Matrix factorizations via Koszul duality

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Abstract

In this paper we prove a version of curved Koszul duality for $\mathbb{Z}/2\mathbb{Z}$-graded curved (dg) (co)algebras. A curved version of the homological perturbation lemma is also obtained as a useful technical tool for studying curved (co)algebras and precomplexes.

The results of Koszul duality can be applied to study the dg category of matrix factorizations $\mathsf{MF}(\mathsf{R}, W)$. We show how Dyckerhoff's generating results fit into the framework of curved Koszul duality theory. One immediate application is the construction of a free dg algebra model for $\mathsf{MF}(\mathsf{R}, W)$. As another application we clarify the relationship between the Borel-Moore Hochschild homology of curved (co)algebras and the ordinary Hochschild homology of the category $\mathsf{MF}(\mathsf{R}, W)$.

The same methods can also be used to study the dg category of equivariant or graded matrix factorizations. Both the Koszul duality property and its applications are generalized to include these cases as well. In particular we obtain an explicit set of (classical) generators for these categories. Our results in the graded case are closely related to Seidel's work on the derived category of coherent sheaves on Calabi-Yau hypersurfaces via the CY/LG correspondence.

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1. Introduction

1.1. Motivation. Let R be a finitely generated commutative regular local ring over a ground field k and let W be an element in R. The data of the pair (R, W) is called a Landau-Ginzburg (LG) model following the terminology from physics. To the data of a LG model, we can associate a differential graded (dg) category $\mathsf{MF}(\mathsf{R}, W)$ consisting of matrix factorizations of finite rank (see Section 2 for details). Its homotopy category $[\mathsf{MF}(\mathsf{R}, W)]$ is called the derived category of matrix factorizations. (For a dg category $\mathscr{D}$ we denote by $[\mathscr{D}]$ its homotopy category.)

Matrix factorizations were first introduced by Eisenbud [6] in the study of singularity theory. More recently matrix factorizations also appeared as boundary conditions for B-branes in LG models in topological string theory. In view of mirror symmetry matrix factorizations are mirror dual of Lagrangians in toric manifolds. The category of graded version of matrix factorizations is intimately related to the derived category of coherent sheaves on projective hypersurfaces as demonstrated by Orlov [10]. By Costello's construction [2] the dg category of matrix factorizations and their Hochschild chain complexes give rise to an interesting class of open-closed topological conformal field theories. It is an interesting and difficult question to carry out explicit computations for his constructions.

1.2. The following fundamental results concerning the structure of the dg category $\mathsf{MF}(\mathsf{R}, W)$ were obtained by Dyckerhoff [5] under the assumption that W has isolated singularities:

— $[\mathsf{MF}(\mathsf{R}, W)]$ is classically generated by a single object $\mathsf{k}^{\mathsf{stab}}$;

— The dg algebra $A := \mathsf{End}_{\mathsf{MF}(\mathsf{R},W)}(\mathsf{k}^{\mathsf{stab}})$ is a model for $\mathsf{MF}(\mathsf{R}, W)$;

— We have $\mathsf{HH}_*(\mathsf{MF}(\mathsf{R}, W)) \cong \mathsf{Jac}(W)[\dim \mathsf{R}]$.

The notions of classical generators for a triangulated category is recalled in Section 2. For a dg algebra A and a dg category $\mathscr{D}$ we say that A is a dg algebra model for $\mathscr{D}$ if a certain completion of the category $[\mathscr{D}]$ is equivalent to the derived category of perfect dg A-modules. We refer to [5] for more details. Finally $\mathsf{Jac}(W)$ denotes the Jacobian ring of W, i.e., the quotient of R by the ideal generated by the partial derivatives of W.

1.3. To a LG model (R, W) we can also associate a curved algebra R_W. Then the category $\mathsf{MF}(\mathsf{R}, W)$ can be identified with the dg category of twisted complexes of finite rank over the curved algebra R_W (see Section 2 for details). We denote the latter category by $\mathsf{Tw}^{\mathsf{b}}(\mathsf{R}_W)$ where the superscript "b" is to indicate finite rank. The category of twisted complexes of possibly infinite rank will be denoted by $\mathsf{Tw}(\mathsf{R}_W)$ which sometimes is also denoted by $\mathsf{MF}^{\infty}(\mathsf{R}, W)$.

1.4. If the algebra R_W were not curved a result of Keller [7] would show that

$$\mathsf{HH}_*(\mathsf{R}_W) \cong \mathsf{HH}_*(\mathsf{MF}(\mathsf{R}, W)).$$

A natural question is whether this isomorphism remains true in the presence of curvature. Unfortunately the answer to this question is negative. In fact it was shown in [4] that $\mathsf{HH}_*(\mathsf{R}_W)$ is always trivial for $W \neq 0$.

Following ideas of Segal [14], Căldăraru and the author [4] introduced a modification of the Hochschild chain complex of the curved algebra R_W. This modified complex is called the Borel-Moore Hochschild chain complex using an analogy with the Borel-Moore complex in algebraic topology. In this modification we take the space of chains to consist of direct products of homogeneous chains, instead of the direct sum in the classical definition. The homology of the Borel-Moore Hochschild chain complex was studied in [4] where an explicit calculation showed that

$$\mathsf{HH}_*^{\mathsf{BM}}(\mathsf{R}_W) \cong \mathsf{Jac}(W)[\dim \mathsf{R}].$$

Note that here the ring R does not have to be local.

1.5. The main goal of the current paper is to understand all the results mentioned above from the perspective of Koszul duality. In particular this point of view allows us to relate $\mathsf{HH}_*^{\mathsf{BM}}(\mathsf{R}_W)$ and $\mathsf{HH}_*(\mathsf{MF}(\mathsf{R}, W))$ directly, without computing both sides. As another application of Koszul duality we generalize the results of Dyckerhoff mentioned above to both the orbifold case and the graded case. The following is a more precise formulation of our main results.

1.6. Koszul duality. For an important technical reason we need to dualize and consider curved coalgebras instead of curved algebras. The main example of interest to us is the symmetric coalgebra $\mathsf{C} := \mathsf{sym}(\mathsf{V})$ on a vector space V over the ground field k. The curvature is given by a k-linear map $\mathsf{M} : \mathsf{C} \to \mathsf{k}$. We assume that M vanishes on scalar and linear terms. To this data we can associate a curved coalgebra denoted by $\mathsf{C_M}$. One can also define dg categories $\mathsf{Tw^b}(\mathsf{C_M})$ and $\mathsf{Tw}(\mathsf{C_M})$ of twisted complexes over $\mathsf{C_M}$ whose objects are called matrix cofactorizations. (These constructions are explained in detail in Section 2).

The k-linear dual of a curved coalgebra $\mathsf{C_M}$ is a curved algebra. For example the dual of the symmetric coalgebra $\mathsf{sym}(\mathsf{V})$ is the commutative algebra $\mathsf{R} := \widehat{\mathsf{sym}(\mathsf{V}^\vee)}$ of formal power series on V expanded at the origin. The dual of the map $\mathsf{M} : \mathsf{C} \to \mathsf{k}$ defines a map $W : \mathsf{k} \to \mathsf{R}$ which gives the curvature element $W \in \mathsf{R}$ (the image of $1 \in \mathsf{k}$).

1.7. Let $\mathsf{C_M}$ be a curved coalgebra as above and let R_W be its dual curved algebra. Then the categories $\mathsf{Tw^b}(\mathsf{C_M})$ and $\mathsf{Tw^b}(\mathsf{R}_W) = \mathsf{MF}(\mathsf{R}, W)$ are related by an equivalence of dg categories

$$\mathsf{D} : \mathsf{Tw^b}(\mathsf{C_M})^{\mathsf{op}} \to \mathsf{Tw^b}(\mathsf{R}_W) = \mathsf{MF}(\mathsf{R}, W)$$

where the functor D is simply the k-linear dualizing operation (for more details see Lemma 2.7). Note that this operation does not extend to an equivalence on infinite rank objects.

1.8. Koszul duality for (curved) coalgebras exhibits a homotopy equivalence between the categories of twisted complexes over a (curved) coalgebra and those over the associated cobar algebra. In this context the theory was developed by Positselski [13]. Note that the classical Koszulness assumption is not necessary here.

For a (coaugmented) curved coalgebra $\mathsf{B_M}$, we can define the associated cobar dg algebra $\Omega(\mathsf{B_M})$ and two natural dg functors

$$\Phi : \mathsf{Tw}(\mathsf{B_M}) \to \mathsf{Tw}(\Omega(\mathsf{B_M})) \text{ and } \Psi : \mathsf{Tw}(\Omega(\mathsf{B_M})) \to \mathsf{Tw}(\mathsf{B_M})$$

between the two categories of twisted complexes (of possibly infinite rank). Both the cobar construction and the definitions for functors Φ, Ψ are recalled in Section 2 where we also prove the following theorem.

1.9. Theorem. *Let $\mathsf{B_M}$ be a coaugmented curved coalgebra and let $\Omega(\mathsf{B_M})$ be its cobar dg algebra. Then the functors Φ and Ψ are inverse homotopy equivalences of dg categories*

$$\mathsf{Tw}(\mathsf{B_M}) \cong \mathsf{Tw}(\Omega(\mathsf{B_M})).$$

If furthermore the coalgebra B *is conilpotent, then the dg algebra* $\Omega(\mathsf{B}_\mathsf{M})$ *itself is a compact generator for* $[\mathsf{Tw}(\Omega(\mathsf{B}_\mathsf{M}))]$ *and* $\Psi(\Omega(\mathsf{B}_\mathsf{M}))$ *is a compact generator for* $[\mathsf{Tw}(\mathsf{B}_\mathsf{M})]$.

Remark: The above theorem is essentially a special case of results in [13] where much more general Koszul duality type theorems are obtained by Positselski. Our proof is simpler and more direct, which results from an interesting technique for studying curved (co)algebra and precomplexes — a curved version of the homological perturbation lemma. The main statement of this lemma is the following. Details can be found in the Appendix A.

1.10. Lemma. *Let* $(\mathsf{i}, \mathsf{p}, \mathsf{H})$ *be a special homotopy retraction data between two complexes* $(\mathsf{L}, \mathsf{b}) \overset{\mathsf{i}}{\hookrightarrow} (\mathsf{M}, \mathsf{d})$. *Let* δ *be a small curved perturbation of* (M, d). *Then there exists a perturbed homotopy retraction data* $(\mathsf{i}_1, \mathsf{p}_1, \mathsf{H}_1)$ *between the perturbed precomplexes.*

1.11. Dyckerhoff's results in view of Koszul duality. We specialize to the curved (co)algebra in (1.6), and we will keep these notations until (1.17). As symmetric coalgebras are conilpotent, all results in Theorem 1.9 apply to this case. The theorem below shows that Dyckerhoff's compact generator $\mathsf{k}^{\mathsf{stab}}$ comes from Koszul duality. Note that for this identification we do not need to assume that W has isolated singularities.

1.12. Theorem. *The object* $\Psi(\Omega(\mathsf{C}_\mathsf{M}))$ *is homotopic to a finite rank object in* $\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})$ *which we will still denote by* $\Psi(\Omega(\mathsf{C}_\mathsf{M}))$. *Hence its* k*-linear dual makes sense and can be identified by a homotopy equivalence*

$$\mathsf{D}\Psi(\Omega(\mathsf{C}_\mathsf{M})) \cong \mathsf{k}^{\mathsf{stab}}.$$

Moreover the dg algebra $\Omega(\mathsf{C}_\mathsf{M})$ *is homotopic to* A^{op} *where* A *is the dg algebra model constructed by Dyckerhoff.*

1.13. It *does not* follow from the above theorem that $\mathsf{k}^{\mathsf{stab}}$ classically generates $[\mathsf{MF}(\mathsf{R}, W)]$. Indeed it follows from Theorem 1.9 that $\Psi(\Omega(\mathsf{C}_\mathsf{M}))$ always classically generates the full compact subcategory of $\mathsf{Tw}(\mathsf{C}_\mathsf{M})$, but the subcategory $[\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})]$ might not be a compact subcategory of $[\mathsf{Tw}(\mathsf{C}_\mathsf{M})]$ for general W. See Section 3 for details.

1.14. Theorem. *The following statements are equivalent:*

(a) *W has isolated singularities;*

(b) $\mathsf{k}^{\mathsf{stab}}$ *classically generates* $[\mathsf{MF}(\mathsf{R}, W)]$*;*

(c) $[\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})]$ *is a compact subcategory of* $[\mathsf{Tw}(\mathsf{C}_\mathsf{M})]$.

Remark: It follows from Theorems 1.12 and 1.14 that the dg algebra $\Omega(\mathsf{C}_{\mathsf{M}})$ can be taken as a dg algebra model for $\mathsf{MF}(\mathsf{R}, W)$ when W has isolated singularities. The advantage of our dg algebra model $\Omega(\mathsf{C}_{\mathsf{M}})$ is that it is a free dg algebra. The associated minimal model A_∞ algebra of it is studied in Section 4.

1.15. Hochschild homology. We can also use Koszul duality to clarify the relationship between the Borel-Moore Hochschild homology of the curved algebra R_W and the ordinary Hochschild homology of $\mathsf{MF}(\mathsf{R}, W)$. The results are summarized in the following theorem.

1.16. Theorem. *Assume that W has isolated singularities. Then there are natural isomorphisms:*

$$\mathsf{HH}_*(\mathsf{MF}(\mathsf{R}, W)) \cong \mathsf{HH}_*(\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})) \cong \mathsf{HH}_*(\mathsf{C}_{\mathsf{M}});$$
$$\mathsf{HH}_*^{\mathsf{BM}}(\mathsf{R}_W) \cong \mathsf{HH}_*(\mathsf{C}_{\mathsf{M}})^\vee.$$

Remark: It is an interesting puzzle to understand why in the second isomorphism the Borel-Moore Hochschild homology is naturally the dual of the Hochschild homology of $\mathsf{MF}(\mathsf{R}, W)$. This might be explained by a relationship between Koszul duality and a natural pairing (generalized Mukai-pairing) on the Hochschild homology.

1.17. LG Orbifolds. In Section 6 we present a generalization of Theorem 1.9 and its applications to the study of LG orbifolds. The main ideas remain the same. We will work over the field of complex numbers $\mathbb{C}$ as we need to consider characters of groups.

Consider a LG model (R, W) with a finite abelian group G acting on it. This means that G acts on the algebra R while preserving the function W. In this situation, one can consider the dg category of equivariant matrix factorizations. Loosely speaking, objects are matrix factorizations with G-actions and all morphisms are required to be G-equivariant. One can think of this category as the category of sheaves on a hypothetical orbifold $[\mathsf{Spec}(\mathsf{R}_W)/\mathsf{G}]$. We denote this category by $\mathsf{Tw}^{\mathsf{b}}([\mathsf{R}_W/\mathsf{G}])$ or $\mathsf{MF}_{\mathsf{G}}(\mathsf{R}, W)$. The main results proved in Section 6 are summarized in the following theorem.

1.18. Theorem. *Assume that W has isolated singularities. Then we have*

— *The category $[\mathsf{Tw}^{\mathsf{b}}([\mathsf{R}_W/\mathsf{G}])]$ or $[\mathsf{MF}_{\mathsf{G}}(\mathsf{R}, W)]$ is classically generated by*

$$\left\{\mathsf{k}^{\mathsf{stab}} \otimes \mathbb{C}_\chi \mid \chi \text{ is a character for the group } \mathsf{G}\right\}$$

where $\mathbb{C}_\chi$ denotes the one dimensional representation associated to the character χ.

— *The smash product dg algebra* $\Omega(C_M)\sharp G$ *is a model for* $MF_G(R, W)$.

— *For the Hochschild homology we have*

$$\begin{aligned} HH_*(MF_G(R, W)) \cong & HH_*(Tw^b([C_M/G])) \cong HH_*(C_M\sharp G); \\ HH_*^{BM}(R_W\sharp G) \cong & HH_*(C_M\sharp G)^\vee. \end{aligned}$$

Remark: In [4] the vector space $HH_*^{BM}(R_W\sharp G)$ was explicitly computed as

$$HH_*^{BM}(R_W\sharp G) = (\oplus_{g\in G} HH_*^{BM}(R_W|_g))^G$$

where $R_W|_g$ denotes the curved algebra associated to the LG model on the g-fixed points of $\mathsf{Spec}(R)$. The commutative ring involved here is the quotient of R by the ideal generated by elements of the form $f - g(f)$ and the curvature is the image of W in this quotient ring.

1.19. Graded matrix factorizations. We still work over the field $k = \mathbb{C}$. Let $C := \mathsf{sym}(V)$ be the symmetric coalgebra and let $M : C \to k$ be a dual potential. This time we consider M to be homogeneous of degree d and assume that $d \geq 2$. The $\mathbf{Z}$-graded dual algebra of C is the (non-complete) symmetric algebra $S := \mathsf{sym}(V^\vee)$ with the ordinary polynomial grading. The curvature $W = M^\vee$ is homogeneous of degree d.

Such data carries an action of the group $G := \mathbf{Z}/d\mathbf{Z}$. Indeed there is a $\mathbb{C}^*$-action on S (C respectively) induced from the $\mathbf{Z}$-grading and the $\mathbf{Z}/d\mathbf{Z}$-action on S (C respectively) is defined through the natural embedding of groups $i + d\mathbf{Z} \mapsto \zeta_d^i$, where ζ_d is a primitive d-th root of unity. As W (M respectively) has degree d, it is preserved by this G-action.

Consider the smash product algebra $S_W\sharp G$ and its dual coalgebra $C_M\sharp G$. One can define a $\mathbf{Z}$-grading on $S_W\sharp G$ ($C_M\sharp G$) so that it becomes a $\mathbf{Z}$-graded curved algebra (coalgebra respectively). With respect to this grading, the curvature term has degree 2. Details of this $\mathbf{Z}$-grading can be found in Section 7, which is also explained in [4] from a categorical point of view. Our main results are included in the following theorem.

