ASYMPTOTICS OF ACH-EINSTEIN METRIC

YOSHIHIKO MATSUMOTO

Abstract. We study the boundary asymptotics of asymptotically complex hyperbolic (ACH) solution of the Einstein equation in terms of the induced partially integrable almost CR structure $T^{1,0}$ on the boundary. Once we prescribe a conformal class $[\Theta]$ of Θ-structures compatible with given $T^{1,0}$, an approximate smooth solution is constructed, which is unique modulo high-order terms and $[\Theta]$-preserving diffeomorphism actions fixing the boundary. A new local CR-invariant tensor naturally arises as the obstruction to construct a better approximation; it vanishes when the boundary structure is integrable. It is shown that there always exist formal solutions to the Einstein equation if we allow logarithmic terms.

Contents

Introduction

Asymptotically complex hyperbolic metrics, or ACH metrics in short, that we study here were introduced by C. L. Epstein, R. B. Melrose and G. A. Mendoza [EMM] as generalizations of complete Kähler metrics of the form $\partial\overline{\partial}\log(1/r)$ on strictly pseudoconvex domains, where r is a boundary defining function. Recently the Einstein equation for ACH metrics is investigated by O. Biquard and M. Herzlich [Bi, BiH]. The purpose of this paper is to discuss the behavior of ACH-Einstein metrics near the boundary.

The boundary behavior of the complete Kähler–Einstein metric on a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^{n+1}$, whose existence was established by S. Y. Cheng and

2010 *Mathematics Subject Classification.* Primary 32V05, Secondary 53A55.
Key words and phrases. Partially integrable almost CR manifolds, ACH metrics, the Einstein equation, obstruction tensor.

Partially supported by Grant-in-Aid for JSPS Fellows (22-6494).

S. T. Yau [CY], is studied by several authors. Following the pioneering work of C. L. Fefferman [Fe], J. M. Lee and R. Melrose [LM] proved that the solution of the zero boundary value problem of the complex Monge–Ampère equation admits at the boundary an asymptotic expansion including logarithmic terms. C. R. Graham [Gr] showed that this expansion is determined by the local CR geometry of the boundary up to the ambiguity of one scalar-valued function on $\partial\Omega$.

In the ACH case, the induced almost CR structure on the boundary is no longer integrable in general; however, it satisfies what S. Tanno [Tno] called the partial integrability condition. By definition a $(2n+1)$-dimensional almost CR manifold $(M, T^{1,0})$ is *partially integrable* if and only if

$$[C^\infty(M, T^{1,0}), C^\infty(M, T^{1,0})] \subset C^\infty(M, H_\mathbb{C}), \qquad \text{where } H_\mathbb{C} := T^{1,0} \oplus \overline{T^{0,1}}. \tag{0.1}$$

The bundle $H_\mathbb{C}$ is the complexification of a certain real subbundle of TM, which is denoted by H. We may also describe an almost CR structure on M by its real expression (H, J), where $J \in \operatorname{End} H$ and $J|_{T^{1,0}} = i \operatorname{id}_{T^{1,0}}$.

The nonintegrablility of $(M, T^{1,0})$ is measured by the *Nijenhuis tensor* $N \in C^\infty(M, H_\mathbb{C}^* \otimes H_\mathbb{C}^* \otimes H_\mathbb{C})$ defined by

$$N(X, Y) := \overline{\Pi^{1,0}}[\Pi^{1,0}X, \Pi^{1,0}Y] + \Pi^{1,0}[\overline{\Pi^{1,0}}X, \overline{\Pi^{1,0}}Y], \qquad X, Y \in C^\infty(M, H_\mathbb{C}), \tag{0.2}$$

where $\Pi^{1,0}$ and $\overline{\Pi^{1,0}}$ are the projections onto $T^{1,0}$ and $\overline{T^{1,0}}$, respectively. Equation (0.2) shows that N is real. Given a local frame $\{ Z_\alpha \} = \{ Z_1, \dots, Z_n \}$ of $T^{1,0}$, we put $Z_{\overline{\alpha}} = \overline{Z_\alpha}$ and write $N(Z_\alpha, Z_\beta) = N_{\alpha\beta}{}^{\overline{\gamma}} Z_{\overline{\gamma}}$.

A partially integrable almost CR manifold $(M, T^{1,0})$ is said to be *nondegenerate* if H is a contact distribution, or equivalently, $\theta \wedge (d\theta)^n$ is a volume form on M for some (hence for any) nowhere vanishing 1-form θ annihilating H. In this case the conormal bundle $E \subset T^*M$ of H is orientable as well as TM. Hence $E^\times := E \setminus$ (zero section) splits into two $\mathbb{R}^+$-bundles; we fix one of them and call its sections *pseudohermitian structures.* A choice of a pseudohermitian structure θ defines the *Levi form* h on $H_\mathbb{C}$ by

$$h(X, Y) := d\theta(X, JY), \qquad X, Y \in C^\infty(M, H_\mathbb{C}). \tag{0.3}$$

Thanks to the nondegeneracy and the partial integrability, the Levi form h is a nondegenerate hermitian form. Furthermore, its signature (p, q), $p + q = n$, is independent of the choice of θ. Once we fix a pseudohermitian structure θ, the Levi form $h_{\alpha\overline{\beta}} = h(Z_\alpha, Z_{\overline{\beta}})$ and its dual $h^{\alpha\overline{\beta}}$ allows us to lower and raise indices.

