot{B}^r_{\infty,\infty}$ for $r > 0$.

Proof. (1) is proved in [5] and (2) is proved in [8] and [10].

To prove (3), in view of the definition of the norm in the homogeneous Besov space, we estimate

$$
\begin{aligned}
\|u\|_{\dot{B}^1_{\infty,1}} &= \sum_{q\geq 0} 2^q\|\dot{\Delta}_q u\|_{L^\infty} + \sum_{q\leq -1} 2^q\|\dot{\Delta}_q u\|_{L^\infty} \\
&\leq \|u\|_{B^1_{\infty,1}} + (\sum_{q\leq -1} 2^q)\|u\|_{L^\infty} \\
&\leq \|u\|_{B^1_{\infty,1}} + \|u\|_{L^\infty} \\
&\lesssim \|u\|_r,
\end{aligned}
$$

where we used the embedding $B^r_{\infty,\infty} \hookrightarrow L^\infty (r > 0)$.

Similarly, we obtain

$$
\begin{aligned}
\|u\|_{\dot{B}^r_{\infty,\infty}} &= \sup_{q\in Z} 2^{qr}\|\dot{\Delta}_q u\|_{L^\infty} = \sup_{q\geq 0} 2^{qr}\|\dot{\Delta}_q u\|_{L^\infty} + \sup_{q\leq -1} 2^{qr}\|\dot{\Delta}_q u\|_{L^\infty} \\
&\leq \|u\|_{B^r_{\infty,\infty}} + \sup_{q\leq -1} \|\dot{\Delta}_q u\|_{L^\infty} \\
&= \|u\|_{B^r_{\infty,\infty}} + \|u\|_{\dot{B}^0_{\infty,\infty}} \\
&\lesssim \|u\|_{B^r_{\infty,\infty}},
\end{aligned}
$$

in which we used the embedding $L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$.

Remark 2.1. As a special case of (2): $B^N_{1,1} \hookrightarrow \dot{B}^N_{1,1}$, which we will use in this paper, the proof is direct:

$$
\begin{aligned}
\|u\|_{\dot{B}^N_{1,1}} &= \sum_{q\in Z} 2^{qN}\|\dot{\Delta}_q u\|_{L^1} \\
&= \sum_{q\geq 0} 2^{qN}\|\dot{\Delta}_q u\|_{L^1} + \sum_{q\leq -1} 2^{qN}\|\dot{\Delta}_q u\|_{L^1} \\
&\leq \sum_{q\geq -1} 2^{qN}\|\Delta_q u\|_{L^1} + (\sum_{q\leq -1} 2^{qN})2^N(2^{-N}\|\Delta_{-1} u\|_{L^1}) \\
&\lesssim \|u\|_{B^N_{1,1}},
\end{aligned}
$$

where we used the fact that for $q \geq 0$, the homogeneous space $(\dot{\Delta}_q)$ and the inhomogeneous space (Δ_q) share the same definition.

Lemma 2.3 Let $s > 0$, $p,\ q \in [1, \infty]$, then $B^s_{p,q} \bigcap L^\infty$ is an algebra and the following inequality holds true

$$
\|uv\|_{B^s_{p,q}} \lesssim \|u\|_{L^\infty}\|v\|_{B^s_{p,q}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,q}}.
$$

The proof can be found in the reference [10], and we omit it here.

Lemma 2.4 Let $r > 0$, then there exists a constant C such that the following inequality holds

$$
\|u \cdot \nabla v\|_r \leq C\|u\|_r\|v\|_{B^1_{\infty,1}}.
$$

Proof. We use Bony's formula for paraproduct of two functions

$$\begin{aligned} u\cdot\nabla v &= T_{u^i}\partial_i v + T_{\partial_i v}u^i + R(u^i,\partial_i v) \\ &= \sum_k S_{k-1}u^i\Delta_k\partial_i v + \sum_k S_{k-1}\partial_i v\Delta_k u^i + \sum_k(\Delta_k u^i\sum_{j=-1}^{1}\Delta_{k+j}\partial_i v). \end{aligned}$$

where we use the denotation of Einstein's sum about i.

From Proposition 2.1, we have

$$\begin{aligned} \Delta_q(u\cdot\nabla v) &= \sum_{|k-q|\le M}\Delta_q(S_{k-1}u^i\Delta_k\partial_i v) + \sum_{|k-q|\le M}\Delta_q(S_{k-1}\partial_i v\Delta_k u^i) \\ &+ \sum_{k\ge q-M}\Delta_q(\Delta_k u^i\sum_{j=-1}^{1}\Delta_{k+j}\partial_i v) \\ &= L_1 + L_2 + L_3. \end{aligned}$$

where M is a finite integer.
Thanks to Hölder inequality and Proposition 2.2, we obtain

$$\begin{aligned} \sup_{q\ge -1}2^{qr}\|L_1\|_{L^\infty} &\le \sup_{q\ge -1}2^{qr}\sum_{|k-q|\le M}\|\Delta_q u^i\|_{L^\infty}\|\Delta_k\partial_i v\|_{L^\infty} \\ &\le C\sup_{q\ge -1}2^{qr}\|\Delta_q u^i\|_{L^\infty}\sum_{k\ge -1}2^k\|\Delta_k v\|_{L^\infty} \\ &\le C\|u\|_r\|v\|_{B^1_{\infty,1}}, \end{aligned} \tag{2.1}$$

and similarly

$$\begin{aligned} \sup_{q\ge -1}2^{qr}\|L_2\|_{L^\infty} &\le \sup_{q\ge -1}2^{qr}\sum_{|k-q|\le M}\|\Delta_q\partial_i v\|_{L^\infty}\|\Delta_k u^i\|_{L^\infty} \\ &\le C\sup_{q\ge -1}2^{qr}\|\Delta_q u^i\|_{L^\infty}\sum_{q\ge -1}2^q\|\Delta_q v\|_{L^\infty} \\ &\le C\|u\|_r\|v\|_{B^1_{\infty,1}}. \end{aligned} \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} \sup_{q\ge -1}2^{qr}\|L_3\|_{L^\infty} &\le \sup_{q\ge -1}2^{qr}\sum_{k\ge q-M}\|\Delta_q u^i\|_{L^\infty}\|\sum_{j=-1}^{1}\Delta_{k+j}\partial_i v\|_{L^\infty} \\ &\le C\sup_{q\ge -1}2^{qr}\|\Delta_q u^i\|_{L^\infty}\sum_{k\ge -1}2^k\|v\|_{L^\infty} \\ &\le C\|u\|_r\|v\|_{B^1_{\infty,1}}. \end{aligned} \tag{2.3}$$

Using (2.1)-(2.3), we get the proof of Lemma 2.4.