1.20. Theorem. *Assume that* W *has isolated singularities. Then we have*

— *The category* $[MF^{gr}(S, W)]$ *is classically generated by*

$$k^{stab}(d-1), k^{stab}(d-2), \cdots, k^{stab}$$

where the shifts in the parentheses are polynomial degree shifts of graded S*-modules.*

— *There is a* $\mathbf{Z}$*-graded smash product algebra* $\Omega(\mathsf{C_M})\sharp\mathsf{G}$ *that can be taken as a dg algebra model for* $\mathsf{MF^{gr}}(\mathsf{S}, W)$.

— *For the Hochschild homology we have the following isomorphisms:*

$$\mathsf{HH}_*(\mathsf{MF^{gr}}(\mathsf{S}, W)) \cong \mathsf{HH}_*(\mathsf{Tw}^{\mathsf{b}}_{\mathsf{Z}}(\mathsf{C_M}\sharp\mathsf{G})) \cong \mathsf{HH}_*(\mathsf{C_M}\sharp\mathsf{G});$$
$$\mathsf{HH}^{\mathsf{BM}}_*(\mathsf{S}_W\sharp\mathsf{G}) \cong \mathsf{HH}_*(\mathsf{C_M}\sharp\mathsf{G})^\vee$$

where the $\vee$ *denotes the graded dual operation.*

Remark: This result is in agreement with the results of Orlov [10] on the relationship between $[\mathsf{MF^{gr}}(\mathsf{S}, W)]$ and the derived category of coherent sheaves on the corresponding projective hypersurface defined by W (see semi-orthogonal decompositions in Section 2 of *loc. cit.*).

Remark: In the Calabi-Yau situation (when $\mathsf{d} = \mathsf{dim}(\mathsf{V})$) Theorem 1.20 is also closely related to the results of Seidel in his unpublished notes [15]. There Seidel obtains an A_∞ algebra structure on the vector space $\wedge^*(\mathsf{V})\sharp\mathsf{G}$ as an A_∞ algebra model for the category of coherent sheaves. As the homology of the dg algebra $\Omega(\mathsf{C_M})\sharp\mathsf{G}$ in Theorem 1.20 is $\wedge^*(\mathsf{V})\sharp\mathsf{G}$ we obtain an A_∞ algebra structure on the same underlying vector space via homological transfer property. Presumably these two A_∞ structures should be at least homotopic to each other if not the same. The explanation for such a coincidence would again be the (dg version of) the CY/LG correspondence [4].

1.21. The paper is organized as follows. In Section 2 we first recall the basic constructions in the Koszul duality theory. After that we prove Theorem 1.9. The (curved) homological perturbation lemma from Appendix A is used in the proofs.

Sections 3, 4, and 5 are applications of the Koszul duality theory to LG models. Section 3 is devoted to the study of generators for $\mathsf{MF}(\mathsf{R}_W)$. We show that the object $\Psi(\Omega(\mathsf{C_M}))$ is homotopic to a matrix cofactorization of finite rank and hence we can apply the dualizing functor D. Then it takes some extra work to match the resulting object with $\mathsf{k}^{\mathsf{stab}}$ from [5]. In Section 4 we compute the homology of the dg model $\Omega(\mathsf{C_M})$. We comment on the minimal model A_∞ algebra structure and its relationship with more standard Koszul duality. Section 5 clarifies the relationship between the Borel-Moore Hochschild homology of the curved algebra R_W and the ordinary Hochschild homology of the category $\mathsf{MF}(\mathsf{R}, W)$.

In Section 6 we generalize the main results to the case of LG orbifolds. The results in Theorem 1.18 are proved. Finally, Section 7 deals with the category of graded matrix factorizations. Theorem 1.20 is proved there.

In Appendix A we recall the homological perturbation lemma and prove the curved version of it. The proof is pretty much the same as that of the case with no curvature. In Appendix B we construct an explicit special homotopy between the cobar algebra $\Omega(C)$ and its homology $\wedge^*(V)$. This follows from a form of Hodge theory on the cobar complex $\Omega(C)$.

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2. Koszul duality for the (co)bar constructions

In this section we recall the bar and cobar constructions and prove a version of Koszul duality between $\mathbb{Z}/2\mathbb{Z}$-graded curved coalgebras and their cobar algebras. Then we prove some useful properties concerning the derived categories of cobar algebras. The results proved in this section are essentially due to Positselski [13], although we give more direct proofs which use the homological perturbation lemma.

Throughout this section we will work over a base field k. Linear algebra operations such as tensor product or homomorphism between vector spaces are all taken over k unless otherwise stated.

2.1. Curved differential graded (dg) algebras. A curved differential graded (dg) algebra structure on a super vector space A is an associative algebra structure on A together with an odd linear map $d : A \to A$ and an even element $W \in A$ such that

- $d(W) = 0$;
- $d^2(a) = [W, a]$;
- $d(a_1 a_2) = d(a_1)a_2 + (-1)^{|a_1|} a_1 d(a_2)$ (Leibniz rule).

Here $[-,-]$ is the graded commutator and $|a|$ is the parity of a. The curved dg algebra obtained from the data above will be denoted by A_W.

Here is an example of a curved differential graded algebra that will be of primary interest in this paper. Let V be a finite dimensional vector space over a field k. Consider the vector space $\mathsf{R} := \widehat{\mathsf{sym}(\mathsf{V}^\vee)}$ of formal power series on V expanded at the origin. As a super vector space R is concentrated the even part. The associative algebra structure is the ordinary multiplication of power series. The differential is trivial and the curvature element can be chosen to be any element $W \in \mathsf{R}$. One easily checks that this data defines a curved algebra R_W.

2.2. Matrix factorizations. We begin by recalling the definition of the category $\mathsf{Tw}(\mathsf{R}_W)$ of twisted complexes over this curved algebra R_W. The objects of this category are pairs (E, Q) where E is a $\mathbf{Z}/2\mathbf{Z}$-graded free R-module and Q is an odd R-linear map such that $\mathsf{Q}^2 = W\,\mathsf{id}$. The morphism space between two objects (E, Q) and (F, P) consists of all R-linear maps from E to F. As such, the Hom space inherits a differential defined by $\mathsf{D}(\varphi) = \mathsf{P} \circ \varphi - (-1)^{|\varphi|}\varphi \circ \mathsf{Q}$. One easily checks that D squares to zero as $W\,\mathsf{id}$ is in the center of any matrix algebra.

This differential makes the category $\mathsf{Tw}(\mathsf{R}_W)$ into a dg category. Note that here we allow possibly infinite rank free R-modules in $\mathsf{Tw}(\mathsf{R}_W)$. We denote by $\mathsf{Tw}^{\mathsf{b}}(\mathsf{R}_W)$ the full subcategory of $\mathsf{Tw}(\mathsf{R}_W)$ consisting of twisted complexes that are of finite rank over R. The category $\mathsf{Tw}^{\mathsf{b}}(\mathsf{R}_W)$ ($\mathsf{Tw}(\mathsf{R}_W)$ respectively) is sometimes also denoted by $\mathsf{MF}(\mathsf{R}, W)$ ($\mathsf{MF}^\infty(\mathsf{R}, W)$ respectively).

As the category $\mathsf{Tw}(\mathsf{R}_W)$ has a dg structure we can define the notion of homotopy between morphisms and objects. More precisely, we say two morphisms f and g are homotopic if $\mathsf{f} - \mathsf{g}$ is exact. We say two objects E and F are homotopic if there are morphisms $\mathsf{f} : \mathsf{E} \to \mathsf{F}$ and $\mathsf{g} : \mathsf{F} \to \mathsf{E}$ such that $\mathsf{f} \circ \mathsf{g}$ is homotopic to id_{F} and $\mathsf{g} \circ \mathsf{f}$ is homotopic to id_{E}.

2.3. We mention some terminologies. For a k-linear category $\mathscr{C}$, recall that a predifferential on a $\mathbf{Z}/2\mathbf{Z}$-graded object L is an odd morphism $\mathsf{d} : \mathsf{L} \to \mathsf{L}$ such that d^2 lies in the center of $\mathsf{End}_{\mathscr{C}}(\mathsf{L})$. The data given by the pair (L, d) is then called a precomplex. For example a matrix factorization structure Q on a free R-module E is in particular a predifferential on E over the k-linear category of R-modules. In general the category of precomplexes can also be endowed with a dg structure.

2.4. Curved differential graded coalgebras. Dualizing the definition for curved dg algebras we arrive at the definition for curved dg coalgebras. A curved dg coalgebra structure on a vector space B is a $\mathbf{Z}/2\mathbf{Z}$-graded coassociative coalgebra structure on B together with an odd map $d : B \to B$ and an even map $M : B \to k$ such that

- $M \circ d = 0$;
- $d^2(x) = M(x^{(1)})x^{(2)} - x^{(1)}M(x^{(2)})$;
- coLeibniz rule.

Here we have used Sweedler's notation for the coproduct. As before, we denote the curved coalgebra obtained from the data above by B_M.

As an example we work out the dual of the example given in the previous subsection. Let $C := \mathsf{sym}(V)$ be the vector space of symmetric tensors on V. Again we consider C as a super vector space concentrated in the even part. There is a natural coalgebra structure on $C = \mathsf{sym}(V)$ whose dual algebra is the commutative algebra R. The differential is trivial again. The curvature term is any linear map $M : C \to k$.

2.5. Matrix cofactorizations. One can construct a category $\mathsf{Tw}(C_M)$ of twisted complexes over C_M. The objects are pairs (E, Q) with E a cofree C-comodule and Q an odd comodule map on E such that the dual of the matrix factorization identity holds,

$$Q^2(x) = M(x^{(1)})x^{(2)}.$$

Here we write the coaction map to be $\rho(x) = \sum x^{(1)} \otimes x^{(2)}$ for $x^{(1)} \in C$ (thus we are using left module structure). The Hom spaces and differentials on Hom spaces are defined in a similar way as for matrix factorizations. Objects in $\mathsf{Tw}(C_M)$ will be called matrix cofactorizations.

As before we can also define a dg structure on $\mathsf{Tw}(C_M)$ using the fact that matrix cofactorizations are also precomplexes. The full subcategory of $\mathsf{Tw}(C_M)$ consisting of matrix cofactorizations that are of finite rank over C will be denoted by $\mathsf{Tw}^b(C_M)$.

2.6. We recall some useful properties of cofree comodules. First of all we consider cofree comodules of the form $C \otimes V$ for some k-vector space V (possibly infinite dimensional) in the twist construction above. Moreover in the abelian category $\mathscr{A}$ of C-comodules, cofree comodules are injective objects and hence is closed under direct product in $\mathscr{A}$. For example we have $\prod(C \otimes V_i) \cong C \otimes (\prod V_i)$. A special property for $\mathscr{A}$ is that the class of injective objects is also closed under direct sum in $\mathscr{A}$.

There is a simple relation between the two dg categories $\mathsf{Tw}^b(R_W)$ and $\mathsf{Tw}^b(C_M)$, made precise in the following lemma.

2.7. Lemma. *Let* R_W *and* C_M *be as above. Assume that the dual map of* M *is* W. *Define a functor* $\mathsf{D} : \mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M}) \to \mathsf{Tw}^\mathsf{b}(\mathsf{R}_W)$ *given by*

$$(\mathsf{E}, \mathsf{Q}) \overset{\mathsf{D}}{\mapsto} (\mathsf{E}^\vee, \mathsf{Q}^\vee)$$

on objects and for any morphism $\mathsf{f} \in \mathsf{Hom}_{\mathsf{Tw}(\mathsf{C}_\mathsf{M})}((\mathsf{E}, \mathsf{Q}), (\mathsf{F}, \mathsf{P}))$

$$\mathsf{D}(\mathsf{f}) := \mathsf{f}^\vee : (\mathsf{F}^\vee, \mathsf{P}^\vee) \to (\mathsf{E}^\vee, \mathsf{Q}^\vee).$$

Then D *is a dg equivalence between* $\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})^{\mathsf{op}}$ *and* $\mathsf{Tw}^\mathsf{b}(\mathsf{R}_W)$.

Proof. Observe that a map $\mathsf{h} : \mathsf{C} \to \mathsf{C}$ of C-comodules is uniquely determined by its composition with the counit map. Conversely, any linear map $\alpha : \mathsf{C} \to \mathsf{k}$ defines a map of C-comodules by

$$\mathsf{C} \to \mathsf{C} \otimes \mathsf{C} \overset{\alpha\otimes\mathsf{id}}{\to} \mathsf{k} \otimes \mathsf{C} = \mathsf{C}.$$

It is easy to check that this defines an isomorphism between $\mathsf{Hom}_\mathsf{C}(\mathsf{C}, \mathsf{C})$ and $\mathsf{Hom}_\mathsf{k}(\mathsf{C}, \mathsf{k}) = \mathsf{R}$. More generally, for two cofree C-comodules $\mathsf{E}_1 = \mathsf{C} \otimes \mathsf{V}_1$ and $\mathsf{E}_2 = \mathsf{C} \otimes \mathsf{V}_2$ with V_1 and V_2 finite dimensional vector spaces over k we have

$$\begin{aligned}\mathsf{Hom}_\mathsf{C}(\mathsf{E}_1, \mathsf{E}_2) = \mathsf{Hom}_\mathsf{C}(\mathsf{C} \otimes \mathsf{V}_1, \mathsf{C} \otimes \mathsf{V}_2) &\cong \mathsf{Hom}_\mathsf{C}(\mathsf{C}, \mathsf{C}) \otimes \mathsf{Hom}_\mathsf{k}(\mathsf{V}_1, \mathsf{V}_2)\\ &\cong \mathsf{R} \otimes \mathsf{Hom}(\mathsf{V}_1, \mathsf{V}_2).\end{aligned}$$

For the Hom space between DE_2 and DE_1, we have

$$\begin{aligned}\mathsf{Hom}_\mathsf{R}((\mathsf{C} \otimes \mathsf{V}_2)^\vee, (\mathsf{C} \otimes \mathsf{V}_1)^\vee) &= \mathsf{Hom}_\mathsf{R}(\mathsf{R} \otimes \mathsf{V}_2^\vee, \mathsf{R} \otimes \mathsf{V}_1^\vee)\\ &= \mathsf{R} \otimes \mathsf{Hom}(\mathsf{V}_2^\vee, \mathsf{V}_1^\vee) = \mathsf{R} \otimes \mathsf{Hom}(\mathsf{V}_1, \mathsf{V}_2)\end{aligned}$$

where the first and the last equality follow from V_1 and V_2 being finite dimensional. Thus we have verified that the functor D is an equivalence. A direct computation shows that it also preserves the differential and hence the lemma is proved. □

2.8. The bar and cobar constructions. We first recall the bar construction for dg algebras. Let A be a unital dg algebra with a k-linear splitting $\epsilon : \mathsf{A} \to \mathsf{k}$ of the unit map. Denote by A^+ the kernel of ϵ. The splitting ϵ induces an isomorphism of k-vector spaces between A and $\mathsf{A}^+ \oplus \mathsf{k}$. We will freely make use of this isomorphism. Moreover we will assume that the map ϵ is *always* a map of algebras (but not necessarily of dg algebras) and hence the space A^+ is closed under the product (but not necessarily closed under the differential). If the map ϵ also preserves the differential it is called an augmentation of A.

Given a splitting ϵ the bar construction produces a curved dg coalgebra $B(A)$ defined as follows. The coalgebra structure is simply the free tensor coalgebra generated by $A^+[1]$. Explicitly we have

$$B(A) := T^c(A^+[1]) = \oplus_{k=0}^{\infty}(A^+[1])^{\otimes k}.$$

The space of coderivations on $B(A)$ can be identified with the space of linear maps $\mathsf{Hom}(B(A), A^+)$. The differential d is a coderivation on $B(A)$, determined by the following two components

$$A^+ \otimes A^+ \hookrightarrow A \otimes A \xrightarrow{\cdot} A \to A^+;$$
$$A^+ \xrightarrow{d_A} A \to A^+.$$

The curvature of $B(A)$ is given by

$$A^+ \hookrightarrow A \xrightarrow{d_A} A \to k$$

on the tensor component A^+ and zero otherwise. Observe that the curvature vanishes if and only if the splitting map ϵ is an augmentation.

2.9. Next we explain the cobar construction for curved coalgebras with a coaugmentation. Let B_M be a curved coalgebra and let $\eta : k \to B$ be a coaugmentation of B_M. This boils down to requiring that η splits the counit map and that it is a map of coalgebras such that

$$M \circ \eta = 0.$$

Denote by B^+ the cokernel of η which can be identified with the kernel of the counit through the direct sum decomposition $B \cong B^+ \oplus k$.

Given a coaugmentation as above, the cobar construction $\Omega(B_M)$ is a dg algebra. Explicitly, as an associative algebra $\Omega(B_M)$ is the free tensor algebra generated by $B^+[-1]$:

$$T(B^+[-1]) = \oplus_{k=0}^{\infty}(B^+[-1])^{\otimes k}.$$

The differential d is a derivation on $\Omega(B_M)$, determined by the following two components

$$B^+ \hookrightarrow B \to B \otimes B \to B^+ \otimes B^+;$$
$$B^+ \hookrightarrow B \xrightarrow{M} k.$$

2.10. An example. We work out an example to illustrate these constructions. Consider the the case of the cocommutative coalgebra $\mathsf{C} := \mathsf{sym}(\mathsf{V})$ with curvature given by a map $\mathsf{M} : \mathsf{sym}(\mathsf{V}) \to \mathsf{k}$. In order to have a coaugmentation, we assume that M vanishes on scalar terms in C. It follows that the inclusion of scalars $\eta : \mathsf{k} \to \mathsf{C}$ is a coaugmentation of C_M. Indeed, it is easy to see that η splits the counit and is a map of coalgebras. The fact that $\mathsf{M} \circ \eta = 0$ follows from our additional assumption that M vanishes on scalars.