There are several versions of the definition of ACH metric in the literature. We use the one by C. Guillarmou and A. Sá Barreto [GuSá]; this is a reformulation of the conditions which [EMM] imposed to Θ-metrics in a study of the resolvent of the Laplacian. Let X be a $(2n+2)$-dimensional smooth manifold-with-boundary with a fixed conformal class $[\Theta]$ of Θ-structures, and $\iota\colon \partial X \hookrightarrow X$ the inclusion map. According to the definition by Guillarmou and Sá Barreto, ACH metrics on X are fiber metrics of a modified tangent bundle, which is denoted by ${}^\Theta TX$, satisfying certain conditions. Over the boundary ∂X, there is a natural filtration $K_2 \subset K_1 \subset {}^\Theta TX|_{\partial X}$ by subbundles, where K_1 is of rank $2n+1$ and K_2 of rank 1. Any ACH metric g induces a decomposition ${}^\Theta TX|_{\partial X} = R \oplus K_2 \oplus L$, $K_1 = K_2 \oplus L$ into subbundles and a complex structure $J \in \operatorname{End} L$, which is identified with a partially integrable almost CR structure endomorphism $J \in \operatorname{End} H$ on $H = \ker \iota^*[\Theta]$. By a *distinguished local frame* $\{\,\xi_\infty, \xi_0, \xi_\alpha, \xi_{\overline{\alpha}}\,\}$ we mean a local frame of ${}^\Theta TX$ near ∂X such that, if restricted to ${}^\Theta T_pX$, $p \in \partial X$, $(\xi_\infty)_p$ generates R_p, $(\xi_0)_p$ generates $K_{2,p}$, $(\xi_1)_p, \dots, (\xi_n)_p$ span the i-eigenspace of $J_p \in \operatorname{End} L_p$, and $\xi_{\overline{\alpha}} = \overline{\xi_\alpha}$. For details see §1.

One of the main theorems of this paper is the following one on the existence of an approximate solution of the Einstein equation. For any ACH metric g, its Ricci tensor Ric is naturally defined as a symmetric 2-tensor over ${}^\Theta TX$. We define the *Einstein tensor* Ein by $\mathrm{Ein} := \mathrm{Ric} + \frac{1}{2}(n+2)g$. A boundary defining function ρ of a manifold-with-boundary X is a real-valued smooth function satisfying $\rho > 0$ in $\mathring{X}$, $\rho = 0$ on ∂X and $d\rho \neq 0$ everywhere on ∂X.

Theorem 0.1. *Let X be a $(2n+2)$-dimensional smooth manifold-with-boundary, $[\Theta]$ a conformal class of Θ-structures, and $T^{1,0}$ a nondegenerate partially integrable almost CR structure on ∂X such that $\iota^*[\Theta]$ determines a conformal class of pseudohermitian structures of $(\partial X, T^{1,0})$. Then there exists an ACH metric g satisfying*

$$
\begin{aligned}
&\mathrm{Ein}_{\infty\infty} = O(\rho^{2n+4}), \qquad \mathrm{Ein}_{\infty 0} = O(\rho^{2n+4}), \qquad \mathrm{Ein}_{\infty\alpha} = O(\rho^{2n+3}),\\
&\mathrm{Ein}_{00} = O(\rho^{2n+4}), \qquad \mathrm{Ein}_{0\alpha} = O(\rho^{2n+3}),\\
&\mathrm{Ein}_{\alpha\overline{\beta}} = O(\rho^{2n+3}), \qquad \mathrm{Ein}_{\alpha\beta} = O(\rho^{2n+2})
\end{aligned}
\tag{0.4}
$$

with respect to any distinguished local frame $\{\,\xi_\infty, \xi_0, \xi_\alpha, \xi_{\overline{\alpha}}\,\}$ of ${}^\Theta TX$ near the boundary, where ρ is any boundary defining function of X.

Note that the condition (0.4) is independent of the choice of a distinguished local frame $\{\,\xi_\infty, \xi_0, \xi_\alpha, \xi_{\overline{\alpha}}\,\}$ and a boundary defining function ρ.

The construction of better approximate solutions is obstructed by a tensor field $\mathcal{O}_{\alpha\beta}$ on the boundary, which is called the *obstruction tensor*. Let g be any ACH metric satisfying the condition of Theorem 0.1 and $\theta \in \iota^*[\Theta]$ a pseudohermitian structure on ∂X. Then there is a special boundary defining function ρ for θ, which satisfies $\|d\rho/\rho\|_g = 1/2$ near ∂X and $\iota^*(\rho^4 g) = \theta^2$. By these ingredients $\mathcal{O}_{\alpha\beta}$ is defined by

$$
\mathcal{O}_{\alpha\beta} := \left(\rho^{-2n-2}\,\mathrm{Ein}_{\alpha\beta}\right)\big|_{\partial X}
$$

in terms of the Einstein tensor of g. This is well-defined, i.e., this does not depend on the choice of g, and is a universal polynomial of pseudohermitian invariants of $(M, T^{1,0}, \theta)$. As expected $\mathcal{O}_{\alpha\beta}$ has some CR-invariant properties. Let ζ be F. Farris' section of the CR canonical bundle K of ∂X associated to θ, i.e., a section of K satisfying

$$
\theta \wedge (d\theta)^n = i^{n^2} n! (-1)^q \theta \wedge (T \lrcorner\, \zeta) \wedge (T \lrcorner\, \overline{\zeta}),
\tag{0.5}
$$

where the signature of the Levi form is (p, q), and define the density-weighted version of the obstruction tensor by

$$
\boldsymbol{\mathcal{O}}_{\alpha\beta} := \mathcal{O}_{\alpha\beta} \otimes |\zeta|^{2n/(n+2)} \in \mathcal{E}_{(\alpha\beta)}(-n, -n).
$$

Then we have the following results.

Proposition 0.2. *(1) The density-weighted obstruction tensor $\boldsymbol{\mathcal{O}}_{\alpha\beta}$ is a CR invariant.*

(2) For an integrable CR manifold, the obstruction tensor vanishes.

(3) Let $P^{\alpha\beta}$ be a differential operator $\mathcal{E}_{(\alpha\beta)}(-n, -n) \to \mathcal{E}(-n-2, -n-2)$ defined by

$$
P^{\alpha\beta} = \nabla^\alpha \nabla^\beta - iA^{\alpha\beta} - N^{\gamma\alpha\beta}\nabla_\gamma - N^{\gamma\alpha\beta}{}_{,\gamma}\,,
$$

where A is the pseudohermitian torsion tensor. Then this is a CR-invariant operator and we have $P^{\alpha\beta}\mathcal{O}_{\alpha\beta} - P^{\overline{\alpha}\overline{\beta}}\mathcal{O}_{\overline{\alpha}\overline{\beta}} = 0$.

