A NEW PERSPECTIVE ON k-TRIANGULATIONS

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Abstract. We connect k-triangulations of a convex n-gon to the theory of Schubert polynomials. We use this connection to prove that the simplicial complex with k-triangulations as facets is a vertex-decomposable triangulated sphere, and we give a new proof of the determinantal formula for the number of k-triangulations.

1. Introduction

Let Δ_n be the simplicial complex with vertices being diagonals in a convex n-gon and facets being triangulations (i.e., maximal subsets of diagonals which are mutually non-crossing). It is well-known that Δ_n is a triangulated sphere and moreover that it is the boundary complex of the *dual associahedron*, see e.g. [Lee89].

This construction can be generalized using an additional positive integer k with $2k+1 \leq n$. Define a $(k+1)$-crossing to be a set of $k+1$ diagonals which are mutually crossing. The simplicial complex $\Delta_{n,k}$ has vertex set

$$\Gamma_{n,k} := \{\overline{ij} : k < |i-j| < n-k\}$$

and facets being k-triangulations, i.e., maximal subsets of diagonals in $\Gamma_{n,k}$ which do not contain a $(k+1)$-crossing. The reason for restricting the set of diagonals is simply that all other diagonals cannot be part of a $(k+1)$-crossing and thus the resulting simplicial complex would be a join of $\Delta_{n,k}$ and an nk-simplex. The complex $\Delta_{n,k}$ was studied by several authors, see e.g. [DKM02, Jon05, JW07, Kra06, Nak00, Rub06]; a very interesting survey of what is known about k-tringulations can be found in [PS09].

A simplicial complex is called pure if all its facets have the same dimension.

Theorem 1.1 (Dress, Koolen, Moulton [DKM02], Nakamigawa [Nak00]). *$\Delta_{n,k}$ is pure of dimension $k(n-2k-1)-1$.*

Theorem 1.2 (Jonsson [Jon05], Krattenthaler [Kra06]). *The number of facets in $\Delta_{n,k}$ (which is the number of k-triangulations of a convex n-gon) is given by the following determinant:*

$$\det\begin{pmatrix} \mathrm{Cat}_{n-2} & \cdots & \mathrm{Cat}_{n-k-1} \\ \vdots & \ddots & \vdots \\ \mathrm{Cat}_{n-k-1} & \cdots & \mathrm{Cat}_{n-2k} \end{pmatrix} = \prod_{1\leq i\leq j<n-2k} \frac{i+j+2k}{i+j}$$

The first proof of Theorem 1.2 by J. Jonsson is rather involved and uses multiple results on moon polyominos whereas the second by C. Krattenthaler uses growth diagrams and proves that the number of k-triangulations equals the number of certain fans of non-intersecting Dyck paths. The number of such fans can be counted

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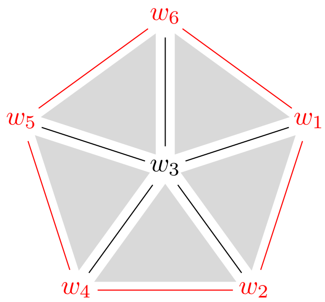


FIGURE 1. $\Delta(Q, \pi)$ as the join of $\Delta(Q', \pi)$ with the centered point w_3.

using the Lindström-Gessel-Viennot Lemma on families of non-intersecting lattice paths, see for example Sections 2.6.1 and 2.6.2. in [Man01]. This gives a natural explanation of the determinantal formula.

For a simplicial complex Δ and a face $F \in \Delta$, define

- the deletion of F from Δ by $\mathrm{del}(\Delta, F) := \{G \in \Delta : G \cap F = \emptyset\}$,
- the link of F in Δ by $\mathrm{link}(\Delta, F) := \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\}$.

Moreover, Δ is called vertex-decomposable if Δ is pure and either (1) $\Delta = \{\emptyset\}$ or (2) $\mathrm{del}(\Delta, v)$ and $\mathrm{link}(\Delta, v)$ are both vertex-decomposable for some vertex $v \in \Delta$. Vertex-decomposability was introduced by L.J. Billera and J.S. Provan in [BP79], where they moreover showed that vertex-decomposability implies shellability.

The following fact about the simplicial complex $\Delta_{n,k}$ was stated by J. Jonsson in an unpublished manuscript communicated in [Jon05] but a proof was never published:

Statement 1.3 (Jonsson)**.** *$\Delta_{n,k}$ is a vertex-decomposable triangulated sphere.*

In this article we want to connect k-triangulations in a surprisingly simple way to the theory of Schubert polynomials and thereby present a proof of Statement 1.3 and as well another viewpoint on C. Krattenthaler's proof of Theorem 1.2.