By the above construction $\Omega(\mathsf{C}_\mathsf{M})$ is the free tensor algebra generated by $\mathsf{sym}(\mathsf{V})^+[-1]$ with differential given by the sum of two components which we denote by d^+ and d^-. The d^+ differential comes from the coproduct on $\mathsf{sym}(\mathsf{V})^+[-1]$ and d^- from the curvature M. Explicitly d^+ and d^- are determined by requiring them to be k-linear and to act on monomials $f_1, \cdots, f_k$ by

$$d^+(f_1|f_2|\cdots|f_k) = \sum_{i=1}^{k} (-1)^{i-1} f_1|\cdots|\Delta(f_i)|\cdots|f_k,$$
$$d^-(f_1|f_2|\cdots|f_k) = \sum_{i=1}^{k} (-1)^{i-1} M(f_i) f_1|\cdots|\widehat{f_i}|\cdots|f_k$$

where Δ is the reduced coproduct given by

$$\Delta(f) = \sum_{(g,h)|gh=f, g\neq 1, h\neq 1} g \otimes h.$$

2.11. Twisting cochains. For a curved dg coalgebra B_M and a unital differential graded algebra A, one can construct a curved dg algebra structure on the space of k-linear maps $\mathsf{Hom}(B, A)$. It is defined by the following formulas:

- Curvature: $W(B, A)$: $B \overset{M}{\to} k \overset{\text{unit}}{\to} A$;
- Differential: $(d\varphi)(x) = d(\varphi(x)) - (-1)^{|\varphi|}\varphi(dx)$;
- Product: $(\varphi * \psi)(x) = (-1)^{|x^{(1)}||\psi|}\varphi(x^{(1)})\psi(x^{(2)})$.

A twisting cochain from B to A is an odd element $\tau \in \mathsf{Hom}(B, A)$ such that

$$\tau * \tau + d\tau + W(B, A) = 0.$$

There are natural twisting cochains associated to the (co)bar constructions. For a dg algebra A we have a natural map from its bar construction $\mathsf{B}(A)$ to itself given by

$$\tau_A : \mathsf{B}(A) \to A^+ \hookrightarrow A.$$

For a curved dg coalgebra B_M, we have

$$\tau_{\mathsf{B}_M} : \mathsf{B}_M \to \mathsf{B}^+ \hookrightarrow \Omega(\mathsf{B}_M).$$

It is easy to verify τ_A and τ_{B_M} are both twisting cochains. The bar-cobar adjunction

$$\mathsf{Hom}(\Omega(C), A) \cong \mathsf{Tw}(C, A) \cong \mathsf{Hom}(C, \mathsf{B}(A))$$

can also generalized to this context with appropriate categories of algebras and coalgebras [13].

2.12. Correspondence of twisted complexes. One use of twisting cochains is to define a correspondence between categories of twisted complexes. We work out the this correspondence for a coaugmented curved coalgebra B_M. The proofs can be easily adapted to include more general cases as well.

2.13. We want to construct dg functors $\Phi : \mathsf{Tw}(\mathsf{B}_M) \to \mathsf{Tw}(\Omega(\mathsf{B}_M))$ and $\Psi : \mathsf{Tw}(\Omega(\mathsf{B}_M) \to \mathsf{Tw}(\mathsf{B}_M)$. We begin by constructing Φ. Let (E, Q) be a matrix cofactorization over B_M (with left B-comodule structure). We will produce a twisted complex denoted by $\Phi(E)$ over the cobar dg algebra $\Omega(\mathsf{B}_M)$ in the following way. As a vector space over k, it is simply $\Omega(\mathsf{B}_M) \otimes E$. The left $\Omega(\mathsf{B}_M)$-module structure is induced from that of $\Omega(\mathsf{B}_M)$. The differential on $\Omega(\mathsf{B}_M) \otimes E$ is defined using the natural twisting cochain τ_{B_M}:

$$d(x \otimes e) = dx \otimes e + (-1)^{|x|} x \otimes Qe + (-1)^{|x|+1} x\tau(y^{(1)}) \otimes e^{(2)}$$

where we have denoted the coaction map $\rho : E \to \mathsf{B} \otimes E$ by $\rho(e) = y^{(1)} \otimes e^{(2)}$ for $y^{(1)} \in \mathsf{B}$. One checks that $d^2 = 0$ and that it is compatible with the left module structure on $\Phi(E)$. Hence $\Phi(E)$ is a free dg module or twisted complex over $\Omega(\mathsf{B}_M)$. We sometimes write $\Phi(E) = \Omega \otimes^\tau E$ where the superscript τ is to indicate that we are using the twisting cochain τ to define the differential on $\Phi(E)$.

However note that $\Phi(E)$ is of infinite rank whenever B is of infinite dimension over k. For this reason we need to consider $\mathsf{Tw}(\Omega(\mathsf{B}_M))$ instead of $\mathsf{Tw}^b(\Omega(\mathsf{B}_M))$.

For a morphism $f : (E, Q) \to (F, P)$ in $\mathsf{Tw}(\mathsf{B}_M)$, define $\Phi(f) = \mathsf{id} \otimes f$ from $\Phi(E)$ to $\Phi(F)$. In this way we have defined a functor Φ from $\mathsf{Tw}(\mathsf{B}_M)$ to $\mathsf{Tw}(\Omega(\mathsf{B}_M))$. One can check that Φ is a dg functor between dg categories.

2.14. In the reverse direction, if (F, d) is a twisted complex over $\Omega(\mathsf{B}_\mathsf{M})$, we define a matrix cofactorization $\Psi(\mathsf{F})$ over B_M. As a vector space it is $\mathsf{B} \otimes \mathsf{F}$. The left B-comodule structure is induced from that of B and the matrix cofactorization map is defined by

$$\mathsf{Q}(\mathsf{x} \otimes \mathsf{f}) = \mathsf{d}\mathsf{x} \otimes \mathsf{f} + (-1)^{|\mathsf{x}|}\mathsf{x} \otimes \mathsf{d}\mathsf{f} + \mathsf{x}^{(1)} \otimes \tau(\mathsf{x}^{(2)})\mathsf{f}$$

where $\tau(\mathsf{x}^{(2)})\mathsf{f}$ is the action of $\Omega(\mathsf{B}_\mathsf{M})$ on F. One checks that Q satisfies the matrix cofactorization identity and hence defines a twisted complex (again of infinite rank) over B_M. Similarly the above construction extends to the morphism space and hence defines a dg functor Ψ in the reverse direction.

2.15. As both the categories $\mathsf{Tw}(\mathsf{B}_\mathsf{M})$ and $\mathsf{Tw}(\Omega(\mathsf{B}_\mathsf{M}))$ are dg categories, one can speak of the notion of homotopy between dg functors. Namely we say a functor F is homotopic to another functor G if they are homotopic when applied to any object. Hence we can also define the notion of homotopy between categories by requiring that there are functors in both ways such that their compositions are homotopic to the identity functors. The following theorem is the Koszul duality property for the cobar construction. Essentially it is duality between the categories of twisted complexes.

2.16. Theorem. *The functors Φ and Ψ are homotopy inverse of each other. Hence the two categories $\mathsf{Tw}(\mathsf{B}_\mathsf{M})$ and $\mathsf{Tw}(\Omega(\mathsf{B}_\mathsf{M}))$ are homotopic.*

2.17. We start by showing that the composition $\Psi \circ \Phi$ is homotopic to the identity functor on $\mathsf{Tw}(\mathsf{B}_\mathsf{M})$. For any object $(\mathsf{E}, \mathsf{Q}) \in \mathsf{Tw}(\mathsf{B}_\mathsf{M})$, consider the morphism η_E between E and $\Psi\Phi(\mathsf{E}) = \mathsf{B} \otimes^\tau \Omega(\mathsf{B}_\mathsf{M}) \otimes^\tau \mathsf{E}$ defined by

$$\mathsf{E} \to \mathsf{B} \otimes \mathsf{E} \hookrightarrow \mathsf{B} \otimes^\tau \Omega(\mathsf{B}_\mathsf{M}) \otimes^\tau \mathsf{E}$$

where the first map is the coaction map and the second map is simply putting unit of $\Omega(\mathsf{B}_\mathsf{M})$ on the middle position of $\mathsf{B} \otimes^\tau \Omega(\mathsf{B}_\mathsf{M}) \otimes^\tau \mathsf{E}$. A direct computation shows that η_E is a map of twisted complexes over B_M. We need the following standard lemma in homological algebra.

2.18. Lemma. *Let $\mathsf{f} : (\mathsf{E}, \mathsf{Q}) \to (\mathsf{F}, \mathsf{P})$ be a closed morphism in $\mathsf{Tw}(\mathsf{B}_\mathsf{M})$. Define the cone of f to be the matrix cofactorization $(\mathsf{E}[1] \oplus \mathsf{F}, \mathsf{T})$ with T given by the matrix*

$$\mathsf{T} = \begin{bmatrix} \mathsf{Q} & 0 \\ \mathsf{f} & \mathsf{P} \end{bmatrix}.$$

Then f is a homotopy equivalence if and only if $\mathsf{cone}(\mathsf{f})$ is contractible.

2.19. Proof of Lemma 2.18. If $\mathsf{cone}(f)$ is contractible, there exists a morphism $H : \mathsf{cone}(f) \to \mathsf{cone}(f)$ such that

$$\mathrm{id} = [T, H].$$

Writing H as a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

after a matrix multiplication, we find that the map b defines a homotopy inverse of f. A similar consideration works for the reversed direction. $\square$

2.20. We apply the above lemma to the morphism η_E constructed above. The cone $\mathsf{cone}(\eta_E)$ is given by $E[1] \oplus (B \otimes^\tau \Omega(B_M) \otimes^\tau E)$ with a predifferential D acting on it (D satisfies the matrix cofactorization identity). This space has a grading by the number of B-tensors it contains. We write $b_0[b_1|\cdots|b_l] \otimes e$ for an element containing $l+1$ B-tensors and e for an element with no B-tensors . On these elements the first type of elements the predifferential D acts by

$$\begin{aligned} D &:= d_\Delta + Q + d_M; \\ d_\Delta(b_0[b_1|\cdots|b_l] \otimes e) &:= \sum_{i=0}^{l} (-1)^i b_0[b_1|\cdots|\Delta(b_i)|\cdots|b_l] \otimes e \\ &\quad + b_0[b_1|\cdots|b_l|b^{(1)}] \otimes e^{(2)}; \\ Q(b_0[b_1|\cdots|b_l] \otimes e) &:= (-1)^{l+1} b_0[b_1|\cdots|b_l] \otimes Qe; \\ d_M(b_0[b_1|\cdots|b_l] \otimes e) &:= \sum_{i=1}^{l} (-1)^i b_0[b_1|\cdots|M(b_i)|\cdots|b_l] \otimes e. \end{aligned}$$

On elements in $E[1]$ the predifferential D acts by

$$\begin{aligned} D &:= d_\Delta + Q + d_M; \\ d_\Delta(e) &:= b^{(1)} \otimes e^{(2)}; \\ Q(e) &:= Q(e); \\ d_M(e) &:= 0. \end{aligned}$$

One recognizes that the differential d_Δ is simply the cobar resolution of the B-comodule E

$$E \to B \otimes E \to B \otimes B^+ \otimes E \to \cdots$$

which is exact. Moreover since the B-comodule E is cofree (hence injective) there exists a B-linear homotopy H on the cobar resolution above that makes the complex contractible over B. Note that the homotopy reduces the number of B-tensor components by one.

The homotopy operator H defines a homotopy retraction data $(0, 0, \mathsf{H})$ between the zero complex and the cobar resolution (see the Appendix A for details on homological perturbation technique). We also want to require H to be special, i.e. $\mathsf{H}^2 = 0$. This can be achieved by making the following transformation

$$\mathsf{H} \mapsto \mathsf{H} \mathsf{d}_\Delta \mathsf{H}.$$

As the maps d_Δ are also B-linear, the new special homotopy retraction H is also B-linear.

2.21. To show that $\mathsf{cone}(\eta_\mathsf{E})$ is contractible, we need to show that there exists a B-linear homotopy for D. For this, we consider D as obtained from d_Δ by a small perturbation $\mathsf{Q} + \mathsf{d}_\mathsf{M}$. Then apply homological perturbation lemma to obtain the homotopy for D.

As mentioned earlier, the map D is not really a differential as D^2 is not zero. Thus the ordinary homological perturbation lemma does not apply to this case. However it is a predifferential, that is D^2 is in the center of B-linear maps from $\mathsf{cone}(\eta_\mathsf{E})$ to itself. In this situation the homological perturbation lemma can still be applied as is explained in the Appendix A. The condition that needs to be checked for this to work is summarized in the following lemma.

2.22. Lemma. *The curved perturbation $\delta := \mathsf{Q} + \mathsf{d}_\mathsf{M}$ is small. That is we can define the operator $(\mathsf{id} - \delta \circ \mathsf{H})^{-1}$ on $\mathsf{cone}(\eta_\mathsf{E})$. In fact the operator $\delta \circ \mathsf{H}$ is locally nilpotent on $\mathsf{cone}(\eta_\mathsf{E})$.*

2.23. Proof of Lemma 2.22. For a $\mathbf{Z}^{\geq 0}$-graded vector space we say an operator on it is locally nilpotent if for any element of bounded degree it is nilpotent. In our case, we consider the space $\mathsf{cone}(\eta_\mathsf{E})$ be graded by the number of B-tensors. Observe that the operator Q preserves the number of B-tensors while d_M reduces the number of B-tensors by one. The homotopy operator also reduces the number of B-tensors by one. Hence the composition $\delta \circ \mathsf{H}$ strictly reduces the number of B-tensors. So it must be locally nilpotent by degree consideration. Since $\delta \circ \mathsf{H}$ is a locally nilpotent operator, one can define the operator $(\mathsf{id} - \delta \circ \mathsf{H})^{-1}$ on the direct sum of each graded components which is $\mathsf{cone}(\eta_\mathsf{E})$. (Note that it is important here that here $\mathsf{cone}(\eta_\mathsf{E})$ is a direct sum rather than a direct product.) $\square$

It follows from the curved homological perturbation lemma A.4 over the linear category of B-linear morphisms that there exists a homotopy H_1 for the operator D. Hence we have finished half of the proof of the theorem, namely the composition $\Psi\Phi$ is homotopic to the identity functor on $\mathsf{Tw}(\mathsf{B}_\mathsf{M})$.

2.24. The other half of the proof, the fact that $\Phi\Psi$ is homotopic to identity on $\mathsf{Tw}(\Omega(\mathsf{B}_\mathsf{M}))$ is relatively easier as we are dealing with actual complexes. The author learned this argument from Positselski. It is an expanded version of the proof given in subsection 6.4 in [13]. For an object $\mathsf{F} \in \mathsf{Tw}(\Omega(\mathsf{B}_\mathsf{M}))$ we consider the natural map $\epsilon_\mathsf{F} : \Phi\Psi(\mathsf{F}) := \Omega(\mathsf{B}_\mathsf{M}) \otimes^\tau \mathsf{B}_\mathsf{M} \otimes^\tau \mathsf{F} \to \mathsf{F}$ defined by

$$a \otimes 1 \otimes f \mapsto af$$

and zero on the other tensors. To show that ϵ_F is a homotopy equivalence it suffice to show that the cocone $\mathsf{K} := \mathsf{cone}(\epsilon_\mathsf{F})[-1]$ is contractible. This dg module as a vector space is $\mathsf{F}[-1] \oplus \Omega(\mathsf{B}_\mathsf{M}) \otimes \mathsf{B} \otimes \mathsf{F}$. Define a finite decreasing filtration on it by

$$\mathsf{F}^0\mathsf{K} := \mathsf{K} \supset \mathsf{F}^1\mathsf{K} := \Omega(\mathsf{B}_\mathsf{M}) \otimes \mathsf{B} \otimes \mathsf{F} \supset \mathsf{F}^2\mathsf{K} := \Omega(\mathsf{B}_\mathsf{M}) \otimes \mathsf{B}^+ \otimes \mathsf{F} \supset \mathsf{F}^3\mathsf{K} := 0.$$

One checks that the differential on K does not preserve this filtration but sends $\mathsf{F}^i\mathsf{K}$ to $\mathsf{F}^{i-1}\mathsf{K}$. Moreover the induced differential on the associated graded components agrees with the canonical resolution

$$0 \to \Omega(\mathsf{C}_\mathsf{M}) \otimes \mathsf{C}^+ \otimes \mathsf{F} \to \Omega(\mathsf{C}_\mathsf{M}) \otimes \mathsf{k} \otimes \mathsf{F} \to \mathsf{F} \to 0$$

which is exact. Then we can define a dg $\Omega(\mathsf{C}_\mathsf{M})$-submodule of K by

$$\mathsf{L} := \mathsf{F}^2\mathsf{K} + \mathsf{d}\mathsf{F}^2\mathsf{K}$$

where d is the differential on K. It follows from the exactness of the above short exact sequence that both L and K/L are contractible. In general this does not imply that K is also contractible. But in our case the dg module K/L is free as $\Omega(\mathsf{B}_\mathsf{M})$-modules, which implies that K admits a direct sum decomposition $\mathsf{L} \oplus \mathsf{K}/\mathsf{L}$ as $\Omega(\mathsf{B}_\mathsf{M})$-modules. Note that this splitting does not necessarily preserve the differential on K, nevertheless it realizes K as the cone of a closed map from $\mathsf{L}[-1]$ to K/L, which implies that K itself is also contractible. $\square$

Remark: More general Koszul duality statements are studies in [13] between various types of derived categories. The theory of Koszul duality for general curved algebras (not necessarily arising as the dual of curved coalgebras) is not yet understood. This is the reason why we need to restrict to curved algebras of the form R_W.

2.25. For a dg category $\mathscr{D}$ we denote by $[\mathscr{D}]$ its homotopy category. Recall that $[\mathscr{D}]$ has the same objects as $\mathscr{D}$, but the morphism spaces between objects are given by the zeroth cohomology of the morphism spaces in $\mathscr{D}$. Our next goal is to have an understanding of the category $[\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))]$.