In spite of (2) above, there actually is a partially integrable almost CR manifold for which $\mathcal{O}_{\alpha\beta}$ is nonzero. This indicates the importance of studying partially integrable almost CR structures.

We shall also investigate how well the solution is improved if we introduce logarithmic terms to ACH metrics. A function $f \in C^0(X) \cap C^\infty(\mathring{X})$ is said to be an element of $\mathcal{A}(X)$ if it admits an asymptotic expansion of the form

$$f \sim \sum_{q=0}^{\infty} f^{(q)}(\log\rho)^q, \qquad f^{(q)} \in C^\infty(X) \tag{0.6}$$

for any boundary defining function ρ. If $f \in \mathcal{A}(X)$, then the Taylor expansions of $f^{(q)}$ at ∂X are uniquely determined. A *singular ACH metric* is a fiber metric g of ${}^\Theta TX$ with $g_{IJ} \in \mathcal{A}(X)$ satisfying the same condition for usual ACH metrics. Then the components of its Ricci tensor also belong to $\mathcal{A}(X)$, and hence so are those of the Einstein tensor. We have the following theorem for such metrics. For any boundary point $p \in \partial X$, we say that $f \in \mathcal{A}(X)$ vanishes to the infinite order at p if and only if all the coefficients $f^{(q)}$ have the vanishing Taylor expansions at p. A tensor over ${}^\Theta TX$ vanishes to the infinite order at p if and only if all of its components vanish to the infinite order at p.

Theorem 0.3. *Let X, $T^{1,0}$, $[\Theta]$ as in Theorem 0.1 and $p \in \partial X$. Then there exists a singular ACH metric whose Einstein tensor vanishes to the infinite order at p. Furthermore, if $\mathcal{O}_{\alpha\beta} = 0$, where $\mathcal{O}_{\alpha\beta}$ is the obstruction tensor for $(\partial X, T^{1,0})$, then there exists such an ACH metric with no logarithmic terms.*

A particularly noteworthy case is when the boundary almost CR structure is integrable. By Proposition 0.2 (2), the second assertion of the theorem above applies to this case. Although Graham [Gr] showed that there is a nontrivial scalar-valued obstruction for the existence of a complete Kähler-Einstein metric on a bounded strictly pseudoconvex domain which is smooth up to the boundary, our result says that in the ACH category we can always erase the logarithmic terms. The author predicts that there is a Kähler-like condition to ACH metrics which revives the scalar-valued obstruction; it might be an interesting topic of further study.

Our result contradicts a work of N. Seshadri [Se], which states that there are a "primary" scalar-valued obstruction function and a "secondary" 1-tensor obstruction to the existence of ACH-Einstein metrics without logarithmic terms. Despite the fact that there is a slight difference in the definition of ACH metrics, the conflict is not because of it. The work [Se] contains some crucial calculation errors in §4, where the computation of the Ricci tensor is carried out. Nevertheless, the influence of Seshadri's paper on our treatment of ACH-Einstein metrics is obvious; if it were not for it, this work should have been much harder to complete.

The paper is organized as follows. We first recall the notion of Θ-structure on a manifold-with-boundary X, the definition of ACH metric and relevant basic facts in §1. In §2 we quickly develop a theory of pseudohermitian geometry for partially integrable almost CR structures. After studying how the Ricci tensor depends on the metric in §3 and §4, we prove Theorem 0.1 and Proposition 0.2 in §5. In §6, we calculate the first variation of $\mathcal{O}_{\alpha\beta}$ with respect to the modification of partially integrable almost CR structure from the flat one and verify that there are abundant examples for which the obstruction tensor does not vanish. The last section §7 is devoted to an investigation of singular ACH metrics and the proof of Theorem 0.3.

In this paper the word "smooth" means infinite differentiability. The Einstein summation convention is used throughout. Parentheses surrounding indices indicate the symmetrization. Our convention for the exterior product $\omega \wedge \eta$ of 1-forms is $(\omega \wedge \eta)(X, Y) = \omega(X)\eta(Y) - \omega(Y)\eta(X)$, while for the symmetric product $\omega\eta$ we observe $(\omega\eta)(X, Y) = \frac{1}{2}(\omega(X)\eta(Y) + \omega(Y)\eta(X))$.

The author would like to express his gratitude to Kengo Hirachi for guidance to this interesting research area and continuous encouragement. He also wishes to thank Takao Akahori, Charles Fefferman, Robin Graham, Colin Guillarmou, Hiraku Nozawa, Raphaël Ponge and Neil Seshadri for useful advice and discussions.

1. Θ-structure and ACH metric

Let X be a smooth manifold-with-boundary. Consider a 1-form $\Theta \in C^\infty(\partial X, T^*X|_{\partial X})$ defined only on the boundary such that $\iota^*\Theta$ is nowhere vanishing, where $\iota\colon \partial X \hookrightarrow X$ is the inclusion map. Then a Lie subalgebra $\mathcal{V}_\Theta$ of $C^\infty(X, TX)$ is defined as follows: for any boundary defining function ρ, a vector field V is an element of $\mathcal{V}_\Theta$ if and only if

$$V \in \rho C^\infty(X, TX), \qquad \tilde{\Theta}(V) \in \rho^2 C^\infty(X).$$

Here $\tilde{\Theta} \in C^\infty(X, T^*X)$ is any extension of Θ. It is clear that the algebra $\mathcal{V}_\Theta$ depends only on the conformal class of Θ. Hence it is reasonable to focus on the conformal class of Θ, which we call a *conformal* Θ*-structure.*