To this end, define, for any permutation $\pi \in \mathcal{S}_n$ and any word Q in the simple generators $s_i := (i, i+1) \in \mathcal{S}_n$, a simplicial complex which was introduced by A. Knutson and E. Miller in the context of Schubert polynomials in [KM05, Definition 1.8.1] and further studied in [KM04]: the subword complex $\Delta(Q, \pi)$ is defined to be the simplicial complex with vertices v_i being labelled by w_i for every letter $w_i \in Q$. Note that $v_i \neq v_j$ for $i \neq j$, even if the letters $w_i = w_j$ coincide. A subword of Q forms a facet of $\Delta(Q, \pi)$ if and only if its complement is a reduced word for π.

Example 1.4 (following Example 1.8.2 in [KM05])**.** Let

$$Q' = w_1 w_2 w_4 w_5 w_6 := s_3 s_2 s_3 s_2 s_3$$

and let $\pi = [1432]$. Consider $\Delta(Q', \pi)$: as π has the two reduced expressions $s_2 s_3 s_2 = s_3 s_2 s_3$, the reduced words in Q' for π are

$$w_1 w_2 w_4,\ w_2 w_4 w_5,\ w_4 w_5 w_6,\ w_1 w_5 w_6,\ \text{and } w_1 w_2 w_6,$$

and therefore, $\Delta(Q', \pi)$ is a pentagon as shown in red in Figure 1 with vertices being the letters w_i and facets being $\{w_i, w_{i+1}\}$ where w_7 and w_1 are identified. If we instead consider

$$Q = w_1 w_2 w_3 w_4 w_5 w_6 = s_3 s_2 s_1 s_3 s_2 s_3,$$

we obviously obtain $\Delta(Q, \pi)$ as shown in Figure 1 to be a 5-piece cake (which is the join of $\Delta(Q', \pi)$ with the centered point $w_3 = s_1$).

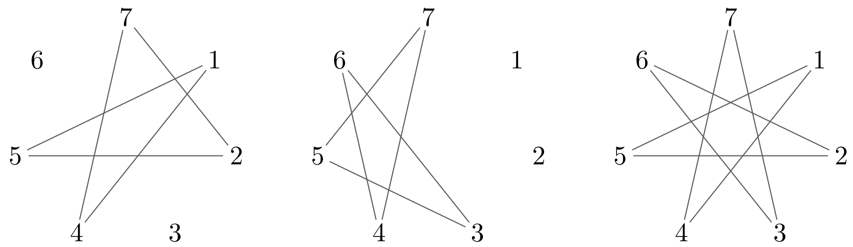


FIGURE 2. Two 2-stars with 5 vertices and a 3-star with 7 vertices.

A. Knutson and E. Miller proved the following two beautiful theorems concerning subword complexes in [KM04, Theorem 2.5, Theorem 3.8].

Theorem 1.5 (Knutson, Miller). *Any subword complex $\Delta(Q,\pi)$ is pure of dimension $l(Q) - l(\pi) - 1$. It is moreover vertex-decomposable, and thus shellable and Cohen-Macaulay.*

In the theorem, $l(Q)$ is simply the number of letters in Q and $l(\pi)$ is the Coxeter length of π which is the length of any minimal expression for π. The pureness and the dimension of $\Delta(Q,\pi)$ then follow immediately, the fact that it is vertex-decomposable is proved by showing that both, the link and the deletion of the first letter in Q, are again subword complexes and thus vertex-decomposable by induction.

Theorem 1.6 (Knutson, Miller). *The $\Delta(Q,\pi)$ is either a triangulated ball or a triangulated sphere.*

2. RESULTS

The main theorem of this article is the following:

Theorem 2.1. *Let*

$$Q_{n-k} := s_{n-k-1} \cdots s_1 s_{n-k-1} \cdots s_2 \cdots s_{n-k-1} s_{n-k-2} s_{n-k-1}$$

be the triangulated reduced word *for $\omega_{0,n-k} := [n-k,\ldots,1] \in \mathcal{S}_{n-k}$ and let*

$$\mathbf{1}_k \times \omega_{0,n-2k} := [1,\ldots,k,n-k,n-k-1,\ldots,k+1].$$

Then

$$\Delta_{n,k} = \Delta(Q'_{n,k}, \mathbf{1}_k \times \omega_{0,n-2k}),$$

where $Q'_{n,k}$ is obtained from $Q_{n,k}$ by deleting all letters s_i for $1 \le i \le k$.

In Example 1.4, the case of $n = 5$ and $k = 1$ is discussed with $Q = Q_4$ and $Q' = Q'_4$.