It is a well-known fact that for a dg algebra A the category $[\mathsf{Tw}(\mathsf{A})]$ is a triangulated category. However it *does not* agree with the derived category of A in general. The reason is that the derived category of A is defined by the localization of $[\mathsf{Tw}(\mathsf{A})]$ with respect to the class of acyclic objects (dg modules with zero cohomology) which might not be trivial in $[\mathsf{Tw}(\mathsf{A})]$. Specifically, there might exist objects in $\mathsf{Tw}(\mathsf{A})$ that are acyclic while not being contractible. One such example is to take $\mathsf{A} = \mathsf{k}[x]/x^2$ and $\mathsf{E} \in \mathsf{Tw}(\mathsf{A})$ to be

$$\cdots \mathsf{A} \to \mathsf{A} \xrightarrow{\cdot x} \mathsf{A} \to \mathsf{A} \cdots$$

where the maps are all given by multiplication by x.

In the following proposition, we will show that for a coaugmented conilpotent coalgebra C with curvature term M acyclic objects are the same as contractible objects in $\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))$. Recall that a coaugmented coalgebra C is conilpotent if C^+ is the union of the kernels of finite iterated coproducts.

2.26. Proposition. *Let C be a coaugmented conilpotent coalgebra and let F be an object in $\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))$. Then F is acyclic if and only if F is contractible.*

Proof. It suffices to prove that if F is acyclic then it is contractible. As F is an acyclic complex there always exists a contracting homotopy for F over the field k. Let H be such a k-linear special homotopy of F.

Consider the Koszul dual $\Psi(\mathsf{F}) = \mathsf{C} \otimes^\tau \mathsf{F}$. The C-linear map $\mathsf{id} \otimes \mathsf{H}$ defines a special contracting homotopy for the complex $(\mathsf{C} \otimes \mathsf{F}, \mathsf{id} \otimes \mathsf{d}_\mathsf{F})$. The predifferential Q on $\Psi(\mathsf{F})$ is given by

$$\mathsf{Q} = \mathsf{id} \otimes \mathsf{d}_\mathsf{F} + \mathsf{d}^\tau$$

where the map d^τ comes from the natural twisting cochain τ associated with the curved coalgebra C_M. We consider $\delta := \mathsf{d}^\tau$ as a curved perturbation of $\mathsf{id} \otimes \mathsf{d}_\mathsf{F}$ and apply the curved homological perturbation lemma as in the proof of the Theorem 2.16.

2.27. Lemma. *The curved perturbation δ is small.*

This is an immediate consequence of the conilpotency condition on C. In fact the conilpotency condition implies that $\delta \circ (\mathsf{id} \otimes \mathsf{H})$ is a locally nilpotent operator. $\square$

Note that all maps involved in the perturbation are C-linear, hence by the curved homological perturbation lemma A.4 the object $\Psi(\mathsf{F})$ is contractible in $\mathsf{Tw}(\mathsf{C}_\mathsf{M})$. It follows that the object $\Phi\Psi(\mathsf{F})$ is also contractible. By Theorem 2.16, $\Phi\Psi(\mathsf{F})$ is homotopic to F and hence F is also contractible. $\square$

2.28. The proposition immediately implies that the dg algebra $\Omega(\mathsf{C}_\mathsf{M})$ itself is a generator for the triangulated category $[\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))]$. To make a more precise statement we recall several distinct notions of generators for triangulated categories. We follow the exposition in [1]. Let $\mathscr{D}$ be a triangulated category. A set of objects $\mathscr{E} := \{\mathsf{E}_i | i \in \mathsf{I}\}$ is said to classically generate $\mathscr{D}$ if the smallest triangulated subcategory of $\mathscr{D}$ containing $\mathscr{E}$ that is closed under isomorphism and direct summands is equal to $\mathscr{D}$ itself. We say that $\mathscr{D}$ is finitely generated if it is classically generated by one object.

The second notion of generation is defined via the orthogonal category of $\mathscr{E}$. Namely, we say that $\mathscr{E}$ weakly generates $\mathscr{D}$ if the right orthogonal $\mathscr{E}^\perp$ is trivial. (The right orthogonal $\mathscr{E}^\perp$ is by definition the full subcategory of $\mathscr{D}$ consisting of objects A such that $\mathsf{Hom}_\mathscr{D}(\mathsf{E}_i[n], \mathsf{A}) = 0$ for all i and all n.) It is clear that classical generators are also weak generators. But the converse is not true in general, often we will drop the adverb "weak" and say that $\mathscr{E}$ generates $\mathscr{D}$ if $\mathscr{E}$ weakly generates it.

If furthermore the category $\mathscr{D}$ admits arbitrary direct sums one can define the notion of compactness for objects. In such a category an object E in $\mathscr{D}$ is said to be compact if the functor $\mathsf{Hom}_\mathscr{D}(\mathsf{E}, -)$ commutes with direct sums. Denote by $\mathscr{D}^c$ the full subcategory consisting of compact objects. We say that $\mathscr{D}$ is compactly generated if $\mathscr{D}^c$ generates $\mathscr{D}$. We need the following result by Ravenel and Neeman [9].

2.29. Theorem. *Assume that a triangulated category $\mathscr{D}$ admitting arbitrary coproduct is compactly generated. Then a set of compact objects classically generates $\mathscr{D}^c$ if and only if it generates $\mathscr{D}$.*

2.30. Corollary. *Let the notations and assumptions be the same as in Proposition 2.26. Then the dg-module $\Omega(\mathsf{C}_\mathsf{M})$ is a compact generator for the category $[\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))]$. Moreover it classically generates the compact subcategory $[\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))]^c$.*

Proof. It is clear that the object $\Omega(\mathsf{C}_\mathsf{M})$ is compact. Moreover if $\mathsf{F} \in [\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))]$ is right orthogonal to $\Omega(\mathsf{C}_\mathsf{M})$, it implies that the object F is acyclic. Then it follows from Proposition 2.26 that F is in fact contractible hence becomes zero in $[\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))]$. The last assertion follows from Theorem 2.29. $\square$

3. Generators for $\mathsf{MF}(\mathsf{R}, W)$

In this section we prove that the image of the cobar algebra $\Omega(\mathsf{C_M})$ itself under the Koszul duality functor lies in $\mathsf{Tw^b}(\mathsf{C_M})$. Hence its k-linear dual makes sense and defines a matrix factorization in $\mathsf{Tw^b}(\mathsf{R}_W) = \mathsf{MF}(\mathsf{R}, W)$. Then we identify this object produced by Koszul duality with Dyckerhoff's $\mathsf{k^{stab}}$.

In [5] it was shown that object $\mathsf{k^{stab}}$ classically generates the category $[\mathsf{MF}(\mathsf{R}, W)]$. The results obtained in this section do not reproduce this generating theorem by Dyckerhoff, despite Corollary 2.30 and Proposition 3.2. The problem is that we do not know *a priori* that finite rank matrix cofactorizations are compact objects in $\mathsf{Tw}(\mathsf{C_M})$. In fact we will show in this section (see Proposition 3.11 that $\mathsf{k^{stab}}$ classically generates $[\mathsf{MF}(\mathsf{R}, W)]$ if and only if $[\mathsf{Tw^b}(\mathsf{C_M})]$ is compact in $[\mathsf{Tw}(\mathsf{C_M})]$, which is useful in the study of equivariant or graded matrix factorizations in Sections 6, 7.

Throughout this section, we specialize to the curved coalgebra $\mathsf{C_M}$ where C is a symmetric coalgebra and $\mathsf{M} : \mathsf{C} \to \mathsf{k}$ is a curvature term that vanishes on scalar and linear terms. As symmetric coalgebras with the canonical coaugmentation are conilpotent coalgebras, hence all the results in the previous section hold for $\mathsf{C_M}$.

3.1. A compact generator for $[\mathsf{Tw}(\mathsf{C_M})]$. We explain the main idea to construct a compact generator for the homotopy category of $\mathsf{Tw}(\mathsf{C_M})$. Note that it is clear that in both the category $\mathsf{Tw}(\mathsf{C_M})$ and $\mathsf{Tw}(\Omega(\mathsf{C_M}))$ arbitrary coproducts exist and hence one can talk about compactness of objects in these categories.

By Theorem 2.16 the two dg categories $\mathsf{Tw}(\mathsf{C_M})$ and $\mathsf{Tw}(\Omega(\mathsf{C_M}))$ are homotopic via the homotopy equivalences Φ and Ψ. Moreover both functors Φ and Ψ preserve coproducts and hence they send compact objects to compact objects. Being homotopy equivalences the functors Φ and Ψ send compact generators to compact generators.

By Corollary 2.30 the object $\Omega(\mathsf{C_M})$ is a compact generator for the homotopy category $[\mathsf{Tw}(\Omega(\mathsf{C_M}))]$. It follows that the matrix cofactorization $\Psi(\Omega(\mathsf{C_M}))$ is a compact generator for the homotopy category of $\mathsf{Tw}(\mathsf{C_M})$. We have proved the following result.

3.2. Proposition. *The homotopy category of* $\mathsf{Tw}(\mathsf{C_M})$ *is compactly generated by* $\Psi(\Omega(\mathsf{C_M}))$*. The same object also classically generates the associated compact subcategory.*

3.3. Next we show that the compact generator $\Psi(\Omega(\mathsf{C_M}))$ is in fact homotopic to an object of $\mathsf{Tw^b}(\mathsf{C_M})$. Again we use the curved homological perturbation lemma A.4 to prove this. Using formulas in Paragraph 2.10, the predifferential Q on $\Psi(\Omega(C_M)) := C \otimes^\tau \Omega(C_M)$ can be split into three parts defined by

$$\begin{aligned} d^+(x \otimes y) &:= x \otimes d^+(y); \\ d^-(x \otimes y) &:= x \otimes d^-(y); \\ d^\tau(x \otimes y) &:= x^{(1)} \otimes \tau(x^{(2)})y; \\ Q &:= d^+ + d^- + d^\tau. \end{aligned}$$

Here we have slightly abused the notation for d^+ and d^-, but no confusion should arise. We consider $\delta := d^- + d^\tau$ as a curved perturbation for the operator d^+. In Appendix B we construct a special homotopy H between $(\wedge^*(V), 0)$ and $(\Omega(C_M), d^+)$ which extends to a special homotopy between

$$(C \otimes \wedge^*(V), 0) \cong (C \otimes \Omega(C_M), d^+)$$

by putting id on the C part. (The notion of special homotopy is reviewed in Appendix A.)

3.4. Lemma. *The map i, p and H (again extend by id on the C part) defines a special homotopy retraction between $(C \otimes \wedge^*(V), 0)$ and $(C \otimes \Omega(C_M), d^+)$. The curved perturbation $\delta := d^- + d^\tau$ is small. In fact, the operator $\delta \circ H$ is locally nilpotent.*

Proof. The first half is clear. The second half follows again from degree considerations. Namely d^- reduces the number of tensor components, d^τ reduces the degree of the C part and H reduces the number of tensor components. □

Remark: It is interesting to observe that if $M = 0$, one can show that the perturbed differential is the Koszul differential on the space $C \otimes \wedge^*(V)$, hence recovering the Koszul complex. In general, d^- and its combination with d^τ is responsible for the other part of Q.

3.5. Lemma 3.4 allows us to apply the curved homological perturbation lemma A.4 to $\Psi(\Omega(C_M))$, which shows that $\Psi(\Omega(C_M))$ is homotopic to a matrix cofactorization on $C \otimes \wedge^*(V)$. From now on, we shall slightly abuse the notation by denoting by $\Psi(\Omega(C_M))$ the finite rank matrix cofactorization obtained via the curved perturbation lemma.

3.6. Relation with Dyckerhoff's generator k^{stab}. In [5] Dyckerhoff defined a matrix factorization on $R \otimes \wedge^*(V^\vee)$ which he denoted by k^{stab}.

The space k^{stab} is a super space with parity determined by the exterior degree. The matrix factorization on k^{stab} is defined as follows.

Choose a basis $x_1, \cdots, x_n$ of $V^\vee$, and write W in the form $\sum_{i=1}^n x_i W_i$. Denote the dual basis for V by $y_1, \cdots, y_n$. Then the matrix map $Q^\vee$ is defined by

$$Q^\vee(f \otimes \alpha) := x_i f \otimes \lrcorner y_i \alpha + W_i f \otimes x_i \wedge \alpha$$

where $\lrcorner y_i$ denotes the contraction operator and repeated indices are implicitly summed.

3.7. As explained above we can get a matrix cofactorization on $C \otimes \wedge^*(V)$ from $\Psi(\Omega(C_M))$ via curved homological perturbation. Applying the functor D yields a matrix factorization on $R \otimes \wedge^*(V^\vee)$ which is the same as the underlying space of k^{stab}. Hence it is natural to ask whether the two matrix factorizations are the same (homotopic). In the following we identify the matrix factorization $D(\Psi(\Omega(C_M)))$ from Koszul duality with Dyckerhoff's k^{stab}.

3.8. Proposition. *With the notations introduced above we have a homotopy equivalence*

$$D(\Psi(\Omega(C_M))) \cong k^{\mathsf{stab}}$$

between objects in $\mathsf{MF}(R, W)$.

Proof. Since the functor D is an equivalence of categories, we denote by $E := (E, Q)$ the matrix cofactorization whose dual is k^{stab}. As Ψ is a homotopy inverse to Φ, it is enough to prove that

$$\Phi \circ \Psi(\Omega(C_M)) \cong \Phi(E).$$

As shown in the proof of Theorem 2.16 the counit of the adjunction map

$$\Phi \circ \Psi(\Omega(C_M)) \stackrel{\epsilon_{\Omega(C_M)}}{\to} \Omega(C_M)$$

is a homotopy equivalence. Hence it suffices to show that $\Phi(E)$ and $\Omega(C_M)$ is homotopic. The object $\Phi(E)$ as a vector space is given by $\Omega(C_M) \otimes C \otimes \wedge^*(V)$. Define a linear map α from $\Omega(C_M)$ to $\Phi(E)$ by

$$[f_1|\cdots|f_k] \mapsto [f_1|\cdots|f_k] \otimes 1 \otimes 1$$

where the middle 1 is the image of the coaugmentation map of $1 \in k$. The last 1 is the unit in $\wedge^*(V)$. The map α clear respects the left $\Omega(C_H)$-module structure. Moreover it is a map of complexes as Q vanishes on $1 \otimes 1$ ($Q^\vee$ increase the polynomial degree on C, Q must decrease the degree). We use homological perturbation to show that α is a homotopy equivalence. Again we split the differential D on $\Phi(E)$ into several parts and use homological perturbation lemma. Explicitly for an element $a \otimes f \otimes y \in \Omega(C_M) \otimes C \otimes \wedge^*(V)$, the map D is the sum of the following four parts:

$$\begin{aligned} d_\Omega(a \otimes f \otimes y) &:= d_\Omega(a) \otimes f \otimes y; \\ d^\tau(a \otimes f \otimes y) &:= a\tau(f^{(1)}) \otimes f^{(2)} \otimes y; \\ Q^+(a \otimes f \otimes y) &:= a \otimes \frac{\partial f}{\partial y_i} \otimes y_i \wedge y; \\ Q^-(a \otimes f \otimes y) &:= a \otimes D_i(f) \otimes \lrcorner x_i \end{aligned}$$

where y_i as before is a basis for the vector space V. The map D_i is defined by

$$C \to C \otimes C \overset{\text{dual of}}{\to}{}^{W_i \otimes \mathrm{id}}\ k \otimes C = C.$$

The map Q^+ is simply the Koszul differential on $C \otimes \wedge^*(V)$. We consider the differential $d := d_\Omega + Q^+$ on the underlying vector space of $\Phi(E)$ and the other part $\delta := d^\tau + Q^-$ as perturbations of d. One can easily write down a special homotopy H for the Koszul differential Q^+ and extend it by id on $\Omega(C_M)$ to give a homotopy retraction data between $\Omega(C_M)$ and $(\Omega(C_M) \otimes C \otimes \wedge^*(V), d)$. Then one verifies that the perturbation δ is small which follows from the conilpotency property of C and the fact that the curvature M vanishes on scalar and linear terms. Moreover observe that both H and δ are $\Omega(C_M)$-linear and

$$\delta \circ \alpha = 0,$$

which implies that the perturbed inclusion is still α and the perturbed differential is still d_Ω on $\Omega(C_M)$ by formulas in Appendix A. Hence the proposition is proved. $\square$

3.9. In view of the two propositions 3.2, 3.8 above it is plausible to expect a new proof of the fact that k^{stab} is a generator for $[\mathsf{MF}(R, W)]$. Indeed it is a direct consequence of these two propositions that k^{stab} weakly generates $[\mathsf{MF}(R, W)]$, i.e. its right orthogonal full subcategory is trivial.

However it does not imply that k^{stab} classically generates $[\mathsf{MF}(R, W)]$ as the category $[\mathsf{MF}(R, W)]$ does not admit arbitrary coproducts (and hence Theorem 2.29 does not apply). The problem here is that the subcategory $\mathsf{Tw}^b(C_M)$ might not be compact in $\mathsf{Tw}(C_M)$. Indeed we show that this is equivalent to the condition that the object k^{stab} classically generates $[\mathsf{MF}(R, W)]$. We need the following theorem (which can be found in [9]) that characterizes compact objects.

3.10. Theorem. *Let $\mathscr{D}$ be a triangulated category with arbitrary coproduct. Moreover, assume that $\mathscr{D}$ is compactly generated by a set of compact objects $\mathscr{E}$. Then an object of $\mathscr{D}$ is compact if and only if it is a direct summand of an iterated extension of copies of objects of $\mathscr{E}$ shifted in both directions.*

3.11. Proposition. *The full subcategory* $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})]$ *of* $[\mathsf{Tw}(\mathsf{C}_{\mathsf{M}})]$ *is compact if and only if* $\mathsf{k}^{\mathsf{stab}} \cong \mathsf{D}\Psi(\Omega(\mathsf{C}_{\mathsf{M}}))$ *classically generates* $[\mathsf{MF}(\mathsf{R}, W)]$.