Now let X be a $(2n + 2)$-dimensional manifold-with-boundary with a conformal Θ-structure $[\Theta]$. There is a canonical vector bundle ${}^\Theta TX$ of rank $2n + 2$ over X, whose sections are the elements of $\mathcal{V}_\Theta$. Over the interior of X it is identified with the usual tangent bundle TX. To illustrate the structure near $p \in \partial X$, let $\{ N, T, Y_j \} = \{ N, T, Y_1, \dots, Y_{2n} \}$ be a local frame of TX in a neighborhood of p dual to a certain local coframe of the form $\{ d\rho, \tilde{\Theta}, \alpha^j \}$, where $\tilde{\Theta}$ is an extension of some $\Theta \in [\Theta]$. Then any $V \in \mathcal{V}_\Theta$ is, near p, expressed as

$$V = a\rho N + b\rho^2 T + c^j \rho Y_j, \qquad a,\ b,\ c^j \in C^\infty(X). \tag{1.1}$$

Hence $\{ \rho N, \rho^2 T, \rho Y_j \}$ extends to a local frame of ${}^\Theta TX$ near $p \in \partial X$. The dual local frame of the bundle ${}^\Theta T^*X := ({}^\Theta TX)^*$ is $\{ d\rho/\rho, \tilde{\Theta}/\rho^2, \alpha^j/\rho \}$. A fiber metric of ${}^\Theta TX$ is called a Θ*-metric;* we consider those of arbitrary signatures.

Example 1.1. Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded strictly pseudoconvex domain. Then the boundary $\partial\Omega$ carries a strictly pseudoconvex CR structure. If $r \in C^\infty(\overline{\Omega})$ is a boundary defining function and $\tilde{\theta} := \frac{i}{2}(\partial r - \overline{\partial} r)$, then $\theta := \iota^*\tilde{\theta}$ is a pseudohermitian structure on $\partial\Omega$, where $\iota\colon \partial\Omega \hookrightarrow \overline{\Omega}$ is the inclusion map. We consider the complete Kähler metric g_0 on Ω with Kähler form $i\,\partial\overline{\partial}(\log(1/r))$, which is regarded as a Riemannian metric on Ω as follows:

$$g_0 = 2\sum_{j,k} \frac{\partial^2}{\partial z^j \partial \overline{z}^k}\left(\log\frac{1}{r}\right)(dz^j \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^j) = \frac{1}{r^2}dr^2 + \frac{4}{r^2}\tilde{\theta}^2 + \frac{2}{r}\tilde{h}.$$

Here we set $\tilde{h} := -(\partial^2 r/\partial z^j \partial \overline{z}^k)(dz^j \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^j)$; note that this is an extension of the Levi form on the boundary. Let $X := \overline{\Omega}_{1/2}$ be the square root of $\overline{\Omega}$ in the sense of [EMM] and $\iota_{1/2} \colon X \to \overline{\Omega}$ the identity map. We define $\Theta := (\iota_{1/2}^* \tilde{\theta})|_{\partial X}$ and take $\rho := \sqrt{r/2}$ as a boundary defining function on X. Then g_0 lifts to the following metric on $\mathring{X}$:

$$g := \iota_{1/2}^* g_0 = \frac{4}{\rho^2} d\rho^2 + \frac{1}{\rho^4} \tilde{\Theta}^2 + \frac{1}{\rho^2} \tilde{H}, \qquad \tilde{\Theta} = \iota_{1/2}^* \tilde{\theta}, \quad \tilde{H} = \iota_{1/2}^* \tilde{h}. \tag{1.2}$$

The expression (1.2) shows that g extends to a positive-definite Θ-metric on X.

Let F_p, $p \in \partial X$, be the set of vector fields of the form (1.1) with $a(p) = b(p) = c^j(p) = 0$. Then there is a natural identification between the fiber ${}^\Theta T_p X$ of ${}^\Theta TX$ at p and the quotient vector space $\mathcal{V}_\Theta / F_p$. Since F_p is an ideal, the fiber ${}^\Theta T_p X$ is a Lie algebra, which is called the *tangent algebra* at p. In the sequel we always further assume that

$$\iota^*[\Theta] \text{ is a conformal class of a contact form on } \partial X;$$

then the derived series of ${}^\Theta T_p X$ consists of the following subalgebras:

$$K_{1,p} := \langle \rho^2 T, \rho Y_1, \dots, \rho Y_{2n} \rangle / F_p, \qquad K_{2,p} := \langle \rho^2 T \rangle / F_p.$$

Collecting these subspaces we obtain the subbundles K_1 and K_2 of ${}^\Theta TX|_{\partial X}$.

ACH metrics generalize the Θ-metrics coming from complete Kähler metrics as illustrated in Example 1.1. The characterizing features are completely described in terms of the boundary value of g. Our first two assumptions are that

$$\left\| \frac{d\rho}{\rho} \right\|_g = \frac{1}{2} \qquad \text{over } \partial X \tag{1.3}$$

and

$$g \text{ is positive-definite on } K_2. \tag{1.4}$$

It is clear that (1.3) is independent of the choice of a boundary defining function ρ. The condition (1.4) implies that if we pull $\rho^4 g$, regarded as a section of $\operatorname{Sym}^2 T^* X$, back to ∂X then it is equal to the square of some contact form in $\iota^*[\Theta]$. If there is a fixed Θ-metric g satisfying these two conditions, then for any $p \in \partial X$ there is a unique orthogonal decomposition

$${}^\Theta T_p X = R_p \oplus K_{2,p} \oplus L_p, \qquad K_{1,p} = K_{2,p} \oplus L_p. \tag{1.5}$$

The subbundle of ${}^\Theta TX|_{\partial X}$ whose fiber at p is L_p is denoted by L.

Let $H \subset T(\partial X)$ be the kernel of $\iota^*[\Theta]$. Given a boundary defining function ρ, there is a vector-bundle isomorphism

$$\lambda_\rho \colon H \to L, \qquad Y_p \to \pi_p(\rho Y \bmod F_p), \tag{1.6}$$

where $Y \in C^\infty(X, TX)$ is any extension of $Y_p \in H_p \subset T_p(\partial X)$ and $\pi_p \colon K_{1,p} \to L_p$ is the projection with respect to the decomposition (1.5). By a *compatible almost CR structure* for $[\Theta]$ we mean any almost CR structure $T^{1,0}$ on ∂X such that $T^{1,0} \oplus \overline{T^{1,0}} = H_{\mathbb{C}}$, where $H_{\mathbb{C}}$ is the complexification of H.