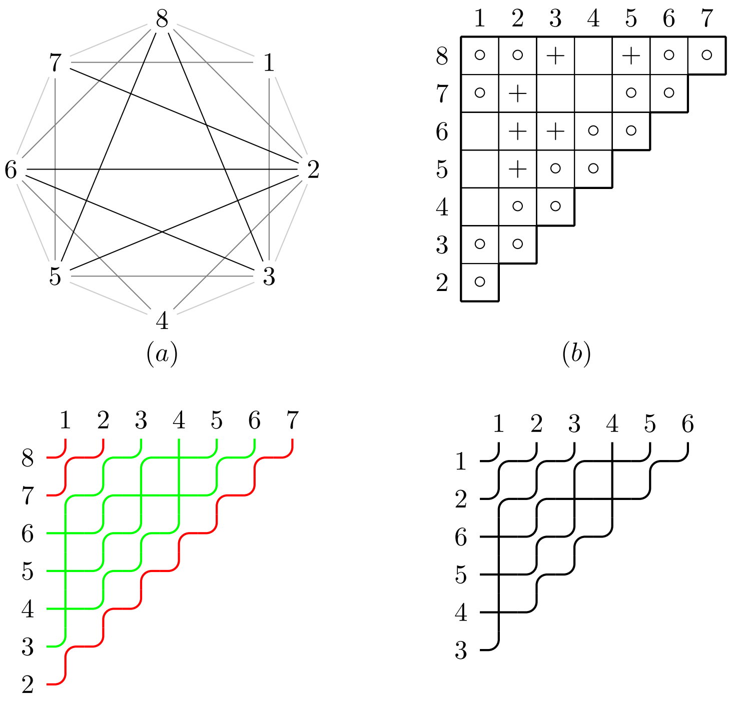
Corollary 2.2. *Statement 1.3 holds.*

Corollary 2.3. *The number of facets in $\Delta_{n,k}$ is given the determinantal expression in Theorem 1.2.*

3. PROOFS

In this section we will provide some further background, proofs and several remarks.

The key ingredient is a property of k-triangulations discovered by V. Pilaud and F. Santos in [PS09, Theorem 1.4]. A k-star is a polygon on $2k+1$ different vertices $v_1,\ldots,v_{2k+1}$ of a convex n-gon (with the condition that $2k+1 \le n$) by connecting v_i and v_j if and only if $|i-j| \in \{k,k+1\}$, see Figure 2 for three examples.



(a) (b) (c) (d)

FIGURE 3. A 2-triangulation and its interpretation in terms of the filling of a staircase diagram. A 2-triangulation, its translation into a pipe diagram and the associated reduced pipe dream for $\pi = [1, 2, 6, 5, 4, 3]$.

Theorem 3.1 (Pilaud, Santos)**.** *A k-triangulation T contains exactly $n-2k$ k-stars. Moreover, a diagonal $\overline{ij}$ in T belongs to one k-star if $|i-j| \in \{k, n-k\}$ and to two k-stars if $k < |i-j| < n-k$.*

We call the construction of a k-triangulation of a convex n-gon from k-stars star property. The next step is to interpret the star property in terms of fillings of a staircase diagram which can be seen as the diagram of the partition $[n-1, n, \ldots, 2, 1]$. It is well-known that a k-triangulation of the n-gon can be encoded as a $(+, \circ, \ )$-filling of the staircase diagram, where a box (i, j) is marked with a

- $+$ if $(i, j+1)$ is a diagonal in the k-triangulation,
- $\circ$ if $(i, j+1)$ is not a diagonal in $\Gamma_{n,k}$,
- and is left blank otherwise.

see Figure 3(a) and (b) for an example, which is the same 2-triangulation of the 8-gon as in [PS09, Figure 19]. The four 2-stars are

$$3-8-2-7-1-3, \quad 4-6-3-5-2-4,$$
$$5-8-3-6-2-5, \quad 6-8-5-7-2-6.$$

To see the star property, we replace every $\circ$ and every $+$ by two turning pipes and every empty box by two crossing pipes as shown in Figure 3(c). Now, every resulting pipe from i to $\pi(i)$ is

- a boundary pipe if $i \leq k$ or if $i > n-k$ and thus $\pi(i) = n+1-i$, or
- an inner pipe if $k < i \leq n-k$ and thus $\pi(i) = i$.

In Figure 3(c), boundary pipes are drawn in red and inner pipes in green. Observe that inner pipes represent exactly the stars in Theorem 3.1 and thus consist of $2k+1$ turning and of $n-2k-1$ crossing pieces and moreover that it connects i on the top with $\pi(i) = i$ on the left. We call this translation of the star property inner pipe property.