As usual, the Riesz operator is defined as

$$R_j(u) = \frac{x_j}{|x|^{N+1}} * u = \mathcal{F}^{-1}(-\frac{i\xi_j}{|\xi|}\cdot\hat{u}),$$

where i,j satisfy $i^2=-1$ and $1\le j\le N$. Hence

$$R_jR_k(u)=\sum_{1\le j,k\le N}\frac{\partial_i\partial_j}{(-\Delta)}(u).$$

The following lemma is about the boundedness of the Riesz operator in the space of $C^r(r>0)$ (see [8]).

Lemma 2.5 Let $r>0$, if $u\in C^r$, then for the Riesz operator, there exists a constant C, such that

$$\|R_jR_k(u)\|_r\le C\|u\|_r \quad i.e. \quad \|\nabla\Delta^{-1}\mathrm{div}u\|_r\le C\|u\|_r.$$

Proof. Note that

$$\begin{aligned}\|\nabla\Delta^{-1}\mathrm{div}u\|_r &= \sup_{q\ge -1}2^{rq}\|\nabla\Delta^{-1}\mathrm{div}\Delta_q u\|_{L^\infty}\\ &\le \sup_{q\ge 0}2^{rq}\|\nabla\Delta^{-1}\mathrm{div}\Delta_q u\|_{L^\infty}+2^{-r}\|\nabla\Delta^{-1}\mathrm{div}\Delta_{-1}u\|_{L^\infty}\\ &\le \sup_{q\in Z}2^{rq}\|\nabla\Delta^{-1}\mathrm{div}\dot{\Delta}_q u\|_{L^\infty}+2^{-r}\|\nabla\Delta^{-1}\mathrm{div}\Delta_{-1}u\|_{L^\infty}\\ &= I+II.\end{aligned}$$

Using Lemma 2.2(3), we have

$$I\le\|u\|_{\dot{C}^r}\le\|u\|_r. \tag{2.4}$$

Since

$$\Delta_{-1}u=\Delta_{-1}\sum_{q\in Z}\dot{\Delta}_q u=\sum_{q\le -1}\Delta_{-1}\dot{\Delta}_q u,$$

we obtain

$$\begin{aligned}II=\|\Delta_{-1}u\|_{L^\infty} &\le \sum_{q\le -1}\|\dot{\Delta}_q u\|_{L^\infty}=\|u\|_{\dot{B}^0_{\infty,1}}\lesssim\|u\|_{\dot{B}^N_{1,1}}\\ &\lesssim\|u\|_{B^N_{1,1}}\lesssim\|u\|_r,\end{aligned} \tag{2.5}$$

where we have used the boundedness property of the singular integral operator (i.e. Riesz operator) from $\dot{B}^s_{p,q}$ into itself in [5].

According to (2.4) and (2.5), we have

$$\|\nabla\Delta^{-1}\mathrm{div}u\|_r\le\|u\|_r,$$

The proof of the lemma is complete.

Our main results of this paper is stated as

Theorem 2.1(local existence and uniqueness) Suppose that the initial data satisfy $\theta_0,u_0\in C^r(r>1)$. Then, there exists $T^*=T(\|\theta_0\|_{C^r},\|u_0\|_{C^r})>0$, such that the system (1.3)-(1.4) has a unique solution (u,θ) satisfying $u\in L^\infty([0,T^*];C^r)$ and $\theta\in L^\infty([0,T^*];C^r)$.

Theorem 2.2 (blow-up criterion) For $r > 1$, if we assume that the solution satisfies

$$\int_0^{T^*} \|\nabla u\|_{L^\infty} < \infty,$$

then the solution can be extended after $t = T^*$. In other word, if the solution blows up at $t = T^*$, then

$$\int_0^{T^*} \|\nabla u\|_{L^\infty} ds = \infty,$$

for any pair of solution (θ, u) in the C^r space.

3. A Priori Estimates

In this section, we will prove the existence part of Theorem 2.1. To this end, we first derive some a priori estimates.

Lemma 3.1 Let $r > 0$, v be a divergence-free vector field belonging to the space $L^1_{loc}((0, +\infty); \mathrm{Lip}(R^N))$ and f be a scalar solution to the following problem

$$\begin{cases} \partial_t f + v \cdot \nabla f = g, \\ f|_{t=0} = f_0. \end{cases}$$

If the initial data $f_0 \in C^r$, then we have for all $t \in (0, +\infty)$

$$\|f(t)\|_r \le \|f_0\|_r + \int_0^t \|g(s)\|_r ds + C(r) \int_0^t \|\nabla v\|_{L^\infty} \|f(s)\|_r ds,$$

where C depends only on the dimension N and r.

Proof. Taking operation Δ_q on both sides of the above system (3.1), we get

$$\begin{cases} \partial_t \Delta_q f + v \cdot \nabla \Delta_q f = \Delta_q g + [v \cdot \nabla, \Delta_q] f, \\ \Delta_q f|_{t=0} = \Delta_q f_0. \end{cases}$$

It's easy to get that

$$\Delta_q f(t) = \Delta_q f_0 + \int_0^t (\Delta_q g + [v \cdot \nabla, \Delta_q] f) ds.$$

According to Lemma 2.1, we obtain

$$\|f(t)\|_r \le \|f_0\|_r + \int_0^t \|g(s)\|_r ds + C(r) \int_0^t \|\nabla v\|_{L^\infty} \|f(s)\|_r ds.$$

The proof of the lemma is finished.

Based on Lemma 3.1, we have

Lemma 3.2 Let $r > 0$. Suppose that u, θ are smooth solutions of (1.3) with initial data $u_0, \theta_0 \in C^r$. Then we have

$$\|\theta(t)\|_r \le \|\theta_0\|_r + C(r) \int_0^t \|\nabla u\|_{L^\infty} \|\theta(s)\|_r ds. \tag{3.1}$$

Remark 3.1 According to Lemma 3.2 and the Gronwall's inequality, we have

$$\|\theta\|_r \leq \|\theta_0\|_r \exp(C(r)\int_0^t \|\nabla u\|_{L^\infty} ds). \tag{3.2}$$

Lemma 3.3 Let $r > 0$. Suppose that u, θ are smooth solutions of (1.3) with initial data $u_0, \theta_0 \in C^r$. Then we have

$$\|u\|_r \leq \|u_0\|_r + 2C(r)\int_0^t \|u\|_r \|\nabla u\|_{L^\infty} ds + (2 + 2^{-r})\int_0^t \|\theta\|_r ds. \tag{3.3}$$

Proof. In view of Lemma 3.1, we get

$$\|u(t)\|_r \leq \|u_0\|_r + \int_0^t \|\nabla \Pi\|_r ds + \int_0^t \|\theta e_N\|_r + C(r)\int_0^t \|\nabla u\|_{L^\infty}\|u(s)\|_r ds,$$

where

$$\nabla \Pi = -\nabla \Delta^{-1}\mathrm{div}(u \cdot \nabla u) + \nabla \Delta^{-1}\partial_N \theta.$$