Definition 1.2. Let $(X, [\Theta])$ a manifold-with-boundary with a conformal Θ-structure. An *ACH metric* on X is a Θ-metric g satisfying (1.3), (1.4) and the following additional conditions:

(i) For any $p \in \partial X$, if $r_p \in R_p$ is the vector such that $(d\rho/\rho)_p(r_p) = 1$, then the map $L_p \to {}^\Theta T_p X$, $Z_p \mapsto [r_p, Z_p]$, is equal to the identity onto L_p;
(ii) There is a compatible nondegenerate partially integrable almost CR structure $T^{1,0}$ such that, for some (hence for any) boundary defining function ρ and a pseudohermitian structure $\theta \in \iota^*[\Theta]$ characterized by $\iota^*(\rho^4 g) = \theta^2$, via (1.6) $g|_L$ agrees with the Levi form on H determined by θ.

The condition (i) above is independent of the choice of ρ. On (ii), the assumptions of partial integrability and nondegeneracy are not restrictive here, since if $\lambda_\rho^*(g|_L) = (d\theta)|_H(\cdot, J\cdot)$ holds for a compatible almost CR structure (H, J) on ∂X, then $(d\theta)|_H(\cdot, J\cdot)$ is symmetric and hence hermitian, which implies that (H, J) is partially integrable, and its nondegeneracy is nothing but the contact condition for $\iota^*[\Theta]$ that we keep imposing. Furthermore, because of the contact condition, (H, J) is unique. We say that this nondegenerate partially integrable almost CR structure is *induced by g*.

Remark 1.3. Let g be a Θ-metric on $(X, [\Theta])$ satisfying (1.3) and (1.4). We further assume that we have fixed a local frame $\{ N, T, Y_j \}$ around $p \in \partial X$, which is dual to $\{ d\rho, \tilde{\Theta}, \alpha^j \}$ for an extension $\tilde{\Theta}$ of some $\Theta \in [\Theta]$, such that $R_p = \langle \rho N \rangle / F_p$ and $[N, T] = [N, Y_j] = 0$. Then, since $r_p = (\rho N)_p$ and $[\rho N, \rho^2 T] = 2\rho^2 T$, $[\rho N, \rho Y_j] = \rho Y_j \mod F_p$, the map $L_p \to {}^\Theta T_p X$, $Z_p \mapsto [r_p, Z_p]$ is the identity if and only if $L_p = \langle \rho Y_1, \dots, \rho Y_{2n} \rangle / F_p$.

A *distinguished local frame* $\{ \xi_\infty, \xi_0, \xi_\alpha, \xi_{\overline{\alpha}} \}$ for an ACH metric g is a local frame of ${}^\Theta TX$ near a point on ∂X such that, if restricted on each $p \in \partial X$, $(\xi_\infty)_p$ generates R_p, $(\xi_0)_p$ generates $K_{2,p}$, $(\xi_1)_p$, ..., $(\xi_n)_p$ span $T_p^{1,0} \subset (H_p)_\mathbb{C}$ of the induced almost CR structure, where H and L are identified via (1.6), and $\xi_{\overline{\alpha}} = \overline{\xi_\alpha}$.

Proposition 1.4. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded strictly pseudoconvex domain. Then, for any choice of boundary defining function $r \in C^\infty(\overline{\Omega})$, the Θ-metric* (1.2) *on the square root X of $\overline{\Omega}$ is an ACH metric.*

Proof. The first two conditions (1.3) and (1.4) are clear from (1.2). To check the other conditions, we identify an open neighborhood of $\partial\Omega \subset \overline{\Omega}$ with $\partial\Omega \times [0, 2\epsilon^2)$, where the coordinate function for the second factor is equal to r. Then an open neighborhood of $\partial X \subset X$ is identified with $\partial X \times [0, \epsilon)_\rho$, where $2\rho^2 = r$. Since $\tilde{\theta} = \theta + O(r)$ and $\tilde{h} = h + O(r)$, where θ and h are extended in such a way that $\theta(\partial_r) = 0$, $h(\partial_r, \cdot) = 0$ and constantly in the r-direction, we have $\tilde{\Theta} = \theta + O(\rho^2)$ and $\tilde{H} = h + O(\rho^2)$. Let T be a vector field on ∂X such that $\theta(T) = 1$ and Y_1, ..., Y_{2n} a local frame of $\ker\theta$, and we extend them in the ρ-direction constantly. If we further set $N := \partial_\rho$, then $\{ N, T, Y_j \}$ is a local frame of $T(\partial X \times [0, \epsilon)_\rho)$ satisfying $[N, T] = [N, Y_j] = 0$. We can see that $(\rho N)_p$ is orthogonal to $\ker(d\rho/\rho)_p$ for each $p \in \partial X$. Hence, by Remark 1.3, (i) of Definition 1.2 holds if and only if $(\rho^2 T)_p$ is orthogonal to $\langle \rho Y_1, \dots, \rho Y_{2n} \rangle / F_p$, which is also easily verified. Finally, again from (1.2) we see that $g|_L$ is identified with the Levi form determined by θ via (1.6). □

For a Θ-metric on X satisfying (1.3) and (1.4), there is a special boundary defining function, which is called a *model boundary defining function,* as shown below.

Lemma 1.5. *Let $(X, [\Theta])$ be a manifold-with-boundary with a conformal Θ-structure and g a Θ-metric on X satisfying* (1.3), (1.4). *Then, for any $\theta \in \iota^*[\Theta]$, there exists a boundary* *defining function ρ such that*

$$\left\| \frac{d\rho}{\rho} \right\|_g = \frac{1}{2} \qquad \text{near } \partial X \tag{1.7}$$

and $\iota^*(\rho^4 g) = \theta^2$. The germ of ρ along ∂X is unique.