We almost reached reduced pipe dreams (or rc-graph) as defined for example in [KM05, Section 1.4]: a pipe dream of size n is a filling of the boxes in the staircase partition $[n-1, \ldots, 1]$ with two crossing pipes $+$ or with two turning pipes ⌟⌜, see Figure 3(c) for an example. The permutation $\pi(D)$ of a pipe dream D is obtained by writing the integers 1 up to n which appear on top again at the end of the associated pipe and read the permutation from top left to bottom left. E.g., the permutation for the pipe dream in Figure 3(d) is [126543]. A pipe dream is reduced if two pipes cross at most once. For a given permutation π, denote the set of all reduced pipe dreams for π by $\mathcal{RP}(\pi)$. Reduced pipe dreams play a central role in the combinatorics of Schubert polynomials, which can be defined as

$$\mathfrak{S}_\pi(x_1, \ldots, x_n) = \sum_{D \in \mathcal{RP}(\pi)} x^D, \tag{1}$$

where $x^D := \prod_{(i,j)\in D} x_i$ and where we say $(i,j) \in D$ if $(i, j+1)$ is filled in D with a $+$, see again e.g. [Man01].

Theorem 3.2. *k-triangulations of the n-gon are in canonical bijection with the set $\mathcal{RP}(\pi)$ of reduced pipe dreams for the permutation $\pi = [1, \ldots, k, n-k, \ldots, k+1]$*

Proof. This theorem follows immediately from the inner pipe property. In particular, observe that $\pi = [1, \ldots, k, n-k, \ldots, k+1]$ is forced by the star property and that the fact that the resulting pipe dream is reduced is equivalent to saying that every inner pipe contains exactly $2k+1$ turns whereas every outer pipe is built completely of turning pipes. □

Observe that we have deleted the outer $k-1$ boundary pipes in the pipe diagram as they do not contribute to the reduced pipe dream, compare Figure 3(b) and (c). Of course, the theorem would stay valid if one would replace π by $[1, \ldots, k, n-k, \ldots, k+1, n-k+1, \ldots, n]$, or, more generally, $\mathcal{RP}(\omega)$ and $\mathcal{RP}(\omega')$ are canonically isomorphic for ω' being obtained from ω by adding fixed points to the end of the one-line notation of ω.

We are now in the position to prove the main theorem of this article.

Proof of Theorem 2.1. Using [KM05, Theorem B] and the definition of $\mathcal{L}_\pi$ therein, the theorem follows from the fact that for $\pi = \mathbf{1}_k \times \omega_{0,n-2k}$, the simplicial complexes $\mathcal{L}_\pi$ and $\Delta(Q'_{n,k}, \pi)$ are equal. □

Note that Theorem 1.1 would follow as well from Theorem 3.2, as the size of any facet of $\Delta_{n,k}$ is given by the number of diagonals in $\Gamma_{n,k}$ minus the length of $\mathbf{1}_k \times \omega_{0,n-2k}$, which equals

$$\binom{n}{2} - nk - \binom{n-2k}{2} = k(n-2k-1).$$

But we use the star property to prove the theorem, and this result is already implicitly contained in the star property.

To prove Corollary 2.2, we need the definition of the Demazure product of simple generators of a Coxeter group. For this, we refer to [KM04, Definition 3.1].

Proof of Corollary 2.2. The fact that $\Delta_{n,k}$ is vertex-decomposable follows immediately from the main theorem together with Theorem 1.3. To prove that $\Delta_{n,k}$ is moreover a triangulated sphere, we use [KM04, Corollary 3.8]: observe that $Q'_{n,k}$

has a suffix which is a reduced expression for $\mathbf{1}_k \times \omega_{0,n-2k}$; thus the Demazure product of $Q'_{n,k}$ equals $\mathbf{1}_k \times \omega_{0,n-2k}$ (as this is the longest element in the standard parabolic subgroup generated by all elements in $Q'_{n,k}$). $\square$

Corollary 3.3. *The number of facets in $\Delta_{n,k}$ is given by the Schubert polynomial $\mathfrak{S}_{\mathbf{1}_k \times \omega_{0,n-2k}}$ evaluated at 1. Moreover, Corollary 2.3 holds.*

Proof. The fact that k-triangulations are counted by $\mathfrak{S}_\pi(1,\ldots,1)$ follows from Theorem 3.2. The counting formula is then immediate with Theorem 1.1 and [FK97, Lemma 1.2]. $\square$

Remark 3.4. A key step in the proof that $\mathfrak{S}_\pi(1,\ldots,1)$ is counted by the determinantal formula is that π is vexillary and thus, $\mathfrak{S}_\pi$ equals a *flagged Schur function*, see [Man01, Corollary 2.6.10]. This proof is related to C. Krattenthaler's proof of the determinantal formula in [Kra06] in the sense that it is as well *not* bijective (as it involves a non-bijective induction) and that it uses the Lindström-Gessel-Viennot Lemma on families of non-intersecting lattice paths. Using a more direct approach, one can as well obtain a purely bijective construction. This will be treated in a joint article with L. Serrano.

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