Proof. Assume that $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})]$ is a compact subcategory of $[\mathsf{Tw}(\mathsf{C}_{\mathsf{M}})]$, i.e. every object of $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})]$ is compact, then it follows from Theorem 3.10 that the object $\Psi(\Omega(\mathsf{C}_{\mathsf{M}}))$ in $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})]$ obtained by perturbation classically generates $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})]$ as it is a compact generator for $[\mathsf{Tw}(\mathsf{C}_{\mathsf{M}})]$. Apply the equivalence functor D implies that $\mathsf{D}\Psi(\Omega(\mathsf{C}_{\mathsf{M}})) = \mathsf{k}^{\mathsf{stab}}$ classically generates $[\mathsf{MF}(\mathsf{R}, W)]$.

Conversely, if $\mathsf{k}^{\mathsf{stab}}$ classically generates $[\mathsf{MF}(\mathsf{R}, W)]$, by Theorem 3.10 we conclude that objects in $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})]$ can be obtained from $\Psi(\Omega(\mathsf{C}_{\mathsf{M}}))$ by taking direct factors of iterated extensions and shifts, which implies that objects in $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})]$ are compact in $[\mathsf{Tw}(\mathsf{C}_{\mathsf{M}})]$ as $\Psi(\Omega(\mathsf{C}_{\mathsf{M}}))$ is a compact generator. $\square$

3.12. We will now show that the homological smoothness of the dg algebra $\Omega(\mathsf{C}_{\mathsf{M}})$ implies that the object $\mathsf{k}^{\mathsf{stab}}$ classically generates $[\mathsf{MF}(\mathsf{R}, W)]$. Recall that a dg algebra A is called homologically smooth if A considered as an $\mathsf{A} \otimes \mathsf{A}$-bimodule is a perfect object, i.e. it is a direct factor of finite rank free $\mathsf{A} \otimes \mathsf{A}$ dg-module.

3.13. Proposition. *If the dg algebra* $\Omega(\mathsf{C}_{\mathsf{M}})$ *is homologically smooth then the full subcategory* $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C}_{\mathsf{M}})]$ *of* $[\mathsf{Tw}(\mathsf{C}_{\mathsf{M}})]$ *is compact.*

Proof. A matrix cofactorization structure on $\mathsf{C} \otimes \mathsf{V}$ is equivalent to a $\Omega(\mathsf{C}_{\mathsf{M}})$ dg-module structure on V. Hence it suffices to show that any finite dimensional dg $\Omega(\mathsf{C}_{\mathsf{M}})$-module is compact in $\mathsf{Tw}(\Omega(\mathsf{C}_{\mathsf{M}}))$. Homological smoothness implies the existence of resolution of diagonal by a perfect complex of $\Omega(\mathsf{C}_{\mathsf{M}}) \otimes \Omega(\mathsf{C}_{\mathsf{M}})$-bimodules. Via integral transform it produces a resolution for any finite dimensional dg module by a perfect complex of $\Omega(\mathsf{C}_{\mathsf{M}})$-modules. $\square$

Remark: By proposition 3.8 the dg algebra $\Omega(\mathsf{C}_{\mathsf{M}})$ is homotopic to the dg algebra $\mathsf{A} := \mathsf{End}_{\mathsf{MF}(\mathsf{R},W)}(\mathsf{k}^{\mathsf{stab}})^{\mathsf{op}}$ as

$$\mathsf{A}^{\mathsf{op}} \cong \mathsf{End}_{\mathsf{Tw}(\mathsf{C}_{\mathsf{M}})}(\Psi(\Omega(\mathsf{C}_{\mathsf{M}}))) \cong \mathsf{End}_{\mathsf{Tw}(\Omega(\mathsf{C}_{\mathsf{M}}))}(\Omega(\mathsf{C}_{\mathsf{M}})) = \Omega(\mathsf{C}_{\mathsf{M}}).$$

In [5] Dyckerhoff showed that A is homological smooth under the assumption that W has isolated singularities, but a more direct proof of this fact is not known. Therefore we have the following implications:

$$\begin{aligned}
&W \text{ has isolated singularities}\\
\Rightarrow&\Omega(C_M) \text{ or } A \text{ is homologically smooth}\\
\Rightarrow&[\mathsf{Tw}^b(C_M)] \text{ is compact in } [\mathsf{Tw}(C_M)]\\
\Leftrightarrow&k^{\mathsf{stab}} \text{ classically generates } [\mathsf{MF}(R, W)].
\end{aligned}$$

In fact these statements are all equivalent in view of [16].

4. A_∞ structures on minimal models.

In this section we study the minimal model A_∞ algebras for $\Omega(C_M)$. As shown in the previous section, the dg algebra $\Omega(C_M)$ is homotopy to $A := \mathsf{End}_{\mathsf{MF}(R,W)}(k^{\mathsf{stab}})^{\mathsf{op}}$ constructed by Dyckerhoff in [5]. The advantage of $\Omega(C_M)$ is that it is a free algebra. Some results in the computational aspects are also presented.

Throughout this section we assume that the curvature M vanishes on scalar and linear terms. We refer the reader to Appendix A for details on homological perturbation lemma.

4.1. Preparations for homological perturbation. Recall the algebra structure of $\Omega(C_M)$ is the free tensor algebra and the differential d is the sum of two parts: d^+ coming from the coproduct of C and d^- coming from the dual potential H. The formula for these maps are given in Paragraph 2.10 in Section 2. Consider $d = d^+ + d^-$ as a deformation of the differential d^+. For the differential d^+, we have the following quasi-isomorphisms

$$i : (\wedge^*(V), 0) \to (\Omega(C), d^+),$$

and

$$p : (\Omega(C), d^+) \to (\wedge^*(V), 0).$$

where i is the anti-symmetrization map and p is the quotient map. It is clear that p splits i. Moreover, an explicit (in principal) homotopy H is constructed in the Appendix B between these two complexes. Namely, we have

$$i \circ p = \mathsf{id} + d^+H + Hd^+.$$

Moreover, the homotopy H is special, that is

$$H \circ i = 0; p \circ H = 0; H^2 = 0.$$

4.2. To do homological perturbation, we just have to show that the operator $\mathrm{id} - \delta H$ is invertible for some perturbation δ (d^- in our case). This is an easy consequence from degree considerations. Indeed, the operator $\delta := d^-$ reduces the tensor degree by 1 and the operator $H := -d^* \circ G$ also reduces the tensor degree by 1. So their composition reduces the tensor degree by 2. It follows that δH is nilpotent on $\Omega(C)$.Thus we have proved the following lemma.

4.3. Lemma. *Let δ be the perturbation operator d^- on $\Omega(C_M)$. The operator δH is locally nilpotent on $\Omega(C_M)$ and hence $\mathrm{id} - \delta H$ is invertible.*

4.4. Homological perturbation. We can apply the homological perturbation technique in the above situation. The perturbed differential on $\wedge(V)$ is again zero. To see this, note that the perturbed differential is given by

$$b := p \circ (\mathrm{id} - \delta H)^{-1} \circ \delta \circ i.$$

The requirement that M vanishes on linear terms implies that

$$\delta \circ i = 0.$$

Hence $b = 0$ by the above formula of b. We can summarize the results in the following corollary.

4.5. Corollary. *Let C_M be as above and assume that the dual potential M vanishes on scalar and linear terms. Then the homology of the dg algebra $\Omega(C_M)$ is $\wedge^*(V)$.*

Remark: If we did not assume that M vanishes on scalar and linear terms, then the cobar construction $\Omega(C_M)$ might be a curved dg algebra. The scalar part of M controls the curvature term of $\Omega(C_M)$ while the linear adds an additional differential on it. Through homological perturbation it adds a differential on the vector space $\wedge * (V)$.

4.6. A_∞ structures. The standard tree formula can be used to transfer the dg algebra structure on $\Omega(C_M)$ to an A_∞ algebra structure on its homology $\wedge^*(V)$. We summarize the computational aspects of this A_∞ structure.

If M vanishes, it is then easy to see that the dg algebra $\Omega(C_M)$ is formal. In fact p defined above is an algebra quasi-isomorphism when putting the exterior algebra structure on $\wedge^*(V)$. This is simply the classical Koszul duality between symmetric algebras and exterior algebras.

If M is only of degree 2, explicit computation can be done in this case. The result is that M deforms the exterior product on $\wedge^*(V)$ into the Clifford product with M the defining quadratic form. This is the standard curved Koszul duality theory as developed by Polishchuk and Positselski in [12].

The higher degree terms of M is more interesting but also more complicated to compute. It produces the higher multiplications (A_∞ algebra structure) on $\wedge(\mathsf{V})$.

Together with the previous remark on scalar and linear terms of M, we see that each homogeneous components of M nicely corresponds to the homotopy associative structures on its Koszul dual. It is reasonable to say that the Koszul dual of C_M is in general a (curved) A_∞ algebra structure on $\wedge^*(\mathsf{V})$.

5. Hochschild invariants

As another application of Theorem 2.16, we show that one can calculate the Hochschild homology of $\mathsf{MF}(\mathsf{R}_W)$ using the Borel-Moore Hochschild chain complex of the curved algebra R_W. The latter was introduced and explicitly computed in [4]. We assume that W has isolated singularities throughout this section.

5.1. Basic ideas. As mentioned in Section 2, the dg category of matrix factorizations $\mathsf{Tw}^\mathsf{b}(\mathsf{R}_W)$ is isomorphic as a dg category to $\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})^{\mathsf{op}}$. As the Hochschild homology is stable under the opposite operation, we have an isomorphism

$$\mathsf{HH}_*(\mathsf{Tw}^\mathsf{b}(\mathsf{R}_W)) \cong \mathsf{HH}_*(\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})).$$

If W has isolated singularities, by Dyckerhoff's generating result and Proposition 3.11 it follows that $[\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})]$ is a compact subcategory of $[\mathsf{Tw}(\mathsf{C}_\mathsf{M})]$ (see Section 3). Thus we have an inclusion of dg categories

$$\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M}) \hookrightarrow \mathsf{Tw}(\mathsf{C}_\mathsf{M})^\mathsf{c}.$$

Moreover Theorem 3.10 implies that every compact object in $\mathsf{Tw}(\mathsf{C}_\mathsf{M})$ is a direct factor of an object in $\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})$ as $\Psi(\Omega(\mathsf{C}_\mathsf{M})) \in \mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})$ compactly generates $[\mathsf{Tw}(\mathsf{C}_\mathsf{M})]$ by Proposition 3.2. This implies the above inclusion of categories is an equivalence up to factors, which yields

$$\mathsf{HH}_*(\mathsf{Tw}^\mathsf{b}(\mathsf{C}_\mathsf{M})) \cong \mathsf{HH}_*(\mathsf{Tw}(\mathsf{C}_\mathsf{M})^\mathsf{c})$$

by Keller's result [7]. The right hand sided category $\mathsf{Tw}(\mathsf{C}_\mathsf{M})^\mathsf{c}$ is homotopic to the category $\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))^\mathsf{c}$ via the coproduct preserving homotopy equivalences Φ and Ψ. As the Hochschild homology is also homotopy invariant, we conclude that

$$\mathsf{HH}_*(\mathsf{Tw}(\mathsf{C}_\mathsf{M})^\mathsf{c}) \cong \mathsf{HH}_*(\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))^\mathsf{c}).$$

Finally the Hochschild homology of $\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}))^\mathsf{c}$ can be calculated by that of the dg algebra $\Omega(\mathsf{C}_\mathsf{M})$ (again by Keller's result [7]).

5.2. Combining all these isomorphisms, we conclude that the Hochschild Homology of $\mathsf{Tw}^\mathsf{b}(\mathsf{R}_W) = \mathsf{MF}(\mathsf{R}, W)$ is isomorphic to that of the dg algebra $\Omega(\mathsf{C}_\mathsf{M})$

$$\mathsf{HH}_*(\mathsf{MF}(\mathsf{R}, W)) \cong \mathsf{HH}_*(\Omega(\mathsf{C}_\mathsf{M})).$$

In the following, we relate the vector space $\mathsf{HH}_*(\Omega(\mathsf{C}_\mathsf{M}))$ with the Borel-Moore Hochschild homology of the curved algebra R_W.

5.3. The case with vanishing curvature. To relate the Hochschild complex of $\Omega(\mathsf{C}_\mathsf{M})$ to that of the curved algebra R_W, we begin with the classical case where the curvature W is not presented. For this, we first recall the Hochschild homology of a coalgebra C.

Namely, for a coalgebra C with an coaugmentation, we form the cobar algebra $\Omega(\mathsf{C})$. The Hochschild chain complex $\mathsf{C}_*(\mathsf{C})$ is by definition given by the complex

$$(\Omega(\mathsf{C}) \otimes^\tau \mathsf{C} \otimes^\tau \Omega(\mathsf{C})) \underset{\Omega(\mathsf{C})\otimes\Omega(\mathsf{C})}{\otimes} \Omega(\mathsf{C}).$$

Here the superscript τ on tensor symbol is again to denote the twisted tensor product using the natural twisting cochain τ. The expression on the left side of the tensor product is a free $\Omega(\mathsf{C})$-bimodule resolution for the diagonal.

After the tensor product operation, $\mathsf{C}_*(\mathsf{C})$ is simply $\mathsf{C}\otimes\Omega(\mathsf{C})$ as a vector space, but the differential is the sum of the differential from Leibniz rule and the one from the twisting cochain. To simply our notations, we denote by $\mathsf{C}\tilde{\otimes}\Omega(\mathsf{C})$ the Hochschild complex $\mathsf{C}_*(\mathsf{C})$.

5.4. The advantage of this definition of the Hochschild complex is that it is quite simple to relate it the Hochschild complex of $\Omega(\mathsf{C})$. Indeed, the latter complex is by definition given by

$$(\Omega(\mathsf{C}) \otimes^\tau \mathrm{B}\Omega(\mathsf{C}) \otimes^\tau \Omega(\mathsf{C})) \underset{\Omega(\mathsf{C})\otimes\Omega(\mathsf{C})}{\otimes} \Omega(\mathsf{C}).$$

Notice that these two complexes only differ by the middle term where twisted tensor products are formed. The fact that they are quasi-isomorphic follows from the following classical lemma, see [8] for example.

5.5. Lemma. *Let* $C_1 \xrightarrow{\tau_1} A$ *be a twisting cochain between a dg coalgebra* C_1 *and an dg algebra* A*. Let* $C_2 \xrightarrow{\gamma} C_1$ *be a quasi-isomorphism of dg coalgebras. Then the composition* τ_2

$$C_2 \to C_1 \to A$$

is also a twisting cochain. Moreover, for any dg A*-module* F*, the map defined by*

$$C_2 \otimes^{\tau_2} F \xrightarrow{\gamma \otimes \mathsf{id}} C_1 \otimes^{\tau_1} F$$

is a quasi-isomorphism.

5.6. Apply the lemma to the unit morphism of the adjunction $\Omega \dashv B$

$$\eta_C : C \to B\Omega(C)$$

and the natural twisting cochain $B\Omega(C) \to \Omega(C)$. The fact the η_C is a quasi-isomorphism is well-know for ordinary (dg) algebras (even non-curved A_∞ algebras).

As a conclusion of the above discussion, we end up with the following quasi-isomorphism between the two Hochschild complexes,

$$C_*(C) := C\tilde{\otimes}\Omega(C) \xrightarrow{\eta_C \otimes \mathsf{id}} C_*(\Omega(C)) := B\Omega(C)\tilde{\otimes}\Omega(C).$$

5.7. Finally, taking the k-linear dual of the complex $C_*(C)$ yields the Hochschild complex of the algebra R. However, it is important to observe that we should be getting the direct product Hochschild chain complex of R as a result of dualizing a direct sum complex. However, there is an obvious $\mathbf{Z}$-grading on $C_*(C)$ by the number of C tensor components. And the homology of C is finite with respect to this grading. It follows from this facts that the Borel-Moore Hochschild homology (the one by taking direct product complex) and the direct sum Hochschild homology coincides for the algebra R.

This explains the main point we wan to make. Indeed, in the curved case, the differential does not preserve the filtration associated to this $\mathbf{Z}$-grading, and we will always end up with the Borel-Moore Hochschild homology.

5.8. Adding the curvature term. We can add the curvature term W (or M) into the previous discussion. All the constructions explained above remain the same as we have already explained the twisting cochain and the twisted tensor products in the curved case in Section 2. However, the proof of Lemma 5.5 does not generalize as the coalgebra $B\Omega(C_M)$ is curved with noncommutative coproduct. Hence the differential does not square to zero in this case. It is even problematic to talk about the notion of quasi-isomorphism for these coalgebras. Nevertheless, the map $\eta_C \otimes \mathsf{id}$ remains a quasi-isomorphism on the associated Hochschild complexes. This is proved in the following proposition.

5.9. Proposition. *The map* $\eta_C \otimes \mathsf{id}$ *is a quasi-isomorphism between the chain complexes* $C_*(C_M)$ *and* $C_*(\Omega(C_M))$.

Proof. Observe the existence of a $\mathbf{Z}$-grading on the space $C_*(C_M)$ by the number of C tensor components. And define the following $\mathbf{Z}$-grading on the space $B\Omega(C_M) \otimes \Omega(C_M)$ by

$$\begin{aligned} \mathsf{deg}(f_1 \otimes \cdots \otimes f_k) &:= k \text{ for an element in } \Omega(C_M); \\ \mathsf{deg}([\alpha_1|\cdots|\alpha_n] \otimes \beta) &:= \mathsf{deg}(\alpha_1) + \cdots + \mathsf{deg}(\alpha_n) + \mathsf{deg}(\beta) - n. \end{aligned}$$

Then one breaks the Hochschild differentials into two parts. The first part is simply the differential when the curvature is not presented. The second part is the differential defined by the curvature term M. For simplicity, we denote them by d^+ and d^- respectively. (We will not bother to distinguish them on the two complexes as we will specify the complex when making statements.) Observe that the first differential increases the degrees defined above by 1 and the second differential decreases the degree by 1. Hence we have a morphism of mixed complexes

$$\eta_C \otimes \mathsf{id} : (C_M \tilde{\otimes} \Omega(C_M), d^+, d^-) \to (B\Omega(C_M) \tilde{\otimes} \Omega(C_M), d^+, d^-).$$

Through the associated bi-complex of these mixed complexes (details of the mixed complex technique is explained in [4]), we can conclude that the $\eta_C \otimes \mathsf{id}$ is a quasi-isomorphism as it is so on the E^1-page. $\square$

Remark: In the proof, it is important that we are dealing with direct sum complexes and d^+ is degree increasing. Because only in this case, the spectral sequences under consideration starts with the differential d^+ for which we know gives an isomorphism.