Proof. This is given in [GuSá], but for readers' convenience we include a proof. Let ρ' be any boundary defining function and set $\rho = e^{\psi}\rho'$. Then $\|d\rho/\rho\|_g = 1/2$ is equivalent to

$$\frac{2X_{\rho'}}{\rho'}\psi + \rho \left\| \frac{d\psi}{\rho'} \right\|_g^2 = \frac{1}{\rho'}\left(\frac{1}{4} - \left\| \frac{d\rho'}{\rho'} \right\|_g^2 \right), \tag{1.8}$$

where $X_{\rho'} = \sharp_g(d\rho'/\rho')$ is the dual of $d\rho'/\rho'$ with respect to g. If we express $X_{\rho'}$ in the form (1.1), then the assumption (1.3) implies that $a = 1/4$ on ∂X. Hence (1.8) is a noncharacteristic first-order PDE. After prescribing the boundary value of ψ so that $\iota^*(\rho^4 g) = \theta^2$ is satisfied, we obtain a unique solution of (1.8) near ∂X. □

Fix any contact form $\theta \in \iota^*[\Theta]$ on ∂X. Let ρ be a model boundary defining function and $X_\rho := \sharp_g(d\rho/\rho)$. We consider the smooth map induced by the flow Fl_t of the vector field $4X_\rho/\rho$, which is transverse to ∂X:

$$\Phi\colon (\text{an open neighborhood of } \partial X \times \{0\} \text{ in } \partial X \times [0,\infty)) \to X, \qquad (p,t) \mapsto \mathrm{Fl}_t(p).$$

This is a diffeomorphism onto its image fixing the boundary (where $\partial X \times \{0\}$ is identified with ∂X). The map Φ can be seen as a map between manifolds-with-boundary carrying conformal Θ-structures; $\partial X \times [0,\infty)_t$ carries a standard Θ-structure, which is also denoted by θ and is given by extending θ in such a way that $\theta(\partial_t) = 0$. Since $\tilde{\Theta}(4X_\rho/\rho) = 4\rho\, g(d\rho/\rho, \tilde{\Theta}/\rho^2) = O(\rho)$, we conclude that Φ preserves the conformal Θ-structures. By this construction $t\partial_t$ is orthogonal to $\ker(dt/t)$ with respect to the induced Θ-metric $\Phi^* g$, and we also remark here the fact that $t = \Phi^*\rho$, which implies that the function t is a model boundary defining function for $\Phi^* g$ and θ. In particular, any ACH metric is identified, via a boundary-fixing diffeomorphism preserving conformal Θ-structures, with an ACH metric g defined near the boundary of $\partial X \times [0,\infty)_\rho$ for which $\rho\partial_\rho \perp_g \ker(d\rho/\rho)$ and the second coordinate function ρ is a model boundary defining function. Hence it is enough to consider the ACH metrics of the following form.

Definition 1.6. Let $(M, T^{1,0}, \theta)$ be a nondegenerate partially integrable almost CR manifold with a fixed pseudohermitian structure, i.e., a pseudohermitian manifold. Then $M \times [0,\infty)_\rho$ carries the standard Θ-structure. Let $\iota\colon M = M \times \{0\} \hookrightarrow M \times [0,\infty)$ be the inclusion map. A *normal-form ACH metric g* is an ACH metric defined near the boundary of $M \times [0,\infty)_\rho$ satisfying the following conditions:

- (i) $\rho\partial_\rho$ is orthogonal to $\ker(d\rho/\rho)$ with respect to g;
- (ii) ρ is a model boundary defining function for g and θ;
- (iii) g induces the partially integrable CR structure $T^{1,0}$ on $M = M \times \{0\}$.

Proposition 1.7. *Let $(M, T^{1,0}, \theta)$ be a pseudohermitian manifold and X an open neighborhood of $M = M \times \{0\}$ in $M \times [0,\infty)$ carrying the standard Θ-structure. Let $\{Z_\alpha\}$ in general* *denote a local frame of $T^{1,0}$ and $\{ \theta^\alpha \}$ a family of 1-forms on M satisfying $\theta^\beta(Z_\alpha) = \delta_\alpha{}^\beta$. Then a Θ-metric g on X is a normal-form ACH metric if and only if it is of the form*

$$g = 4\left(\frac{d\rho}{\rho}\right)^2 + g_{00}\left(\frac{\theta}{\rho^2}\right)^2 + 2g_{0\alpha}\frac{\theta}{\rho^2}\frac{\theta^\alpha}{\rho} + 2g_{0\overline{\alpha}}\frac{\theta}{\rho^2}\frac{\theta^{\overline{\alpha}}}{\rho} + 2g_{\alpha\overline{\beta}}\frac{\theta^\alpha}{\rho}\frac{\theta^{\overline{\beta}}}{\rho} + g_{\alpha\beta}\frac{\theta^\alpha}{\rho}\frac{\theta^\beta}{\rho} + g_{\overline{\alpha}\overline{\beta}}\frac{\theta^{\overline{\alpha}}}{\rho}\frac{\theta^{\overline{\beta}}}{\rho}, \tag{1.9}$$

where θ and θ^α are extended in such a way that $\theta(\partial_\rho) = 0$, $\theta^\alpha(\partial_\rho) = 0$ and constantly in the ρ-direction, and satisfies

$$g_{00}|_{\rho=0} = 1, \qquad g_{0\alpha}|_{\rho=0} = 0, \qquad g_{\alpha\overline{\beta}}|_{\rho=0} = h_{\alpha\overline{\beta}} \quad \text{and} \quad g_{\alpha\beta}|_{\rho=0} = 0, \tag{1.10}$$

where $h_{\alpha\overline{\beta}}$ is the Levi form associated to θ.