Note that

$$\nabla \Delta_q \Pi = -\Delta_q \nabla \Delta^{-1}\mathrm{div}(u \cdot \nabla u) + \Delta_q \nabla \Delta^{-1}\partial_N \theta.$$

Then one has

$$\|\nabla \Delta \Pi\|_{L^\infty} \leq \|\Delta_q \nabla \Delta^{-1}\mathrm{div}(u \cdot \nabla u)\|_{L^\infty} + \|\Delta_q \nabla \Delta^{-1}\partial_N \theta\|_{L^\infty},$$

and

$$\sup_{q\geq -1} 2^{qr}\|\nabla \Delta \Pi\|_{L^\infty} \leq \sup_{q\geq -1} 2^{qr}\|\Delta_q \nabla \Delta^{-1}\mathrm{div}(u\cdot\nabla u)\|_{L^\infty} + \sup_{q\geq -1} 2^{qr}\|\Delta_q \nabla \Delta^{-1}\partial_N \theta\|_{L^\infty}.$$

Using Lemma 2.5 , Lemma 2.3 and Lemma 2.2 (3), one has (see [8]):

$$\sup_{q\geq -1} 2^{qr}\|\Delta_q \nabla \Delta^{-1}\mathrm{div}(u \cdot \nabla u)\|_{L^\infty} \leq C(r)\|u\|_r \|\nabla u\|_{L^\infty}.$$

Concerning the term $\sup_{q\geq -1} 2^{qr}\|\Delta_q \nabla \Delta^{-1}\partial_N \theta\|_{L^\infty}$, we have

$$\begin{aligned} \sup_{q\geq -1} 2^{qr}\|\Delta_q \nabla \Delta^{-1}\partial_N \theta\|_{L^\infty} &\leq \sup_{q\geq 0} 2^{qr}\|\Delta_q \nabla \Delta^{-1}\partial_N \theta\|_{L^\infty} + 2^{-r}\|\Delta_{-1}\nabla \Delta^{-1}\partial_N \theta\|_{L^\infty} \\ &= I + II. \end{aligned} \tag{3.4}$$

Direct estimates give

$$\begin{aligned} I &= \sup_{q\geq 0} 2^{qr}\|\Delta_q \nabla \Delta^{-1}\partial_N \theta\|_{L^\infty} \leq \sup_{q\in Z} 2^{qr}\|\dot{\Delta}_q \nabla \Delta^{-1}\partial_N \theta\|_{L^\infty} \\ &\leq \|\theta\|_{\dot{C}^r} \leq \|\theta\|_r, \end{aligned} \tag{3.5}$$

where we used the embedding $B^r_{\infty,\infty} \hookrightarrow \dot{B}^r_{\infty,\infty} (r > 1)$ in Lemma 2.2 (3). Here we could also get (3.7) by using directly Lemma 2.5.

By the dyadic decomposition in the homogeneous space, we have

$$
\begin{aligned}
II &= 2^{-r}\|\Delta_{-1}\nabla\Delta^{-1}\partial_N\theta\|_{L^\infty} = 2^{-r}\|\Delta_{-1}\sum_{j\in Z}\dot{\Delta}_j\nabla\Delta^{-1}\partial_N\theta\|_{L^\infty} \\
&\le 2^{-r}\sum_{j\in Z}\|\dot{\Delta}_j\nabla\Delta^{-1}\partial_N\theta\|_{L^\infty} \\
&\le 2^{-r}\|\theta\|_{\dot{B}^0_{\infty,1}} \le 2^{-r}\|\theta\|_{\dot{B}^N_{1,1}},
\end{aligned}
$$

where we used the boundedness of the Riesz operator in any homogeneous Besov spaces $\dot{B}^s_{p,r}$ in [5].

Then in view of the embedding inequalities $\|\theta\|_{\dot{B}^N_{1,1}} \lesssim \|\theta\|_{B^N_{1,1}} \lesssim \|\theta\|_r (r > 0)$, we have $II \lesssim 2^{-r}\|\theta\|_r$.

Putting (3.7) and (3.8) into (3.6), one has

$$
\sup_{q\ge -1} 2^{qr}\|\triangle_q\nabla\triangle^{-1}\partial_N\theta\|_{L^\infty} \le (1+2^{-r})\|\theta\|_r.
$$

So about the pressure term, we have

$$
\|\nabla\Pi\|_r = \sup_{q\le -1} 2^{qr}\|\nabla\Delta_q\Pi\|_{L^\infty} \le C(r)(\|u\|_r\|\nabla u\|_{L^\infty} + \|\theta\|_r).
$$

Thus we get

$$
\|u\|_r \le \|u_0\|_r + 2C(r)\int_0^t \|u\|_r\|\nabla u\|_{L^\infty}ds + (2+2^{-r})\int_0^t \|\theta\|_r ds.
$$

The proof of the lemma is finished.

4. Proof of Main Results

In this section we will prove one of our main results, Theorem 2.1.

4.1 The approximate solutions

In order to establish the local in time existence of solution, we construct the approximate sequences $\{(\theta^n, u^n)\}_{n\in Z^+\cup\{0\}}$ as follows

$$
\begin{cases}
\partial_t\theta_{n+1} + u_n\cdot\nabla\theta_{n+1} = 0, \\
\partial_t u_{n+1} + u_n\cdot\nabla u_{n+1} + \nabla\Pi_{n+1} = \theta_{n+1}e_N, \\
\text{div}u_n = 0 = \text{div}u_{n+1},
\end{cases} \tag{4.1}
$$

with the initial data

$$
\begin{cases}
\theta_1 = S_2\theta_0, \\
\theta_{n+1}|_{t=0} = S_{n+2}\theta_0, \\
u_1 = S_2 u_0, \\
u_{n+1}|_{t=0} = S_{n+2}u_0.
\end{cases} \tag{4.2}
$$

Taking Δ_q on both sides of $(4.1)_1$ and $(4.2)_2$, we have

$$\begin{cases} \partial_t \Delta_q \theta_{n+1} + u_n \cdot \nabla \Delta_q \theta_{n+1} = [u_n \cdot \nabla, \Delta_q] \theta_{n+1}, \\ \Delta_q \theta_{n+1}|_{t=0} = \Delta_q S_{n+2} \theta_0. \end{cases}$$

By Lemma 3.1, we get

$$\|\theta_{n+1}\|_r \leq \|\theta_0\|_r + C(r) \int_0^t \|\theta_{n+1}\|_r \|\nabla u_n\|_{L^\infty} ds.$$