5.10. The last step is to dualize the Hochschild complex $C_*(C_M)$. It is easy to see that the Borel-Moore Hochschild complex $C_*^{\mathsf{BM}}(R_W)$ of R_W is quasi-isomorphic to $C_*(C_M)^\vee$. To see this observe that through the natural inclusions

$$R \otimes R^+ \cdots R^+ \otimes R^+ \hookrightarrow (C \otimes C^+ \cdots C^+ \otimes C^+)^\vee,$$

we get a map of mixed complexes

$$C_*^{\mathsf{BM}}(R_W) \xhookrightarrow{i} (C_*(C_M))^\vee.$$

Consider the associated double complexes, it is easy to see that the map induces an isomorphism on the E^1-term by the classical HKR isomorphism. Hence the inclusion map i is a quasi-isomorphism of mixed complexes. We summarize the main results in the following proposition.

5.11. Proposition. *We have the following isomorphisms:*

$$HH_*(MF(R, W)) \cong HH_*(Tw^b(C_M)) \cong HH_*(C_M);$$
$$HH_*^{BM}(R_W) \cong HH_*(C_M)^\vee.$$

Remark: When W has isolated singularities, the vector space $HH_*(C_M)$ is finite dimensional. Moreover, the generalized Mukai pairing induces an isomorphism of $HH_*(Tw^b(R_W))$ with its dual vector space. In view of the proposition, it is natural to ask how the Koszul duality phenomena might be related to the Chern character theory and Riemann-Roch type theorems. The author would like to return to this aspect in future.

6. Equivariant matrix factorizations

In this section, we study the orbifold version of the Theorem 2.16 and its applications to the category of equivariant matrix factorizations. Throughout the section we work over the ground field $k = \mathbb{C}$ as we need to consider characters of groups.

6.1. Orbifold Koszul duality. Let $C := \mathsf{sym}(V)$ to be the symmetric coalgebra over a vector space V and let $M : C \to k$ be a linear map on C that vanishes on scalar and linear terms. Consider a finite abelian group G acting on C via coalgebra morphisms and that the action preserves the linear map M, i. e. the composition

$$C \xrightarrow{g} C \xrightarrow{M} k$$

is equal to M for any element $g \in G$.

Given such data we would like to consider the dg category of equivariant twisted complexes over the curved coalgebra C_M. The objects are pairs (E, Q) where E is a cofree C-comodule with a G-action of the form

$$E := \oplus_i C \otimes \mathbb{C}_{\chi_i}.$$

Here $\mathbb{C}_{\chi_i}$ denotes the one dimensional G-representation associated to a given character χ_i and we allow repeated indices in the direct sum above. The linear map Q is a C-comodule morphism on E which is also G-equivariant. Moreover Q satisfies the matrix cofactorization identity. The morphism spaces between objects would be G-equivariant C-comodule maps. We denote this category by $\mathsf{Tw}([C_M/G])$ to mimic the orbifold notation. As before we denote by $\mathsf{Tw}^b([C_M/G])$ the full subcategory consisting of finite rank objects.

Since the cobar construction is functorial, we also have a G-action on the cobar algebra $\Omega(C_M)$. Thus we can define the category $\mathsf{Tw}([\Omega(C_M)/G])$ in a similar way.

6.2. The Koszul duality functors Φ and Ψ are defined in the same way as before. Namely, for an equivariant matrix cofactorization (E, Q) define

$$\Phi(E) := \Omega(C_M) \otimes^{\tau} E$$

where $\Phi(E)$ inherits the tensor product G-representation. Observe that the differential on $\Phi(E)$ is G-equivariant as the differential on $\Omega(C_M)$, the predifferential Q on E and the twisting cochain τ are all equivariant maps.

One easily checks from direct inspection that the functors Φ and Ψ send equivariant objects to equivariant objects and equivariant morphisms to equivariant morphisms. Moreover the homotopy constructed to prove that Φ and Ψ are homotopy inverses can be made G-equivariant by averaging if necessary. Thus we have shown the following theorem.

6.3. Theorem. *The functors Φ and Ψ are homotopy inverses between $\mathsf{Tw}([C_M/G])$ and $\mathsf{Tw}([\Omega(C_M)/G])$.*

6.4. Change of category. To make better use of the above Theorem 6.3, we first need to make a change of category. Namely, we will switch from equivariant categories to categories of twisted complexes over a smash product algebra. These two types of categories are closely related.

The equivariant category construction can be reduced to the ordinary twisting construction by the smash product construction. Namely as G acts on the curved coalgebra C_M, we could form the smash product curved coalgebra $C_M\sharp G$. As a vector space it is $C \otimes k[G]$ and the coproduct is defined by

$$x \otimes g \mapsto \sum_{g_1 g_2 = g} (x^{(1)} \otimes g_1) \otimes (g_1^{-1}(x^{(2)}) \otimes g_2)$$

The curvature of $C_M\sharp G$ is defined by M on the component $C \otimes \mathrm{id}_G$ and zero otherwise.

6.5. The dg category $\mathsf{Tw}(\mathsf{C_M}\sharp\mathsf{G})$ is closely related to the equivariant dg category $\mathsf{Tw}([\mathsf{C_M}/\mathsf{G}])$. Observe that the smash product coalgebra $\mathsf{C_M}\sharp\mathsf{G}$ carry natural G-action and $\mathsf{C_M}\sharp\mathsf{G}$-linear maps are equivalent to C-linear maps that are also G-equivariant. Thus the category $\mathsf{Tw}(\mathsf{C_M}\sharp\mathsf{G})$ is a fully faithful subcategory of $\mathsf{Tw}([\mathsf{C_M}/\mathsf{G}])$ consists of objects that are free $\mathbb{C}_\mathsf{M}\sharp\mathsf{G}$-comodules. Conversely every objects of $\mathsf{Tw}([\mathsf{C_M}/\mathsf{G}])$ is a direct summand of an object in $\mathsf{Tw}(\mathsf{C_M}\sharp\mathsf{G})$ through the fully faithful embedding. To see this observe that for any object $(\mathsf{E},\mathsf{Q}) \in \mathsf{Tw}([\mathsf{C_M}/\mathsf{G}])$ form the object

$$\mathsf{g}^*(\mathsf{E},\mathsf{Q}) := (\oplus_{\mathsf{g}\in\mathsf{G}}\mathsf{g}^*\mathsf{E}, \oplus_{\mathsf{g}\in\mathsf{G}}\mathsf{g}^*\mathsf{Q}).$$

One easily checks that $\mathsf{g}^*(\mathsf{E},\mathsf{Q})$ is an object of $\mathsf{Tw}(\mathsf{C_M}\sharp\mathsf{G})$. Such a relation between the two categories are called equivalence up to factors (from [7]). If two categories are equivalent up to factors, then lots of properties of them are the same. For example (classical) generators of the smaller category are also (classical) generators of the bigger one. It is also proved by Keller [7] that the Hochschild type invariants are isomorphic for these two categories.

6.6. It is easy to see that Φ and Ψ restrict to homotopy equivalences

$$\mathsf{Tw}(\Omega(\mathsf{C_M})\sharp\mathsf{G}) \cong \mathsf{Tw}(\mathsf{C_M}\sharp\mathsf{G}).$$

We can summarize the previous discussion in the following commutative diagram.

$$\begin{tikzcd}
\mathsf{Tw}(\Omega(\mathsf{C_M})\sharp\mathsf{G}) \arrow[r, "\text{Koszul duality}"] \arrow[d, "\text{inclusion}"] & \mathsf{Tw}(\mathsf{C_M}\sharp\mathsf{G}) \arrow[d, "\text{inclusion}"] \\
\mathsf{Tw}([\Omega(\mathsf{C_M})/\mathsf{G}]) \arrow[r, "\text{Koszul duality}"] & \mathsf{Tw}([\mathsf{C_M}/\mathsf{G}]).
\end{tikzcd}$$

The vertical inclusions are all equivalences up to factors.

6.7. Generators. The advantage of the smash product construction is that it is clear in this description the object $\Omega(\mathsf{C_M})\sharp\mathsf{G}$ compactly generates the homotopy category of $\mathsf{Tw}(\Omega(\mathsf{C_M})\sharp\mathsf{G})$. Indeed for an object $\mathsf{F} \in \mathsf{Tw}(\Omega(\mathsf{C_M})\sharp\mathsf{G})$ we have

$$\mathsf{Hom}_{\mathsf{Tw}(\Omega(\mathsf{C_M})\sharp\mathsf{G})}(\Omega(\mathsf{C_M})\sharp\mathsf{G},\mathsf{F}) = \mathsf{Hom}_{\mathsf{Tw}(\mathsf{C_M})}(\Omega(\mathsf{C_M}),\mathsf{F})$$

through the inclusion mentioned above. By Corollary 2.30 if the latter is acyclic, then the dg-module F is contractible over $\Omega(\mathsf{C_M})$. Averaging the contracting homotopy yields a contraction over $\Omega(\mathsf{C_M})\sharp\mathsf{G}$. Hence arguing as in Corollary 2.30 shows that the object $\Omega(\mathsf{C_M})\sharp\mathsf{G}$ compactly generates

$[\mathsf{Tw}(\Omega(\mathsf{C_M})\sharp \mathsf{G})]$. As the categories $\mathsf{Tw}(\Omega(\mathsf{C_M})\sharp \mathsf{G})$ and $\mathsf{Tw}([\Omega(\mathsf{C_M})/\mathsf{G}])$ are equivalent up to factors, the object $\Omega(\mathsf{C_M})\sharp \mathsf{G}$ (through the inclusion functor) also compactly generates the homotopy category of the latter one.

Apply the Koszul duality functor Ψ yields compact generators for the homotopy category of $\mathsf{Tw}([\mathsf{C_M}/\mathsf{G}])$. Moreover, one can easily identify the generators by observing that the object $\Omega(\mathsf{C_M})\sharp \mathsf{G}$ when considered as objects in $\mathsf{Tw}([\Omega(\mathsf{C_M})/\mathsf{G}])$ is isomorphic to the direct sum

$$\oplus_\chi \Omega(\mathsf{C_M}) \otimes \mathbb{C}_\chi$$

over the characters of G. Hence its image under Ψ is the direct sum

$$\oplus_\chi \Psi(\Omega(\mathsf{C_M})) \otimes \mathbb{C}_\chi.$$

Moreover twisting by characters does not change the homology of $\Omega(\mathsf{C_M})^\chi$ and hence Lemma 3.4 still applies, which proves that the objects $\Psi(\Omega(\mathsf{C_M})^\chi)$ are all homotopic to matrix cofactorizations of finite rank. By Proposition 3.8 the k-linear dual of these cofactorizations yields the following collection of matrix factorizations:

$$\left\{ \mathsf{k}^{\mathsf{stab}} \otimes \mathbb{C}_\chi \mid \chi \text{ is a character for the group } \mathsf{G} \right\}.$$

6.8. Theorem. *Let notations be as above and assume that W has isolated singularities. Then the category $[\mathsf{MF_G}(\mathsf{R}, W)]$ is classically generated by objects of the form $\mathsf{k}^{\mathsf{stab}} \otimes \mathbb{C}_\chi$ defined above.*

Proof. It is enough to show that the subcategory $[\mathsf{Tw}^{\mathsf{b}}(\mathsf{C_M}\sharp \mathsf{G})]$ is compact in $[\mathsf{Tw}(\mathsf{C_M}\sharp \mathsf{G})]$ in view of Proposition 3.11. For this observe that taking cohomology commutes with taking G-invariants and hence for a finite rank object E we have

$$\begin{aligned}
\mathsf{Hom}_{[\mathsf{Tw}(\mathsf{C_M}\sharp \mathsf{G})]}(\mathsf{E}, \oplus \mathsf{E_i}) &:= \mathsf{H}^0(\mathsf{Hom}_{\mathsf{Tw}(\mathsf{C_M}\sharp \mathsf{G})}(\mathsf{E}, \oplus \mathsf{E_i})) \\
&= \mathsf{H}^0(\mathsf{Hom}_{\mathsf{Tw}(\mathsf{C_M})}(\mathsf{E}, \oplus \mathsf{E_i}))^{\mathsf{G}} \\
&= [\oplus \mathsf{H}^0(\mathsf{Hom}_{\mathsf{Tw}(\mathsf{C_M})}(\mathsf{E}, \mathsf{E_i}))]^{\mathsf{G}} \\
&= \oplus \mathsf{Hom}_{[\mathsf{Tw}(\mathsf{C_M}\sharp \mathsf{G})]}(\mathsf{E}, \mathsf{E_i}).
\end{aligned}$$

Here we have used the fact that E is of finite rank and the group G is finite, which implies that E viewed as an object in $[\mathsf{Tw}(\mathsf{C_M})]$ is compact by Proposition 3.11. □

6.9. Hochschild homology. We argue in the same way as in Section 5. Again for the Hochschild homology we need to assume that W has isolated singularities. We have

$$\mathsf{HH}_*(\mathsf{MF}_\mathsf{G}(\mathsf{R},W)) \cong \mathsf{HH}_*(\mathsf{Tw}^\mathsf{b}([\mathsf{C}_\mathsf{M}/\mathsf{G}]))$$

as these two dg categories are opposite to each other by the k-linear dual functor D. Since the compact generators $\Psi(\Omega(\mathsf{C}_\mathsf{M}))\otimes\mathbb{C}_\chi$ of $\mathsf{Tw}([\mathsf{C}_\mathsf{M}/\mathsf{G}])$ lies inside $\mathsf{Tw}^\mathsf{b}([\mathsf{C}_\mathsf{M}/\mathsf{G}])$ which is compact under the assumption of W having isolated singularities, we have

$$\mathsf{HH}_*(\mathsf{Tw}^\mathsf{b}([\mathsf{C}_\mathsf{M}/\mathsf{G}])) \cong \mathsf{HH}_*(\mathsf{Tw}([\mathsf{C}_\mathsf{M}/\mathsf{G}])^\mathsf{c}) \cong \mathsf{HH}_*(\mathsf{Tw}(\mathsf{C}_\mathsf{M}\sharp\mathsf{G})^\mathsf{c})$$

as these categories are equivalent up to factors. Finally, we invoke the Koszul duality of the curved coalgebra $\mathsf{C}_\mathsf{M}\sharp\mathsf{G}$ itself to calculate the Hochschild homology of $\mathsf{Tw}(\mathsf{C}_\mathsf{M}\sharp\mathsf{G})^\mathsf{c}$. This gives a homotopy equivalence

$$\mathsf{Tw}(\mathsf{C}_\mathsf{M}\sharp\mathsf{G})^\mathsf{c} \cong \mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}\sharp\mathsf{G}))^\mathsf{c}$$

between dg categories. From this homotopy equivalence, we conclude that

$$\mathsf{HH}_*(\mathsf{Tw}(\mathsf{C}_\mathsf{M}\sharp\mathsf{G})^\mathsf{c}) \cong \mathsf{HH}_*(\mathsf{Tw}(\Omega(\mathsf{C}_\mathsf{M}\sharp\mathsf{G}))^\mathsf{c}) \cong \mathsf{HH}_*(\Omega(\mathsf{C}_\mathsf{M}\sharp\mathsf{G}))$$

where the last isomorphism is again by Keller's result. Combining the above isomorphisms yields the following isomorphism

$$\mathsf{HH}_*(\mathsf{MF}_\mathsf{G}(\mathsf{R},W)) \cong \mathsf{HH}_*(\Omega(\mathsf{C}_\mathsf{M}\sharp\mathsf{G})).$$

Then the same proof as in Section 5 implies the following proposition.

6.10. Proposition. *Let the notations be as above and assume that W has isolated singularities. Then we have the following isomorphisms:*

$$\mathsf{HH}_*(\mathsf{MF}_\mathsf{G}(\mathsf{R},W)) \cong \mathsf{HH}_*(\mathsf{Tw}^\mathsf{b}([\mathsf{C}_\mathsf{M}/\mathsf{G}])) \cong \mathsf{HH}_*(\mathsf{C}_\mathsf{M}\sharp\mathsf{G});$$
$$\mathsf{HH}_*^{\mathsf{BM}}(\mathsf{R}_W\sharp\mathsf{G}) \cong \mathsf{HH}_*(\mathsf{C}_\mathsf{M}\sharp\mathsf{G})^\vee.$$

Remark: Explicit computation of $\mathsf{HH}_*^{\mathsf{BM}}(\mathsf{R}_W\sharp\mathsf{G})$ has been obtained in [4].

7. Graded matrix factorizations

In this section, we study the category of graded matrix factorizations via Koszul duality. The main ideas is the same as in the orbifold case. The results obtained are closely related to the work of Orlov [10] (on the relationship between graded matrix factorizations and derived category of coherent sheaves) and Seidel [15] (on the A_∞ category of coherent sheaves on Calabi-Yau hypersurfaces). Throughout the section we work over the ground field $\mathsf{k} = \mathbb{C}$.

7.1. Graded matrix factorizations. For a graded commutative ring S and a homogeneous curvature element $W \in R$, one can define the dg category of graded matrix factorizations $\mathsf{MF}^{\mathsf{gr}}(R, W)$ (see [4] for a definition). As is explained in *loc. cit.* this category is closely related to certain orbifold construction.