Proof. The condition $\rho\partial_\rho \perp_g \ker(d\rho/\rho)$, together with (1.3), implies that g is of the form (1.9). The second coordinate ρ is a model boundary defining function for g and θ if and only if $g_{00}|_{\rho=0} = 1$. By Remark 1.3, the condition (i) in Definition 1.2 is equivalent to $g_{0\alpha}|_{\rho=0} = 0$ in the current setting. Since g induces $T^{1,0}$, the condition (ii) in Definition 1.2 is equivalent to $g_{\alpha\overline{\beta}}|_{\rho=0} = h_{\alpha\overline{\beta}}$ and $g_{\alpha\beta}|_{\rho=0} = 0$. Conversely, if $g_{\alpha\overline{\beta}}|_{\rho=0} = h_{\alpha\overline{\beta}}$ and $g_{\alpha\beta}|_{\rho=0} = 0$ then g induces $T^{1,0}$ by the uniqueness of induced partially integrable almost CR structures. □

2. Pseudohermitian geometry

Let $(M, T^{1,0})$ be a nondegenerate partially integrable almost CR manifold. In the presence of a fixed pseudohermitian structure θ, there is a canonical direct sum decomposition of $T_{\mathbb{C}}M$:

$$T_{\mathbb{C}}M = \mathbb{C}T \oplus T^{1,0} \oplus T^{0,1}.$$

Here T, the *Reeb vector field,* is characterized by

$$\theta(T) = 1, \qquad T \rfloor d\theta = 0.$$

If $\{ Z_\alpha \}$ is a local frame of $T^{1,0}$, the *admissible coframe* $\{ \theta^\alpha \}$ is defined in such a way that $\theta^\alpha(Z_\beta) = \delta_\beta{}^\alpha$ and $\theta^\alpha|_{\mathbb{C}T\oplus T^{0,1}} = 0$. This makes $\{ \theta, \theta^\alpha, \theta^{\overline{\alpha}} \}$ into the dual coframe of $\{ T, Z_\alpha, Z_{\overline{\alpha}} \}$. The index 0 is used for components corresponding with T or θ.

The *Tanaka–Webster connection* is described by the following proposition. The proof goes in the same manner as in the integrable case, e.g., Proposition 3.1 in [Tnk].

Proposition 2.1. *On a nondegenerate partially integrable almost CR manifold $(M, T^{1,0})$ with a fixed pseudohermitian structure θ, there is a unique connection ∇ on TM satisfying the following conditions:*

(i) *H, T, J, h are all parallel with respect to ∇;*
(ii) *The torsion tensor $\Theta(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ satisfies*

$$\begin{cases} \Theta(X,Y) + \Theta(JX,JY) = 2\,d\theta(X,Y)T, & X, Y \in \Gamma(H), \\ \Theta(T,JX) = -J\Theta(T,X), & X \in \Gamma(H). \end{cases} \tag{2.1}$$

The components $\Theta_{\alpha\beta}{}^{0}$, $\Theta_{\alpha\beta}{}^{\gamma}$, $\Theta_{\alpha\beta}{}^{\overline{\gamma}}$ of the torsion are not visible in (2.1). Following the argument in the integrable case the first two are shown to be zero. One immediately sees from the definition that the last one is related to the Nijenhuis tensor by

$$\Theta_{\alpha\beta}{}^{\overline{\gamma}} = -N_{\alpha\beta}{}^{\overline{\gamma}} \qquad (\text{and } \Theta_{\overline{\alpha}\overline{\beta}}{}^{\gamma} = -N_{\overline{\alpha}\overline{\beta}}{}^{\gamma}).$$

The other nonzero components of the torsion are

$$\Theta_{\alpha\overline{\beta}}{}^{0} = ih_{\alpha\overline{\beta}}, \qquad \Theta_{0\alpha}{}^{\overline{\beta}} = -\Theta_{\alpha 0}{}^{\overline{\beta}} =: A_{\alpha}{}^{\overline{\beta}}$$

and their complex conjugates. We call $A_{\alpha}{}^{\overline{\beta}}$ the *Tanaka–Webster torsion tensor.*

Remark 2.2. There is another generalization of the Tanaka–Webster connection to the partially integrable case given by Tanno [Tno], which is also used in [BaD], [BlD] and [Se]. Our generalization is different from it in that ours preserves J, which facilitates the whole argument below, and that $\Theta_{\alpha\beta}{}^{\overline{\gamma}}$ is generally nonzero instead. It seems that our connection is first considered by R. I. Mizner [Mi].

The first structure equation is as follows:

$$d\theta = ih_{\alpha\overline{\beta}}\theta^{\alpha} \wedge \theta^{\overline{\beta}}, \tag{2.2}$$

$$d\theta^{\gamma} = \theta^{\alpha} \wedge \omega_{\alpha}{}^{\gamma} - A_{\overline{\alpha}}{}^{\gamma}\theta^{\overline{\alpha}} \wedge \theta - \tfrac{1}{2}N_{\overline{\alpha}\overline{\beta}}{}^{\gamma}\theta^{\overline{\alpha}} \wedge \theta^{\overline{\beta}}. \tag{2.3}$$

Let $\{\, \omega_{\alpha}{}^{\beta} \,\}$ be the connection forms of the Tanaka–Webster connection. Without any modification the proof of Lemma 2.1 in [L2] applies to the partially integrable case and we obtain the following lemma.

Lemma 2.3. *In a neighborhood of any point $p \in M$ there exists a frame $\{\, Z_{\alpha} \,\}$ of $T^{1,0}$ for which $\omega_{\alpha}{}^{\beta}(p) = 0$ holds.*

With such a local frame, it is easy to relate exterior derivatives with covariant derivatives. For example, one immediately sees that the exterior derivative of a $(1,0)$-form $\sigma = \sigma_{\alpha}\theta^{\alpha}$ is given by

$$d\sigma = \sigma_{\alpha,\beta}\theta^{\beta} \wedge \theta^{\alpha} + \sigma_{\alpha,\overline{\beta}}\theta^{\overline{\beta}} \wedge \theta^{\alpha} + \sigma_{\alpha,0}\theta \wedge \theta^{\alpha} - A_{\overline{\beta}}{}^{\alpha}\sigma_{\alpha}\theta^{\overline{\beta}} \wedge \theta - \tfrac{1}{2}N_{\overline{\beta}\overline{\gamma}}{}^{\alpha}\sigma_{\alpha}\theta^{\overline{\beta}} \wedge \theta^{\overline{\gamma}}.$$

Here covariant derivatives of tensors are denoted by indices after commas. This notation will be used in the sequel. In the case of covariant derivatives of a scalar-valued function we omit the comma; e.g., $\nabla_{\alpha}u = u_{\alpha}$ and $\nabla_{\overline{\beta}}\nabla_{\alpha}u = u_{\alpha\overline{\beta}}$.