Using the Gronwall's inequality, we have

$$\|\theta_{n+1}(t)\|_r \leq \|\theta_0\|_r \exp(C(r) \int_0^t \|\nabla u_n\|_{L^\infty} ds). \tag{4.3}$$

Taking Δ_q on both sides of $(4.1)_2$ and $(4.2)_4$, we get

$$\begin{cases} \partial_t \Delta_q u_{n+1} + u_n \cdot \nabla \Delta_q u_{n+1} = [u_n \cdot \nabla, \Delta_q] u_{n+1} - \nabla \Delta_q \Pi_{n+1} + \Delta_q \theta_n e_N, \\ \Delta_q u_{n+1}|_{t=0} = \Delta_q S_{n+2} u_0. \end{cases}$$

Similar to Lemma 3.1, we get

$$\Delta_q u_{n+1}(t) = \Delta_q u_{n+1}(t=0) + \int_0^t ([u_n \cdot \nabla, \Delta_q] u_{n+1} - \nabla \Delta_q \Pi_{n+1} + \Delta_q \theta_n e_N) ds,$$

here the pressure term

$$\nabla \Pi_{n+1} = -\nabla \Delta^{-1} \mathrm{div}(u_n \cdot \nabla u_{n+1}) + \nabla \Delta^{-1} \partial_N \theta_n.$$

So we have

$$\begin{aligned} \|u_{n+1}(t)\|_r &\leq \|u_{n+1}(t=0)\|_r + \int_0^t (\sup_{q \geq -1} 2^{qs} \|[u_n \cdot \nabla, \Delta_q] u_{n+1}\|_{L^\infty} \\ &+ \sup_{q \geq -1} 2^{qs} \|\nabla \Delta_q \Pi_{n+1}\|_{L^\infty} + \sup_{q \geq -1} 2^{qs} \|\Delta_q \theta_{n+1} e_N\|_{L^\infty}) ds. \end{aligned}$$

In view of Lemma 2.1, the following estimates hold

$$\sup_{q \geq -1} 2^{qs} \|[u_n \cdot \nabla, \Delta_q] u_{n+1}\|_{L^\infty} \leq C(r) \|\nabla u_n\|_{L^\infty} \|u_{n+1}\|_r,$$

and

$$\sup_{q \geq -1} 2^{qs} \|\Delta_q \theta_{n+1} e_N\|_{L^\infty} \leq \|\theta_{n+1}\|_r.$$

Concerning the pressure term, we have

$$\begin{aligned} \|\nabla \Pi_{n+1}\|_r &= \sup_{q \geq -1} 2^{qs} \|\nabla \Delta_q \Pi_{n+1}\|_{L^\infty} \leq \sup_{q \geq -1} 2^{qs} \|\Delta_q \nabla \Delta^{-1} \mathrm{div}(u_n \cdot \nabla u_{n+1})\|_{L^\infty} \\ &+ \sup_{q \geq -1} 2^{qs} \|\Delta_q \nabla \Delta^{-1} \partial_N \theta_{n+1}\|_{L^\infty} = J_1 + J_2. \end{aligned}$$

Similar as in [8] Proposition 2.5.1, we obtain

$$\begin{aligned} J_1 &= \sup_{q\geq -1} 2^{qs}\|\Delta_q\nabla\Delta^{-1}\mathrm{div}(u_n\cdot\nabla u_{n+1})\|_{L^\infty} \\ &\leq C(r)(\|\nabla u_n\|_{L^\infty}\|u_{n+1}\|_r + \|\nabla u_{n+1}\|_{L^\infty}\|u_n\|_r), \end{aligned}$$

and

$$\begin{aligned} J_2 &\leq \sup_{q\in Z}\|\dot{\Delta}_q\nabla\Delta^{-1}\partial_N\theta_{n+1}\| + 2^{-r}\|\Delta_{-1}\nabla\Delta^{-1}\partial_N\theta_{n+1}\| \\ &\leq \|\theta_{n+1}\|_{\dot{C}^r} + \|\theta_{n+1}\|_{\dot{B}^0_{\infty,1}} \lesssim \|\theta_{n+1}\|_r. \end{aligned}$$

Putting (4.6) and (4.7) into (4.5), we have

$$\begin{aligned} \|u_{n+1}(t)\|_r &\leq \|u_{n+1}(0)\|_r \\ &+\int_0^t (2C(r)\|\nabla u_n\|_{L^\infty}\|u_{n+1}\|_r + C(r)\|\nabla u_{n+1}\|_{L^\infty}\|u_n\|_r + 2\|\theta_n\|_r)ds \\ &\leq \|u_{n+1}(0)\|_r + 3C(r)\int_0^t \|u_n\|_r\|u_{n+1}\|_r ds + 2\int_0^t\|\theta_n\|_r ds \end{aligned}$$

Applying the Gronwall's inequality yields

$$\|u_{n+1}\|_r \leq [\|S_{n+2}u_0\|_r + 2\int_0^t\|\theta_{n+1}\|_r ds]\exp(\int_0^t 3C(r)\|u_n\|_r ds),$$

i.e.

$$\|u_{n+1}\|_r \leq [\|u_0\|_r + 2\int_0^t\|\theta_{n+1}\|_r ds]\exp(\int_0^t 3C(r)\|u_n\|_r ds). \tag{4.4}$$

Now we define $a_0 = \|\widetilde{h}\|_{L^1}$(the function $\widetilde{h}$ is given in the notation). We will establish that, for all initial data θ_0, u_0,

$$\|\theta_n\|_{C([0,T_1];C^r)} \leq Pa_0\|\theta_0\|_r \quad \|u_n\|_{C([0,T_1];C^r)} \leq Qa_0\|u_0\|_r \tag{4.5}$$

with some constants P and Q which may be large enough (for example, $P = Q = 32$), and $T_1 > 0$ which will be determined later.

Using (4.4) and (4.8), we have

$$\|\theta_1\|_r \leq \|\theta_0\|_r\exp(C(r)\int_0^t\|u_0\|_r ds) \leq Pa_0\|\theta_0\|_r.$$

Let

$$T_1^{(1)} = \frac{1}{C(r)\|u_0\|_r}\ln(Pa_0),$$

Then we obtain that $\|\theta_1\|_{C([0,T_1^{(1)}],C^r)} \leq Pa_0\|\theta_0\|_r$, when $t < T_1^{(1)}$.