For this we specialize the group G to be $\mathbf{Z}/d\mathbf{Z}$. The symmetric algebra $S := \mathsf{sym}(V^\vee)$ (non-complete) has a $\mathbf{Z}$-grading by the ordinary polynomial degrees which defines a $\mathbb{C}^*$-action on the vector space S. We consider the G-action on S as the induced representation by embedding the group G into $\mathbb{C}^*$ via

$$i + d\mathbf{Z} \mapsto \zeta_d^i$$

for $\zeta_d := \exp(2\pi\sqrt{-1}/d)$. Moreover we consider a homogeneous curvature element $W \in S$ of polynomial degree d. We will denote by the polynomial degree of an element $f \in S$ by $|f|$.

Clearly the G-action on S preserves the curvature element W. This implies that the G-action in fact acts on the curved algebra S_W. Thus we can form the smash product curved algebra $S_W \sharp G$. Note that this smash product algebra inherits a $\mathbf{Z}$-grading from that of S. The problem is that with respect to this grading $S_W \sharp G$ *does not* form a $\mathbf{Z}$-graded curved algebra in which the curvature element must be of degree 2.

7.2. A new Z-grading. To fix this problem we need to introduce a new $\mathbf{Z}$-grading on $S_W \sharp G$. Note that the underlying vector space of $S_W \sharp G$ is $S \otimes k[G]$. The group algebra $k[G]$ has a special basis coming from characters of G. Explicitly we denote by χ_i for i an integer between $[0, d-1]$ the characters of the group G. They are defined by

$$\chi_i(j + d\mathbf{Z}) := (\zeta_d)^{i \cdot j}.$$

Then the elements

$$U_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g) \sharp g$$

indexed by characters form an orthogonal idempotent basis for the group algebra $k[G]$. Using this basis we can define a new $\mathbf{Z}$-grading on the vector space $S \otimes k[G]$. The homogeneous elements are of the form

$$f \otimes U_{\chi_j}$$

for some homogeneous polynomial $f \in S$. Define an integer $i \in [0, d-1]$ by

$$i \equiv j - |f| \pmod{d}.$$

Then the new grading of $f \otimes u_{\chi_j}$ is defined by

$$\deg(f \otimes u_{\chi_j}) := \frac{2}{d}(|f| - j + i).$$

7.3. Relation between $\mathrm{Tw}^b_{\mathbf{Z}}(S_W \sharp G)$ and $\mathrm{MF}^{gr}(S, W)$. We mention some important properties for this $\mathbf{Z}$-grading on $S_W \sharp G$. First of all, as promised the curvature term $W \sharp \mathrm{id}_G$ has degree 2 with respect to this grading. This follows the following expression

$$W \sharp \mathrm{id}_G = \sum_{\chi_j} W \otimes u_{\chi_j}.$$

Since $|W| = d$ we have $i = j$ and hence

$$\deg(W \otimes u_{\chi_j}) = \frac{2}{d} \cdot |W| = \frac{2}{d} \cdot d = 2.$$

Secondly the category of $\mathbf{Z}$-graded twisted complexes over $S_W \sharp G$ is closely related to the category of graded matrix factorizations. In fact, it is shown in [4] that they are equivalent up to factors. (There we considered $S_W \sharp G$ as a category, then the twist construction would yields in fact an equivalence. Here we prefer to consider $S_W \sharp G$ as a curved algebra.) Namely, there is an inclusion

$$\mathrm{Tw}^b_{\mathbf{Z}}(S_W \sharp G) \hookrightarrow \mathrm{MF}^{gr}(S, W)$$

which is fully faithful and an equivalence up to factors.

7.4. Dualizing. Next we dualize the $\mathbf{Z}$-graded curved algebra to consider a $\mathbf{Z}$-graded curved coalgebra $C_M \sharp G$ where C is the symmetric coalgebra $\mathrm{sym}(V)$. We denote the polynomial degree for a homogeneous $f \in \mathrm{sym}(V)$ by $|f|$. If we identify the vector space $C_M \sharp G$ with $C \sharp k[G]$, then the homogeneous elements in $C_M \sharp G$ are of the form

$$f \otimes u_{\chi_j}$$

and the degree of it is given by

$$\deg(f \otimes u_{\chi_j}) := -\frac{2}{d}(|f| - j + i)$$

for the same i as defined above.

7.5. With respect to this $\mathbf{Z}$-grading the map $M : C \to k$ has degree 2. Hence it forms a $\mathbf{Z}$-graded curved coalgebra. When forming the category $\mathsf{Tw}^b_{\mathbf{Z}}(C_M \sharp G)$ we do not want to allow arbitrary arbitrary coalgebra maps but only the direct sums of the homogeneous ones. We introduce a notation to deal with such situations. Let E be a vector space with a $\mathbb{C}^*$-action, we denote by E^{gr} the vector space defined by

$$E^{\mathsf{gr}} := \oplus E_i$$

where E_i is the subspace of E on which $\mathbb{C}^*$ acts by λ^i. With this notation we have

$$\mathsf{Hom}_{\mathsf{Tw}_{\mathbf{Z}}(C_M \sharp G)}(-,-) := [\mathsf{Hom}_{\mathsf{Tw}(C_M \sharp G)}(-,-)]^{\mathsf{gr}}.$$

Then the $\mathbf{Z}$-graded k-linear dual operation D defines an equivalence

$$\mathsf{Tw}^b_{\mathbf{Z}}(C_M \sharp G)^{\mathsf{op}} \cong \mathsf{Tw}^b_{\mathbf{Z}}(S_W \sharp G)$$

between dg categories.

7.6. Z-graded Koszul duality. From the above discussion, we would like to understand the curved Koszul duality for the $\mathbf{Z}$-graded curved coalgebra $C_M \sharp G$. Motivated by the discussion in the orbifold case, we consider the algebra $\Omega(C_M)\sharp G$. We define a $\mathbf{Z}$-grading on $\Omega(C_M)\sharp G$ as follows. The homogeneous elements are of the form

$$[f_1|\cdots|f_k] \otimes U_{\chi_j}$$

for some character χ_j of the group G. Its degree is defined by

$$\mathsf{deg}([f_1|\cdots|f_k] \otimes U_{\chi_j}) := -\frac{2}{d}(\sum_l |f_l| - j + i) + k$$

where the integer $i \in [0, d-1]$ is defined by

$$i \equiv j - \sum_l |f_l| \pmod{d}.$$

7.7. Define the $\mathbf{Z}$-graded Koszul duality functors by the same formula as before

$$E \in Tw_{\mathbf{Z}}(C_M \sharp G) \overset{\Phi}{\mapsto} \Omega(C) \otimes^{\tau} E \text{ and}$$
$$F \in Tw_{\mathbf{Z}}(\Omega(C_M)\sharp G) \overset{\Psi}{\mapsto} C \otimes^{\tau} F.$$

The degrees on $\Phi(E)$ can be defined by

$$\deg([f_1|\cdots|f_k]\otimes f_0\otimes U_{\chi_j}) := -\frac{2}{d}(\sum_{l=0}^{k}|f_l| - j + i) + k$$

where the integer $i \in [0, d-1]$ is defined by

$$i \equiv j - \sum_{l=0}^{k}|f_l| \pmod d.$$

Similar one can also define degrees for $\Psi(F)$. With respect to these gradings the twisted differentials on $\Phi(E)$ or $\Psi(F)$ have degree one. Moreover it is easy to see that Φ and Ψ are homotopy equivalences by observing that the homotopy equivalences used in the proof of Theorem 2.16 respects this grading (the homotopies are of degree -1).

7.8. Theorem. *The functors Φ and Ψ are homotopy inverses between dg categories*

$$\mathsf{Tw}_{\mathbf{Z}}(\Omega(C_M)\sharp G) \cong \mathsf{Tw}_{\mathbf{Z}}(C_M\sharp G).$$

7.9. Generators. We assume that W has isolated singularities from now on. One can argue in the same way as in the orbifold case that $\Omega(C_M)\sharp G$ compactly generates $[\mathsf{Tw}_{\mathbf{Z}}(\Omega(C_M)\sharp G)]$. Through the $\mathbf{Z}$-graded Koszul duality functor, $\Psi(\Omega(C_M)\sharp G)$ defines a compact generator for $[\mathsf{Tw}_{\mathbf{Z}}(C_M\sharp G)]$. The same proof as in Section 3 shows that the object $\Psi(\Omega(C_M)\sharp G)$ in fact is homotopic to an object in $\mathsf{Tw}^b_{\mathbf{Z}}(C_M\sharp G)$. Thus its k-linear graded dual object in $\mathsf{MF}^{\mathsf{gr}}(S, W)$ makes sense.

To identify this object we consider the natural forgetful functor from $\mathsf{Tw}_{\mathbf{Z}}(C_M\sharp G)$ to $\mathsf{Tw}(C_M\sharp G)$. Note that this is well-defined as the new $\mathbf{Z}$-grading on $C_M\sharp G$ is in $2\mathbf{Z}$ and hence its reduction modulo 2 reduces to the curved coalgebra $C_M\sharp G$. Using the forgetful functor we see that as matrix factorizations the object $D\Psi(\Omega(C_M)\sharp G)$ is given by

$$\oplus_i k^{\mathsf{stab}} \otimes \chi_i.$$

Through the correspondence

$$\mathsf{Tw}^b_{\mathbf{Z}}(R_W\sharp G) \hookrightarrow \mathsf{MF}^{\mathsf{gr}}(S, W)$$

defined in [4], twisting by characters χ_j corresponds to twisting (j) of ordinary graded S-modules. Hence if we assume any lifting of the $\mathbf{Z}$-grading on

$\mathsf{k}^{\mathsf{stab}}$, we conclude that the object $D\Psi(\Omega(\mathsf{C_M}))$ in the category $\mathsf{MF^{gr}}(\mathsf{S}, W)$ given by the direct sum of the objects

$$\mathsf{k}^{\mathsf{stab}}(\mathsf{d}-1), \mathsf{k}^{\mathsf{stab}}(\mathsf{d}-2), \cdots, \mathsf{k}^{\mathsf{stab}}.$$

7.10. Theorem. *Let notations be as above and assume that W has isolated singularities. Then the above collection of objects classically generates* $[\mathsf{MF^{gr}}(\mathsf{R}, W)]$.

Proof. For this it is enough to prove that the category $\mathsf{Tw}^{\mathsf{b}}_{\mathbf{Z}}(\mathsf{C_M}\sharp\mathsf{G})$ is compact in $\mathsf{Tw}_{\mathbf{Z}}(\mathsf{C_M}\sharp\mathsf{G})$ which follows from the fact that taking cohomology of a differential of $\mathbf{Z}$-degree 1 commutes with both taking G-invariants and the operation $- \mapsto -^{\mathsf{gr}}$. $\square$

Remark: In the CY situation, i.e. when $\mathsf{dim}(\mathsf{S}) = \mathsf{d} = \mathsf{deg}(W)$ this collection of generators is in fact well-know in algebraic geometry. They correspond to the collection

$$\mathscr{O}_{\mathsf{X}}, \mathscr{O}_{\mathsf{X}}(1), \cdots, \mathscr{O}_{\mathsf{X}}(\mathsf{d}-1)$$

on the CY hypersurface defined by the equation W via CY/LG correspondence.

7.11. Minimal model A_∞ algebras. The homology of the dg algebra $\Omega(\mathsf{C_M})\sharp\mathsf{G}$ is easily seen to be $\wedge^*(\mathsf{V})\sharp\mathsf{G}$. This latter notation is slightly misleading because we did not mean the smash product algebra. It is simply the smash product vector space. The presence of the curvature term puts A_∞ structure on $\wedge^*(\mathsf{V})\sharp\mathsf{G}$ via homotopy transfer property.

However this computation quickly gets complicated. The author has not been able to describe it even in the case of elliptic curves. We mention two closely related results in these directions. In an unpublished notes [15], Seidel has obtained the above picture for an A_∞ structure on $\wedge^*(\mathsf{V})\sharp\mathsf{G}$ via quite different methods. Explicit calculations for A_∞ structures on elliptic curves have been obtained by Polishchuk in [11], again through other methods. In latter case even the underlying vector space is different.

7.12. Hochschild homology. In the graded case, the Hochschild homology of the dg category $\mathsf{MF^{gr}}(\mathsf{S}, W)$ can also be related with the Borel-Moore Hochschild homology of a curved algebra. The proof is the same as the orbifold case except that we use graded k-linear dualizing functor. We omit the proof here. The precise results are stated in the following proposition.

7.13. Proposition. *Let the notations be as above and assume that W has isolated singularities. Then we have the following isomorphisms:*

$$\mathsf{HH}_*(\mathsf{MF}^{\mathsf{gr}}(S,W)) \cong \mathsf{HH}_*(\mathsf{Tw}^b_{\mathbf{Z}}([C_M/G])) \cong \mathsf{HH}_*(C_M\sharp G);$$
$$\mathsf{HH}^{\mathsf{BM}}_*(S_W\sharp G) \cong \mathsf{HH}_*(C_M\sharp G)^\vee$$

where the $\vee$ denotes the graded dual operation.

Remark: Again the groups $\mathsf{HH}^{\mathsf{BM}}_*(S_W\sharp G)$ has been computed in [4] via certain localization formula. What's new here is the existence of a $\mathbf{Z}$-grading on these homology groups. In the Calabi-Yau situation, the dg version of CY/LG correspondence shows that this computation provides an alternative way to compute the Hochschild homology of CY hypersurfaces.

A. Curved homological perturbation lemma

In this appendix we recall the homological perturbation technique as studied in [3]. Then we prove that the homological perturbation lemma remains true when curvatures are presented. This is useful to study homotopy between precomplexes.

We will work over a fixed k-linear category over a ground field k. For the purposes of this paper, we are primarily concerned with the category of twisted complexes $\mathsf{Tw}(B)$ over some coalgebra B.

A.1. Deformation retractions. The homological perturbation technique deals with perturbations of homotopy equivalences. Let (L, b) and (M, d) be two complexes. Consider a special type of homotopy equivalence — deformation retractions between them. Explicitly this means maps of complexes

$$i : (L, b) \to (M, d) \text{ and } p : (M, d) \to (L, b)$$

such that

$$p \circ i = \mathrm{id}_L .$$

Moreover there is a homotopy H between $i \circ p$ and id_M:

$$i \circ p = \mathrm{id} + dH + Hd.$$

The data (i, p, H) is then called a deformation retraction. If in additional these maps also satisfy

$$Hi = 0,\ pH = 0,\ \text{and } H^2 = 0, \tag{1}$$

Then it is called a special homotopy retraction.

A.2. Perturbations. A perturbation of the complex (M, d) is an odd map $\delta : \mathsf{M} \to \mathsf{M}$ such that $(\mathsf{d} + \delta)^2 = 0$. Following the terminologies in [3], we call δ small if $(\mathsf{id} - \delta \mathsf{H})$ is invertible. For a small perturbation δ, define the operator

$$\mathsf{A} := (\mathsf{id} - \delta\mathsf{H})^{-1}\delta$$

and define the perturbed homotopy retraction operators by

$$\mathsf{b}_1 := \mathsf{b} + \mathsf{p}\mathsf{A}\mathsf{i}, \ \ \mathsf{i}_1 := \mathsf{i} + \mathsf{H}\mathsf{A}\mathsf{i}, \ \ \mathsf{p}_1 := \mathsf{p} + \mathsf{p}\mathsf{A}\mathsf{H}, \ \ \mathsf{H}_1 := \mathsf{H} + \mathsf{H}\mathsf{A}\mathsf{H}. \tag{2}$$

Homological perturbation lemma states that the data $(\mathsf{i}_1, \mathsf{p}_1, \mathsf{H}_1)$ defines a new special deformation retraction between the perturbed complexes $(\mathsf{L}, \mathsf{b}_1)$ and $(\mathsf{M}, \mathsf{d}+\delta)$. This simple lemma plays an important role in the homotopy theory of algebras.

A.3. Curved homological perturbation lemma. Next we state and prove a curved version of the homological perturbation lemma. Namely we assume the same initial conditions for i, p, H. But for the perturbation, we do not assume that $(\mathsf{d} + \delta)^2 = 0$. Instead we assume that that

$$(\mathsf{d} + \delta)^2 \text{ lies in the center of the algebra } \mathsf{End}(\mathsf{M}).$$

We denote this central element by $\mathsf{F} \in \mathsf{End}(\mathsf{M})$ and call δ a curved perturbation. Note that here End is taken inside a pre-fixed linear category.

The differential $\mathsf{d}_1 := \mathsf{d}+\delta$ no longer squares to zero but lies in the center of $\mathsf{End}(\mathsf{M})$. The data $(\mathsf{M}, \mathsf{d}_1)$ is called a precomplex. It is easy to see that the category of precomplexes also form a dg category. Hence one can speak of the homotopy between precomplexes. What curved homological perturbation achieves is the fact one can obtain a homotopy between precomplexes by perturbing ordinary complexes. The main result of this appendix is the following lemma.

A.4. Lemma. *(Curved homological perturbation lemma.) For a special homotopy retraction data $(\mathsf{i}, \mathsf{p}, \mathsf{H})$ and a curved perturbation δ, the formula 2 defines a new special homotopy retract between the precomplexes $(\mathsf{L}, \mathsf{b}_1)$ and $(\mathsf{M}, \mathsf{d}_1)$. Explicitly we have*

- $(\mathsf{L}, \mathsf{b}_1)$ *is a precomplex;*
- $\mathsf{d}_1 \circ \mathsf{i}_1 = \mathsf{i}_1 \circ \mathsf{b}_1$ *(i_1 is a map of precomplexes);*
- $\mathsf{b}_1 \circ \mathsf{p}_1 = \mathsf{p}_1 \circ \mathsf{d}_1$ *(p_1 is a map of precomplexes);*
- $\mathsf{p}_1 \circ \mathsf{i}_1 = \mathsf{id}_\mathsf{L}$ *and* $\mathsf{i}_1 \circ \mathsf{p}_1 = \mathsf{id}_\mathsf{M} + \mathsf{d}_1\mathsf{H}_1 + \mathsf{H}_1\mathsf{d}_1$ *(homotopy retract);*
- $\mathsf{H}_1 \circ \mathsf{i}_1 = 0, \mathsf{p}_1 \circ \mathsf{H}_1 = 0$ *and* $\mathsf{H}_1^2 = 0$ *(specialness).*

A.5. The proof is analogous to the proof of the ordinary perturbation lemma in [3]. We basically only need to check the above formulas with a weaker condition that F is in the center (weaker as 0 is in the center). We begin with the following lemma.