Then we let

$$\begin{aligned} \|u_1\|_r &\leq \|u_0\|_r[1 + 2tPa_0\frac{\|\theta_0\|_r}{\|u_0\|_r}]\exp(3tC(r)\|u_0\|_r) \\ &\leq \|u_0\|_r\exp(3tC(r)\|u_0\|_r + 2tPa_0\frac{\|\theta_0\|_r}{\|u_0\|_r}) \leq Qa_0\|u_0\|_r, \end{aligned}$$

Let

$$T_1^{(2)} = \frac{\ln(Qa_0)}{3C(r)\|u_0\|_r + 2Pa_0\frac{\|\theta_0\|_r}{\|u_0\|_r}}.$$

Then we get $\|u_1\|_{C([0,T_1^{(2)}],C^r)} \leq Qa_0\|u_0\|_r$, when $t \leq T_1^{(2)}$.

Now we apply the induction of n to obtain (4.9). We assume the estimates (4.9) to be true for every $j \leq n$.

Firstly, we have

$$\|\theta_{n+1}\|_r \leq \|\theta_0\|_r \exp(C(r)\int_0^t \|u_n\|_r ds) \leq \|\theta_0\|_r \exp(tC(r)\|u_n\|_r) \leq Pa_0\|\theta_0\|_r.$$

Then we set

$$T_1^{(3)} = \frac{\ln(Pa_0)}{C(r)Qa_0\|u_0\|_r}.$$

If $t \leq T_1^{(3)}$, the inequality $\|\theta_{n+1}\|_{C([0,T_1^{(3)}],C^r)} \leq Pa_0\|\theta_0\|_r$ is obtained.

Then for the term u_{n+1}, we let

$$\begin{aligned} \|u_{n+1}\|_r &\leq [\|u_0\|_r + 2\int_0^t \|\theta_{n+1}\|_r ds]\exp(\int_0^t 3C(r)\|u_n\|_r ds) \\ &\leq [\|u_0\|_r + 2tPa_0\|\theta_0\|_r]\exp(3tC(r)Qa_0\|u_0\|_r) \\ &\leq \|u_0\|_r \exp\|u_0\|_r \exp(3tC(r)Qa_0\|u_0\|_r + 2tPa_0\frac{\|\theta_0\|_r}{\|u_0\|_r}) \leq Qa_0\|u_0\|_r. \end{aligned}$$

Set

$$T_1^{(4)} = \frac{\ln(Qa_0)}{3C(r)Qa_0\|u_0\|_r + 2Pa_0\frac{\|\theta_0\|_r}{\|u_0\|_r}}.$$

If $t \leq T_1^{(4)}$, there holds $\|u_{n+1}\|_{C([0,T_1^{(4)}],C^r)} \leq Qa_0\|u_0\|_r$.

Thus the sequences $\{(\theta_n, u_n)\}_{n\in Z^+\cup\{0\}}$ are constructed, and they are also bounded in the space $C([0,T_1],C^r)$, where $T_1 = \min\{T_1^i\}_{i=1}^4$.

4.2 The Cauchy sequences

Now we prove that there exists a $T_2 > 0$, such that the sequences $(\theta_n)_{n\in Z^+\cup\{0\}}$, $(u_n)_{n\in Z^+\cup\{0\}}$ are the Cauchy sequences in the space $C([0,T_2];C^{r-1})(r>1)$. To do so, we estimate the quantities $\|\theta_{n+1}(t)-\theta_n(t)\|_{r-1}$ and $\|u_{n+1}(t)-u_n(t)\|_{r-1}$. For conciseness, we set $\bar{u}_{n+1}(t) = u_{n+1}(t) - u_n(t)$ and $\bar{\theta}_{n+1}(t) = \theta_{n+1}(t) - \theta_n(t)$.

By constructions of the approximate solutions, we know that

$$\begin{cases} \partial_t\theta_{n+1} + u_n\cdot\nabla\theta_{n+1} = 0, \\ \theta_{n+1}|_{t=0} = S_{n+2}\theta_0, \end{cases}$$

and

$$\begin{cases} \partial_t\theta_n + u_{n-1}\cdot\nabla\theta_n = 0, \\ \theta_n|_{t=0} = S_{n+1}\theta_0. \end{cases}$$

Subtracting (4.11) from (4.10), we get

$$\begin{cases} \partial_t(\theta_{n+1}-\theta_n)+u_n\cdot\nabla(\theta_{n+1}-\theta_n)=-(u_n-u_{n-1}\cdot\nabla\theta_n), \\ (\theta_{n+1}-\theta_n)|_{t=0}=S_{n+2}\theta_0-S_{n+1}\theta_0=\Delta_{n+1}\theta_0. \end{cases}$$

In view of Lemma 3.2, we obtain

$$\begin{aligned} \|\bar\theta_{n+1}\|_{r-1} &\le \|\Delta_{n+1}\theta_0\|_{r-1}+\int_0^t\|\bar u_n\cdot\nabla\theta_n\|_{r-1}ds \\ &\quad +C(r)\int_0^t\|\nabla u_n\|_{L^\infty}\|\bar\theta_{n+1}\|_{r-1}ds. \end{aligned}$$

Thanks to Lemma 2.3, we have

$$\begin{aligned} \|\bar u_n\cdot\nabla\theta_n\|_{r-1} &\le C(\|\bar u_n\|_{L^\infty}\|\nabla\theta_n\|_{r-1}+\|\nabla\theta_n\|_{L^\infty}\|\bar u_n\|_{r-1}) \\ &\le C(\|\bar u_n\|_{r-1}\|\theta_n\|_r+\|\theta_n\|_r\|\bar u_n\|_{r-1}) \\ &\le 2C\|\theta_n\|_r\|\bar u_n\|_{r-1}, \end{aligned}$$

where we used the embedding inequalities: $\|\nabla u\|_{B^{s-1}_{p,q}}\lesssim\|u\|_{B^s_{p,q}}$ and $\|\nabla u\|_{L^\infty}\lesssim\|u\|_{\dot B^1_{\infty,1}}\lesssim\|u\|_r,(r>1)$.

Therefore we get the estimate

$$\begin{aligned} \|\bar\theta_{n+1}\|_{r-1} &\le \|\Delta_{n+1}\theta_0\|_{r-1}+\int_0^t 2C\|\theta_n\|_r\|\bar u_n\|_{r-1}ds \\ &\quad +C(r)\int_0^t\|u_n\|_r\|\bar\theta_{n+1}\|_{r-1}ds. \end{aligned}$$

Since

$$\begin{aligned} \|\Delta_{n+1}\theta_0\|_{r-1} &= \sup_{q\in Z}2^{q(r-1)}\|\dot\Delta_q\Delta_{n+1}\theta_0\|_{L^\infty}=\sup_{|q-n|\le3}2^{q(r-1)}\|\dot\Delta_q\theta_0\|_{L^\infty} \\ &= \sup_{|q-n|\le3}2^{qr}\|\dot\Delta_q\theta_0\|_{L^\infty}2^{-q}\lesssim 2^{-n}\|\theta_0\|_r, \end{aligned}$$

we obtain

$$\begin{aligned} \|\bar\theta_{n+1}\|_{r-1} &\le 2^{-n}\|\theta_0\|_r+\int_0^t 2C\|\theta_n\|_r\|\bar u_n\|_{r-1}ds \\ &\quad +C(r)\int_0^t\|u_n\|_r\|\bar\theta_{n+1}\|_{r-1}ds. \end{aligned} \tag{4.6}$$

Using the Gronwall's inequality, we have

$$\begin{aligned} \|\bar\theta_{n+1}\|_{r-1} &\le 2^{-n}\|\theta_0\|_r\exp(C(r)\int_0^t\|u_n\|_rds) \\ &\quad +2C\int_0^t\|\theta_n\|_r\|\bar u_n\|_{r-1}ds\exp(C(r)\int_0^t\|u_n\|_rds). \end{aligned}$$

Now we estimate $\|\bar{u}_{n+1}\|_{r-1}$. Noting that

$$\begin{cases} \partial_t u_{n+1} + u_n \cdot \nabla u_{n+1} = -\nabla \Pi_{n+1} + \theta_{n+1} e_N, \\ u_{n+1}|_{t=0} = S_{n+2} u_0, \\ \mathrm{div} u_{n+1} = 0, \end{cases}$$

and that

$$\begin{cases} \partial_t u_n + u_{n-1} \cdot \nabla u_n = -\nabla \Pi_n + \theta_n e_N, \\ u_n|_{t=0} = S_{n+1} u_0, \\ \mathrm{div} u_n = 0, \end{cases}$$

subtracting (4.16) from (4.15), we get

$$\begin{cases} \partial_t \bar{u}_{n+1} + u_n \cdot \nabla \bar{u}_{n+1} = -\bar{u}_n \cdot \nabla u_n - \nabla \Pi_{n+1} + \nabla \Pi_n + \bar{\theta}_{n+1} e_N, \\ \bar{u}_{n+1}|_{t=0} = S_{n+2} u_0 - S_{n+1} u_0 = \Delta_{n+1} u_0. \end{cases}$$

In view of Lemma 3.2, we obtain

$$\begin{aligned} \|\bar{u}_{n+1}\|_{r-1} &\leq \|\Delta_{n+1} u_0\|_{r-1} + \int_0^t \|\nabla \Pi_{n+1} - \nabla \Pi_n\|_{r-1} ds + \int_0^t \|\bar{\theta}_{n+1}\|_{r-1} ds \\ &+ C(r) \int_0^t \|\nabla u_n\|_{L^\infty} \|\bar{u}_{n+1}\|_{r-1} ds, \end{aligned}$$

where

$$\begin{aligned} \nabla \Pi_{n+1} &= -\nabla \Delta^{-1} \mathrm{div}(u_n \cdot \nabla u_{n+1}) + \nabla \Delta^{-1} \partial_N \theta_{n+1}, \\ \nabla \Pi_n &= -\nabla \Delta^{-1} \mathrm{div}(u_{n-1} \cdot \nabla u_n) + \nabla \Delta^{-1} \partial_N \theta_n. \end{aligned}$$

It follows that

$$\begin{aligned} \nabla \Pi_{n+1} - \nabla \Pi_n &= -\nabla \Delta^{-1} \mathrm{div}(u_n \cdot \nabla u_{n+1}) + \nabla \Delta^{-1} \partial_N \theta_{n+1} \\ &+ \nabla \Delta^{-1} \mathrm{div}(u_{n-1} \cdot \nabla u_n) - \nabla \Delta^{-1} \partial_N \theta_n \\ &= -\nabla \Delta^{-1} \mathrm{div}(u_n \cdot \nabla \bar{u}_{n+1}) - \nabla \Delta^{-1} \mathrm{div}(\bar{u}_n \cdot \nabla u_n) + \nabla \Delta^{-1} \partial_N \bar{\theta}_{n+1} \end{aligned}$$

When $1 < r < 2$, we use an estimate in [8], which says that, if $r \in (-1, 1)$, then

$$\|\pi(v, w)\|_r \leq C(\frac{1}{1+r} + \frac{1}{1-r}) \min\{\|v\|_{Lip} \|w\|_r, \|v\|_r \|w\|_{Lip}\}, \tag{4.7}$$

where π is viewed as the term $-\nabla \Delta^{-1} \mathrm{div}(u \cdot \nabla u)$, i.e. $\pi = \nabla \Pi - \nabla \Delta^{-1} \partial_N \theta$.

It follows from (4.18) that

$$\|\nabla \Pi_{n+1} - \nabla \Pi_n\|_{r-1} \leq C(\|\bar{u}_{n+1}\|_{r-1} \|u_n\|_r + \|\bar{u}_n\|_{r-1} \|u_n\|_r) + \|\bar{\theta}_{n+1}\|_{r-1}. \tag{4.8}$$

When $r > 2$, using the inequality $\|\nabla u\|_{L^\infty} \leq \|u\|_{r-1}$, and using Lemma 2.3 and Lemma 2.5, we get

$$\begin{aligned} \|\nabla \Pi_{n+1} - \nabla \Pi_n\|_{r-1} &\leq \|u_n \cdot \nabla \bar{u}_{n+1}\|_{r-1} + \|\bar{u}_n \cdot \nabla u_n\|_{r-1} + \|\bar{\theta}_{n+1}\|_{r-1} \\ &\leq C(\|u_n\|_{L^\infty} \|\nabla \bar{u}_{n+1}\|_{r-1} + \|\nabla \bar{u}_{n+1}\|_{L^\infty} \|u_n\|_{r-1} \\ &+ \|\bar{u}_n\|_{L^\infty} \|\nabla u_n\|_{r-1} + \|\nabla u_n\|_{L^\infty} \|\bar{u}_n\|_{r-1}) + \|\bar{\theta}_{n+1}\|_{r-1}. \end{aligned}$$

we can also obtain (4.19).

For $r=2$, using Lemma 2.4 and Lemma 2.3, we have

$$\begin{aligned}\|\nabla\Pi_{n+1}-\nabla\Pi_n\|_1 \leq\ & \|u_n\cdot\nabla\bar{u}_{n+1}\|_1+\|\bar{u}_n\cdot\nabla u_n\|_1+\|\bar{\theta}_{n+1}\|_1\\ & \leq C\|u_n\|_1\|\bar{u}_{n+1}\|_{B^1_{\infty,1}}+\|\bar{\theta}_{n+1}\|_1\\ & +C(\|\bar{u}_n\|_{L^\infty}\|\nabla u_n\|_1+\|\bar{u}_n\|_1\|\nabla u_n\|_{L^\infty})\\ & \leq C(\|\bar{u}_{n+1}\|_1\|u_n\|_2+\|\bar{u}_n\|_1\|u_n\|_2)+\|\bar{\theta}_{n+1}\|_1.\end{aligned}$$

(4.19) is proved for $r=2$. And we have proved that for all $r>1$, (4.19) holds true.