A.6. Lemma. *We have*

- $\delta HA = AH\delta = A - \delta$;
- $(\mathrm{id} - \delta H)^{-1} = \mathrm{id} + AH$ *and* $(\mathrm{id} - H\delta)^{-1} = \mathrm{id} + HA$;
- $AipA + Ad + dA = F + FAH + FHA$.

Proof. The first two equations are direct computations and is the same as in [3]. For the last one, we have

$$
\begin{aligned}
&AipA + Ad + dA = A(\mathrm{id} + dH + Hd)A + Ad + dA \\
&= A^2 + AdHA + AHdA + Ad + dA \\
&= A^2 + Ad(HA + \mathrm{id}) + (AH + \mathrm{id})dA \\
&= A^2 + Ad(\mathrm{id} - H\delta)^{-1} + (\mathrm{id} - \delta H)^{-1} dA \\
&= (\mathrm{id} - \delta H)^{-1}[(\mathrm{id} - \delta H)A^2(\mathrm{id} - H\delta) + (\mathrm{id} - \delta H)Ad + dA(\mathrm{id} - H\delta)](\mathrm{id} - H\delta)^{-1} \\
&= (\mathrm{id} - \delta H)^{-1}[\delta^2 + \delta d + d\delta](\mathrm{id} - H\delta) - 1 \\
&= F(\mathrm{id} - \delta H)^{-1}(\mathrm{id} - H\delta)^{-1} \\
&= F(\mathrm{id} + AH)(\mathrm{id} + HA) \\
&= F + FAH + FHA. \quad \square
\end{aligned}
$$

A.7. With these preparations, the proof of the main lemma follows easily as an extension of the case without curvature. We have

$$
\begin{aligned}
b_1^2 &= (b + pAi)(b + pAi) \\
&= bpAi + pAib + p(AipA)i \\
&= bpAi + pAib + p(F + FAH + FHA - Ad - dA)i \\
&= pFi + pFAHi + pFHAi \\
&= pFi + pFAHi + pHFAi \text{ (F is central)} \\
&= pFi \text{ (specialness) .}
\end{aligned}
$$

Thus b_1^2 is simply the restriction of F on its subspace L (via i and p). And hence it is in the center of $\mathrm{End}(L)$. This proves the first assertion that (L, b_1) is a precomplex.

A.8. To check that i_1 is a map of precomplexes, we have

$$\begin{aligned}
i_1 b_1 - (d+\delta) i_1 &= (i + HAi)(b + pAi) - (d+\delta)(i + HAi) \\
&= ib + ipAi + HAib + H(AipA)i - di - dHAi - \delta i - \delta HAi \\
&= ipAi + HAib + H(F + FAH + FHA - dA - Ad)i - dHAi \\
&\quad - \delta i - (A - \delta)i \\
&= ipAi - HdAi - dHAi - Ai + HFi + HFAHi + HFHAi \\
&= (ip - Hd - dH - \mathsf{id})Ai + HFi + HFAHi + HFHAi \\
&= FHi + HFAHi + FHHAi \\
&= 0 \text{ (by specialness and } \mathsf{F} \text{ is central)} .
\end{aligned}$$

A.9. Similarly we check that p_1 is map of precomplexes:

$$\begin{aligned}
b_1 p_1 - p_1 (d+\delta) &= (b + pAi)(p + pAH) - (p + pAH)(d + \delta) \\
&= bpAH + pAip + p(AipA)H - p\delta - pAHd - p(AH\delta) \\
&= bpAH + pAip - p(Ad + dA)H + p(F + FAH + FHA)H \\
&\quad - p\delta - pAHd - p(A - \delta) \\
&= pAip - pAdH - pAHd - pA + pFH + pFAHH + pFHAH \\
&= pFH + pFAHH + pFHAH \\
&= 0 \text{ (by specialness and } \mathsf{F} \text{ is central)} .
\end{aligned}$$

A.10. To show that the data forms a deformation retraction, we have

$$\begin{aligned}
p_1 i_1 &= (p + pAH)(i + HAi) \\
&= pi + pHAi + pAHi + pAHHAi \\
&= \mathsf{id} \text{ (by specialness)} .
\end{aligned}$$

A.11. In the reversed direction, we have

$$\begin{aligned}
&\mathsf{id} + H_1 d_1 + d_1 H_1 - i_1 p_1 = \\
&\quad = \mathsf{id} + (H + HAH)(d + \delta) + (d + \delta)(H + HAH) - (i + HAi)(p + pAH) \\
&\quad = H\delta + HAHd + H(AH\delta) + \delta H + dHAd + (\delta HA)H \\
&\quad - ipAH - HAip - H(AipA)H \\
&\quad = H\delta + HAHd + H(A - \delta) + \delta H + dHAd + (A - \delta)H \\
&\quad - ipAH - HAip + H(Ad + dA)H - H(F + FAH + FHA)H \\
&\quad = HA(Hd + \mathsf{id} + dH - ip) + (dH + \mathsf{id} - ip + Hd)AH \\
&\quad - HFH - HFAHH - HFHAH \\
&\quad = 0 \text{ (again by specialness and } \mathsf{F} \text{ is central)} .
\end{aligned}$$

A.12. Thus we have shown that (i_1, p_1, H_1) forms a deformation retraction. It still remains to show that it is special. This is again a computation:

$$\begin{aligned} H_1 \circ i_1 &= (H + HAH)(i + HAi) \\ &= Hi + HHAi + HAHi + HAHHAi \\ &= 0; \end{aligned}$$

$$\begin{aligned} p_1 \circ H_1 &= (p + pAH)(H + HAH) \\ &= pH + pHAH + pAHH + pAHHAH \\ &= 0; \end{aligned}$$

$$\begin{aligned} H_1 \circ H_1 &= (H + HAH)(H + HAH) \\ &= HH + HAHH + HHAH + HAHHAH \\ &= 0. \end{aligned}$$

Thus the lemma is proved. $\square$

B. Hodge theory on the cobar complex

In this appendix we study the cobar construction $\Omega(C)$ of the coalgebra $C = \mathsf{sym}(V)$. Motivated from the classical Hodge theory of elliptic complexes, we construct a Green's operator on $\Omega(C)$ with which one can easily write down a homotopy H between $\Omega(C)$ and its cohomology. Such homotopy is by construction $O(V)$-invariant.

B.1. The coalgebra C. We first recall the coalgebra structure on $C = \mathsf{sym}(V)$. We explicitly write the coproduct using a basis $(e_1, e_2, \cdots, e_d)$ of the vector space V. Let $(f_1, f_2, \cdots, f_d)$ be the dual basis for V^*. These gives basis $(e_1^{k_1} \cdots e_d^{k_d})$ for the vector space $\mathsf{sym}(V)$. We call these terms monomials. To simplify the notation, let $K := (k_1, \cdots, k_d)$ be multi-index and denote by e^K the corresponding monomial. The degree of a multi-index is defined by

$$|K| = \sum_{i=1}^{d} k_i.$$

Using these notations, the coproduct is given by

$$\Delta(e^K) = \sum_{(I,J)|I+J=K} e^I \otimes e^J.$$

Clearly the coproduct Δ is the k-linear dual of the ordinary product of polynomials,

$$\mathsf{f}^I \otimes \mathsf{f}^J \mapsto \mathsf{f}^{I+J}.$$

B.2. The cobar algebra of C. The cobar construction $\Omega(\mathsf{sym}(\mathsf{V}))$ (see Section 2) is a dg algebra, and hence in particular a complex of vector spaces. We denote the differential on $\Omega(\mathsf{sym}(\mathsf{V}))$ by d.

On the other hand, $\mathsf{C} = \mathsf{sym}(\mathsf{V})$ is also a commutative algebra, the bar construction $\mathsf{B}(\mathsf{sym}(\mathsf{V}))$ is a dg coalgebra which as a vector space is the same as $\Omega(\mathsf{sym}\,\mathsf{V})$. The dg structure hence defines another differential on the same graded vector space. We denote this differential by d^*.

B.3. Metrics. Our notation indicates that d and d^* are adjoint operators on $\Omega(\mathsf{C})$ (from now on, we denote by $\Omega(\mathsf{C})$ the underlying vector space). To state this property precisely, we need to define a metric on $\Omega(\mathsf{C})$.

Observe that there are several gradings on the vector space $\Omega(\mathsf{C})$. We have the grading by the number of tensor component. We also have the grading by the degree of polynomials. The differential d increases the tensor degree and preserves the polynomial degree. The differential d^* decreases the tensor degree and also preserves the polynomial degree.

Hence for a fixed positive integer N, we denote the subspace in $\Omega(\mathsf{C})$ of polynomial degree N by $\Omega^{\mathsf{N}}(\mathsf{C})$. Note that in the cobar construction we removed the scalar part of C, this implies that the vector space $\Omega^{\mathsf{N}}(\mathsf{C})$ is finite dimensional. Moreover the previous discussion on degrees shows that both operators d and d^* restrict to $\Omega^{\mathsf{N}}(\mathsf{C})$.

The choice of a basis $(e_1, \cdots, e_d)$ endows a canonical metric structure on $\Omega^{\mathsf{N}}(\mathsf{C})$ by requiring that the canonical basis of $\Omega^{\mathsf{N}}(\mathsf{C})$ formed by $(e^{K_1}|\cdots|e^{K_l})$ such that K_j are multi-index with

$$\sum_{j=1}^{l} |K_j| = N$$

is orthonormal.

B.4. Lemma. *With the metric on $\Omega^{\mathsf{N}}(\mathsf{C})$ defined as above, the operators d and d^* are adjoint operators.*

Proof. It suffice to prove this for the coproduct map and product map itself.

For that, we have

$$\begin{aligned}\langle d(e^K), e^I|e^J\rangle &= \sum_{(R,S)|R+S=K} \langle e^R|e^S, e^I|e^J\rangle \\ &= \sum_{(R,S)|R+S=K} \delta_{I=R,J=S} \\ &= \delta_{I+J=K}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle e^K, d^*(e^I|e^J)\rangle &= \langle e^K, e^{I+J}\rangle \\ &= \delta_{I+J=K}.\end{aligned}$$

Thus the lemma is proved. $\square$

B.5. Hodge theory on $\Omega(C)$. Hence we can form the Hodge Laplacian operator

$$\square := d \circ d^* + d^* \circ d.$$

Elements in the kernel of $\square$ are called harmonic. The subspace of harmonic elements in $\Omega^N(C)$ is denoted by H^N.

The complex $(\Omega^N(C), d)$ is graded by the tensor grading. Denote by the component of tensor degree l by $\Omega^N_l(C)$. And the space of harmonic elements of tensor degree l in $\Omega^N(C)$ is denoted by H^N_l.

The Hodge decomposition property can be proved in the same way as in the usual Hodge theory on manifolds since the space $\Omega^N(C)$ is finite dimensional, Fredholm property is trivially satisfied.

B.6. Theorem. *We have the following orthogonal Hodge decomposition for $\Omega^N_l(C)$:*

$$\Omega^N_l(C) = H^N_l \oplus \mathsf{Im}(d) \oplus \mathsf{Im}(d^*).$$

As a consequence of the above decomposition, we have an isomorphism

$$H^l(\Omega^N(C), d) \cong H^N_l.$$

In other words, every cohomology class admits a unique harmonic representative.

Proof. It is easy to verify if x is harmonic, then $dx = 0$ and $d^*x = 0$. It follows that the subspaces H^N_l is orthogonal to $\mathsf{Im}(d)$ and $\mathsf{Im}(d^*)$. For instance, we have

$$\langle x, dy\rangle = \langle d^*x, y\rangle = 0.$$

The two subspaces $\mathsf{Im}(d)$ and $\mathsf{Im}(d^*)$ are also orthogonal as d and d^* are differentials (square to zero). Hence the image of $\Box$ is perpendicular to harmonic elements. So the map

$$\Box : (H_l^N)^\perp \to (H_l^N)^\perp,$$

being an endomorphism of finite dimensional vector spaces with zero kernel, is an isomorphism. Hence in particular it is surjective. It follows that $(H_l^N)^\perp = \mathsf{Im}(d) + \mathsf{Im}(d^*)$. Moreover these subspaces are also perpendicular, thus it is a direct sum decomposition. Hence the Hodge decomposition is proved. For the second part, let [x] be a cohomology class that is represented by $x \in \Omega_l^N(C)$. So we have

$$dx = 0.$$

Moreover from the Hodge decomposition, we can assume that

$$x = x_H + dy + d^*z.$$

Then we have

$$dx = dd^*z = 0,$$

which implies that $d^*z = 0$ by adjoint property. And hence

$$x = x_H + dy,$$

which shows that every cohomology class can be represented by some harmonic element. The uniqueness follows from the fact the space H_l^N is perpendicular to $\mathsf{Im}(d)$ and hence their intersection is trivial. $\Box$

B.7. A homotopy of $\Omega(C)$. It is well known that for the coalgebra $C = \mathsf{sym}(V)$, the cohomology of the cobar complex $\Omega(C)$ is the exterior algebra $\bigwedge(V)$. In fact, there is a quasi-isomorphism of complexes

$$p : (\Omega(C), d) \to (\bigwedge(V), 0)$$

defined by the canonical quotient map from the tensor algebra to the exterior algebra. In fact, this is a map of dg algebras (hence the dg algebra $\Omega(C)$ is formal).

However we need to consider another map in the reverse direction. Let us assume that k has characteristic zero. Consider the anti-symmetrization map

$$i : (\bigwedge(V), 0) \to (\Omega(C), d)$$

defined by

$$v_1 \wedge \cdots v_k \overset{i}{\mapsto} \frac{1}{k!} \sum_{\sigma} (-1)^{\sigma} v_{\sigma(1)} | \cdots | v_{\sigma(k)}.$$

This map splits the canonical quotient map and hence is also a quasi-isomorphism of complexes. Moreover $\mathsf{Im}(i)$ is in fact harmonic. As d^* involves applying symmetric products of $\mathsf{sym}(V)$ which vanishes on anti-symmetric tensors.

Define the Green's operator $G : \Omega(C) \to \Omega(C)$ to be the zero operator on the harmonic subspace $H(C)$, and the inverse of the Laplacian operator $\square$ on the orthogonal complement of $H(C)$. Note that this is well-defined as $\square$ is invertible on the complement.

B.8. Theorem. *Let the notations be as above. Then the image of* i *is the harmonic space* $H(C)$*. And the image of the projection operator* $\mathsf{id} - i \circ p$ *is the orthogonal complement of* $H(C)$*. Define the homotopy operator*

$$H := -d^* \circ G$$

where G *is the Green's operator defined as above. Then we have*

$$i \circ p = \mathsf{id} + dH + Hd.$$

If fact, the data (i, p, H) *defines a special homotopy retraction (see the Appendix A).*

Proof. First, it is easy to see the image of i is inside $H(C)$ by a direct calculation that

$$d \circ i = 0, \text{ and } d^* \circ i = 0.$$

Moreover, by the Hodge decomposition B.6 and the fact that i is a quasi-isomorphism, we know that the image of i is the whole space $H(C)$. Next we show that $\mathsf{Im}(\mathsf{id} - i \circ p)$ is orthogonal to $H(C)$. We have

$$\langle e_1 | \cdots | e_k, \sum_{\sigma} (-1)^{\sigma} e_{\sigma(1)} | \cdots | e_{\sigma(k)} \rangle = 1$$

and

$$\begin{aligned}\langle \frac{1}{k!} \sum_{\mu} (-1)^{\mu} e_{\mu(1)} | \cdots | e_{\mu(k)}, \sum_{\sigma} (-1)^{\sigma} e_{\sigma(1)} | \cdots | e_{\sigma(k)} \rangle &= \frac{1}{k!} \sum_{\mu=\sigma} (-1)^{\mu} (-1)^{\sigma} \\ &= 1.\end{aligned}$$

Thus the orthogonality follows. For the last identity, let $i(x)$ be a harmonic elements, we have

$$\begin{aligned} i \circ p(i(x)) &= i(x), \text{ and} \\ (\mathrm{id} + dH + Hd)(i(x)) &= i(x) - dd^*G(i(x)) - d^*dG(i(x)) \\ &= i(x). \end{aligned}$$

For an element y in the orthogonal complement of $H(C)$, we have

$$\begin{aligned} (\mathrm{id} - i \circ p)(y) &= y, \text{ and} \\ (dH + Hd)(y) &= -dd^*G(y) - d^*Gd(y) \\ &= -dd^*G(y) - d^*dG(y) \text{ (d commutes with G)} \\ &= -(\Box)(\Box)^{-1}(y) \\ &= -y. \end{aligned}$$

The identity is proved. Finally we check that the homotopy H is special. The identity $H \circ i = 0$ is by definition of G. The identity $p \circ H = 0$ follows from the fact the the image of d^* is orthogonal to $H(C)$. And H squares to zero as G commutes with d^* and d^* squares to zero. $\Box$

B.9. An example. Let V be the one dimensional vector space spanned by x. Then one easily checks that the Green's operator G on $\Omega(C)$ is given by

$$x^{i_1}|\cdots|x^{i_k} \mapsto \frac{1}{i_1 + \cdots + i_k - 1} x^{i_1}|\cdots|x^{i_k}.$$

Hence the homotopy operator in this case is given by

$$H(x^{i_1}|\cdots|x^{i_k}) := \sum_{j=1}^{k-1} (-1)^j \frac{1}{i_1 + \cdots + i_k - 1} x^{i_1}|\cdots|x^{i_j + i_{j+1}}|\cdots|x^{i_k}.$$

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