According to the inequality about the pressure term and some embedding inequalities, we have

$$\begin{aligned}\|\bar{u}_{n+1}\|_{r-1} & \leq\|\Delta_{n+1}u_0\|_{r-1}+C(r)\int_0^t\|u_n\|_r\|\bar{u}_n\|_{r-1}ds\\ & +2C(r)\int_0^t\|u_n\|_r\|\bar{u}_{n+1}\|_{r-1}ds+2\int_0^t\|\bar{\theta}_{n+1}\|_{r-1}ds.\end{aligned}$$

Since $\|\Delta_{n+1}u_0\|_{r-1}\leq C2^{-n}\|u_0\|_r$, we deduce that

$$\begin{aligned}\|\bar{u}_{n+1}\|_{r-1} & \leq C2^{-n}(\|u_0\|_r+\|\theta_0\|_r\exp(C(r)\int_0^t\|u_n\|_r ds))\\ & +C(r)\int_0^t\|u_n\|_r\|\bar{u}_n\|_{r-1}ds+2C(r)\int_0^t\|u_n\|_r\|\bar{u}_{n+1}\|_{r-1}ds\\ & +2C\int_0^t\int_0^\tau\|\theta_n\|_r\|\bar{u}_n\|_{r-1}ds\exp(C(r)\int_0^\tau\|u_n\|_r ds)d\tau.\end{aligned}$$

Then we use the Gronwall's inequality to give that

$$\begin{aligned}&\|\bar{u}_{n+1}\|_{C([0,T_2];C^{r-1})}\\ &\leq C2^{-n}(\|u_0\|_r+\|\theta_0\|_r\exp(C(r)\int_0^t\|u_n\|_r ds))\exp(2C(r)\int_0^t\|u_n\|_r ds)\\ &+C(r)\int_0^t\|u_n\|_r\|\bar{u}_n\|_{r-1}ds\exp(2C(r)\int_0^t\|u_n\|_r ds)\\ &+2C\int_0^t\int_0^\tau\|\theta_n\|_r\|\bar{u}_n\|_{r-1}ds\exp(C(r)\int_0^t\|u_n\|_r ds)d\tau\exp(2C(r)\int_0^t\|u_n\|_r ds)\\ &\leq C2^{-n}(\|u_0\|_r+\|\theta_0\|_r\exp(C(r)tQa_0\|u_0\|_r))\exp(2C(r)tQa_0\|u_0\|_r)\\ &+\|\bar{u}_n\|_{C([0,T_2];C^{r-1})}(C(r)Qa_0\|u_0\|_r\exp(2C(r)tQa_0\|u_0\|_r)\\ &+\frac{2CPa_0\|\theta_0\|_r}{C(r)Qa_0\|u_0\|_r}\exp(C(r)tQa_0\|u_0\|_r)t\exp(2C(r)tQa_0\|u_0\|_r)\\ &+\frac{2CPa_0\|\theta_0\|_r}{(C(r)Qa_0\|u_0\|_r)^2}[\exp(C(r)tQa_0\|u_0\|_r)-1])\exp(2C(r)tQa_0\|u_0\|_r)\\ &=\sum_{i=1}^4 I_i.\end{aligned}$$

Now we deal with the four terms $I_i(i=1,2,3,4)$ one by one.

Concerning I_1, we choose

$$T_2^{(1)} = \min\{\frac{1}{2C(r)Qa_0\|u_0\|_r}\ln(\frac{S}{\|u_0\|_r}), \frac{1}{3C(r)Qa_0\|u_0\|_r}\ln(\frac{S}{\|\theta_0\|_r})\},$$

to obtain

$$\|u_0\|_r\exp(2tC(r)Qa_0\|u_0\|_r) \leq S \quad \text{and} \quad \|\theta_0\|_r\exp(3tC(r)Qa_0\|u_0\|_r) \leq S$$

where S is a real constant large enough.

Then we get $I_1 \leq CS2^{-n+1}$.

Concerning I_2, we choose

$$T_2^{(2)} = \frac{1}{3C(r)Qa_0\|u_0\|_r}\ln(\frac{Pa_0}{5C(r)Q\|u_0\|_r}),$$

to get

$$C(r)Qa_0\|u_0\|_r\exp(3C(r)tQa_0\|u_0\|_r) \leq \frac{1}{5}Pa_0^2.$$

Then we get $I_2 \leq (1/5)\|\bar{u}\|_{C([0,T_2];C^{r-1})}$.

Concerning I_3, we choose

$$\frac{2CPa_0\|\theta_0\|_r}{C(r)Qa_0\|u_0\|_r}\exp(3C(r)tQa_0\|u_0\|_r)t \leq \frac{1}{5}Qa_0^2.$$

Then there exists a constant $T_2^{(3)}$ such that when $t \leq T_2^{(3)}$ the inequality $I_3 \leq (1/5)\|\bar{u}\|_{C([0,T_2];C^{r-1})}$ holds.

Concerning I_4, we set

$$T_2^{(4)} = \frac{1}{C(r)Qa_0\|u_0\|_r}\ln(1 + \frac{Pa_0(C(r)Qa_0\|u_0\|_r)^2}{5CPa_0\|\theta_0\|_r}),$$

such that

$$\frac{2CPa_0\|\theta_0\|_r}{(C(r)Qa_0\|u_0\|_r)^2}[\exp(C(r)tQa_0\|u_0\|_r) - 1] \leq \frac{1}{5}Pa_0.$$

Then we get $I_4 \leq (1/5)\|\bar{u}\|_{C([0,T_2];C^{r-1})}$.

Choosing $T_2 = \min\{T_2^{(i)}\}_{i=1}^4$, we obtain

$$\|u_{n+1} - u_n\|_{C([0,T_2];C^{r-1})} \lesssim 2^{-n} + \frac{3}{5}\|u_n - u_{n-1}\|_{C([0,T_2];C^{r-1})}.$$

Therefore, for $r > 1$, the sequence $(u_n)_{n\in Z^+\cup\{0\}}$ is a Cauchy sequence in $C([0,T_2];C^{r-1})$. Furthermore using (4.14), we obtain that the sequence $(\theta_n)_{n\in Z^+\cup\{0\}}$ is a Cauchy sequence in $C([0,T_2];C^{r-1})$.

Let $T^* = \min\{T_1, T_2\}$, and denote the limit of sequences $(\theta_n)_{n\in Z^+\cup\{0\}}, (u_n)_{n\in Z^+\cup\{0\}}$ by $\theta(t,x), u(t,x)$ respectively, we obtain that $\theta(t,x) \in L([0,T^*];C^r)$, $u(t,x) \in L([0,T^*];C^r)$, where $r > 1$, are solutions of (1.3)-(1.4).

4.3 Uniqueness

Suppose that $(\theta_i, u_i) \in L^\infty([0,T]; C^r(R^N; R^N))$ $(i = 1, 2)$ are two solutions of the system (1.3)-(1.4).

We set $\theta = \theta_1 - \theta_2, u = u_1 - u_2, \Pi = \Pi_1 - \Pi_2$, where Π_1 and Π_2 are pressure functions respectively. Then we have

$$\begin{cases} \partial_t \theta + u_1 \cdot \nabla \theta = -u \cdot \nabla \theta_2, \\ \theta|_{t=0} = 0, \end{cases}$$

and

$$\begin{cases} \partial_t u + u_1 \cdot \nabla u = -u \cdot \nabla u_2 - \nabla \Pi + \theta e_N, \\ \mathrm{div} u = 0, \\ u|_{t=0} = 0. \end{cases}$$

From Lemma 3.2, we have

$$\|\theta(t)\|_{r-1} \le \int_0^t \|u \cdot \nabla \theta_2\|_{r-1} ds + C(r) \int_0^t \|\nabla u_1\|_{L^\infty} \|\theta\|_{r-1} ds.$$

Due to Lemma 2.3, we have

$$\begin{aligned} \|u \cdot \nabla \theta_2\|_{r-1} &\le C(\|u\|_{L^\infty} \|\nabla \theta_2\|_{r-1} + \|\nabla \theta_2\|_{L^\infty} \|u\|_{r-1}) \\ &\le C \|\theta_2\|_r \|u\|_{r-1}. \end{aligned}$$

Then we get

$$\|\theta(t)\|_{r-1} \le C \int_0^t \|u\|_{r-1} \|\theta_2\|_r ds + C(r) \int_0^t \|\nabla u_1\|_{L^\infty} \|\theta\|_{r-1} ds.$$

Using the Gronwall's inequality, we obtain

$$\|\theta(t)\|_{r-1} \le C \int_0^T \|\theta_2\|_r \|u\|_{r-1} ds \exp(C(r) \int_0^T \|\nabla u_1\|_{L^\infty} ds). \tag{4.9}$$

From Lemma 3.3, the following estimate holds true

$$\begin{aligned} \|u\|_{r-1} &\le \|u_0\|_{r-1} + C(r) \int_0^T \|\nabla u_1\|_{L^\infty} \|u\|_{r-1} + \|u \cdot \nabla u_2\|_{r-1} \\ &\quad + \|\nabla(\Pi_1 - \Pi_2)\|_{r-1} + \|\theta\|_{r-1} ds, \end{aligned}$$

where

$$\begin{aligned} \nabla(\Pi_1 - \Pi_2) &= -\nabla \Delta^{-1} \mathrm{div}(u_1 \cdot \nabla(u_1 - u_2)) - \nabla \Delta^{-1} \mathrm{div}((u_1 - u_2) \cdot \nabla u_2) \\ &\quad + \nabla \Delta^{-1} \partial_N (\theta_1 - \theta_2) \end{aligned}$$

i.e.

$$\nabla \Pi = -\nabla \Delta^{-1} \mathrm{div}(u_1 \cdot \nabla u) - \nabla \Delta^{-1} \mathrm{div}(u \cdot \nabla u_2) + \nabla \Delta^{-1} \partial_N \theta.$$

Using (4.18) when $1 < r < 2$ and using the inequality $\|\nabla u\|_{L^\infty} \leq \|u\|_{r-1}$ when $r > 2$, by Lemma 2.3, we get

$$\begin{aligned}\|\nabla\Pi\|_{r-1} &\leq C(\|\nabla u_1\|_{L^\infty}\|u\|_{r-1} + \|\nabla u\|_{L^\infty}\|u_1\|_{r-1} \\ &+\|\nabla u\|_{L^\infty}\|u_2\|_{r-1} + \|\nabla u_2\|_{L^\infty}\|u\|_{r-1} + \|\theta\|_{r-1}).\end{aligned} \tag{4.10}$$

then we have

$$\|\nabla\Pi\|_{r-1} \leq C(2\|u\|_{r-1}(\|u_1\|_r + \|u_2\|_r) + \|\theta\|_{r-1}). \tag{4.11}$$

for $1 < r < 2$ and $r > 2$.

For $r = 2$, using Lemma 2.4 and Lemma 2.3, we have

$$\begin{aligned}\|\nabla\Pi\|_1 &\leq \|u_1 \cdot \nabla u\|_1 + \|u \cdot \nabla u_2\|_1 + \|\theta\|_1 \\ &\leq C\|u_1\|_1\|u\|_{B^1_{\infty,1}} + \|\theta\|_1 \\ &+C(\|u\|_1\|\nabla u_2\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla u_2\|_1) \\ &\leq C(\|u_1\|_1\|u\|_2 + \|u\|_1\|u_2\|_2) + \|\theta\|_1.\end{aligned}$$

So (4.25) is also true for $r = 2$.

Putting (4.25) into (4.23), we get

$$\|u\|_{r-1} \leq \|u_0\|_{r-1} + C(r)\int_0^t 3\|u\|_{r-1}(\|u_1\|_r + \|u_2\|_r) + 2\|\theta\|_{r-1}ds.$$

Using (4.22), we have

$$\begin{aligned}\|u\|_{r-1} &\leq 3C(r)\int_0^T \|u\|_{r-1}(\|u_1\|_r + \|u_2\|_r) \\ &+2C(r)T\int_0^T \|\theta_2\|_r\|u\|_{r-1}ds\exp(C(r)\int_0^T \|u_1\|_r ds).\end{aligned}$$

In view of Gronwall's inequality, we obtain that $u \equiv 0$. Applying (4.22) again, we directly get $\theta \equiv 0$.

This completes the proof of the uniqueness in Theorem 2.1

5. Blow-up criterion

Now we prove Theorem 2.2, which is about the blow-up criterion

Proof of Theorem 2.2. Applying (3.4), (3.6) and the Gronwall's inequality, we have

$$\begin{aligned}\|u\|_r &\leq \|u_0\|_r\exp(C(r)\int_0^t \|\nabla u\|_{L^\infty}ds) \\ &+(2 + 2^{-r})\|\theta_0\|_r\int_0^t \exp(C(r)\int_0^\tau \|\nabla u\|_{L^\infty}d\tau)ds\exp(C(r)\int_0^t \|\nabla u\|_{L^\infty}ds).\end{aligned}$$

Using (3.4) again and the above inequality (5.1), we get the proof of Theorem 2.2.

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