Proposition 2.4. *We have*

$$A_{\alpha\beta} = A_{\beta\alpha}, \tag{2.4}$$

$$N_{\alpha\beta\gamma} + N_{\beta\alpha\gamma} = 0, \qquad N_{\alpha\beta\gamma} + N_{\beta\gamma\alpha} + N_{\gamma\alpha\beta} = 0. \tag{2.5}$$

Proof. By differentiating (2.2) and considering types we obtain (2.4) and $N_{[\alpha\beta\gamma]} = 0$ (where the square brackets denotes skew-symmetrization). The first identity of (2.5) is obvious from the definition of the Nijenhuis tensor, and it thereby proves the second one. □

Lemma 2.5. *The second covariant derivatives of a scalar-valued function u satisfy the following:*

$$u_{\alpha\overline{\beta}} - u_{\overline{\beta}\alpha} = ih_{\alpha\overline{\beta}}u_0, \qquad u_{\alpha\beta} - u_{\beta\alpha} = -N_{\alpha\beta}{}^{\overline{\gamma}}u_{\overline{\gamma}}, \qquad u_{0\alpha} - u_{\alpha 0} = A_{\alpha}{}^{\overline{\beta}}u_{\overline{\beta}}. \tag{2.6}$$

Proof. The same argument as the one in [L2] applies to our case. □

Next we shall study the curvature $R^{\mathrm{TW}}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. If we set $\Pi_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta$, it holds that $R^{\mathrm{TW}}(X,Y)Z_\alpha = \Pi_\alpha{}^\beta(X,Y)Z_\beta$. We put

$$\Pi_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}\sigma\overline{\tau}}\theta^\sigma \wedge \theta^{\overline{\tau}} + W_{\alpha\overline{\beta}\gamma}\,\theta^\gamma \wedge \theta + W_{\alpha\overline{\beta}\overline{\gamma}}\theta^{\overline{\gamma}} \wedge \theta + V_{\alpha\overline{\beta}\sigma\tau}\,\theta^\sigma \wedge \theta^\tau + V_{\alpha\overline{\beta}\overline{\sigma}\overline{\tau}}\,\theta^{\overline{\sigma}} \wedge \theta^{\overline{\tau}}, \tag{2.7}$$

where $V_\alpha{}^\beta{}_{(\sigma\tau)} = V_\alpha{}^\beta{}_{(\overline{\sigma}\overline{\tau})} = 0$. Since $\nabla h = 0$ we have $\Pi_{\alpha\overline{\beta}} + \Pi_{\overline{\beta}\alpha} = 0$, and hence

$$R_{\alpha\overline{\beta}\sigma\overline{\tau}} = R_{\overline{\beta}\alpha\overline{\tau}\sigma}, \qquad W_{\alpha\overline{\beta}\overline{\gamma}} = -W_{\overline{\beta}\alpha\overline{\gamma}}, \qquad V_{\alpha\overline{\beta}\sigma\tau} = -V_{\overline{\beta}\alpha\sigma\tau}\,. \tag{2.8}$$

We substitute (2.7) into the exterior derivative of (2.3) and compare the coefficients to obtain

$$R_{\alpha\overline{\beta}\sigma\overline{\tau}} - R_{\sigma\overline{\beta}\alpha\overline{\tau}} = -N_{\alpha\sigma}{}^{\overline{\gamma}}N_{\overline{\tau}\overline{\gamma}\overline{\beta}}, \tag{2.9a}$$

$$W_{\alpha\overline{\beta}\gamma} = A_{\alpha\gamma,\overline{\beta}} - N_{\gamma\sigma\alpha}A_{\overline{\beta}}{}^{\sigma}, \qquad V_{\alpha\overline{\beta}\sigma\tau} = \tfrac{i}{2}(h_{\sigma\overline{\beta}}A_{\alpha\tau} - h_{\tau\overline{\beta}}A_{\alpha\sigma}) + \tfrac{1}{2}N_{\sigma\tau\alpha,\overline{\beta}}\,. \tag{2.9b}$$

The component $R_{\alpha\overline{\beta}\rho\overline{\sigma}}$ is called the *Tanaka–Webster curvature tensor*. We put $R_{\alpha\overline{\beta}} := R_\gamma{}^\gamma{}_{\alpha\overline{\beta}}$ and $R := R_\alpha{}^\alpha$. It is seen from the first identity of (2.8) that $R_{\alpha\overline{\beta}} = R_{\overline{\beta}\alpha}$, and from (2.9a) we have

$$R_\alpha{}^\gamma{}_{\gamma\overline{\beta}} = R_{\alpha\overline{\beta}} - N_{\alpha\sigma\tau}N_{\overline{\beta}}{}^{\tau\sigma}. \tag{2.10}$$

As we have discussed above, a choice of a pseudohermitian structure θ defines the Tanaka–Webster connection and supplies various pseudohermitian invariants. If a certain pseudohermitian invariant is also conserved by any change of pseudohermitian structure, it is called a *CR invariant* (rigorously speaking we should say "partially-integrable-almost-CR invariant," but we prefer the shorter expression). To investigate such invariants, we need the transformation law of the connection and relevant quantities.

Proposition 2.6. *Let θ and $\hat\theta = e^{2u}\theta$, $u \in C^\infty(M)$, be two pseudohermitian structures on a nondegenerate partially integrable almost CR manifold $(M, T^{1,0})$. Then, the Tanaka–Webster connection forms, the torsions and the Ricci tensors are related as follows:*

$$\hat\omega_\alpha{}^\beta = \omega_\alpha{}^\beta + 2(u_\alpha\theta^\beta - u^\beta\theta_\alpha) + 2\delta_\alpha{}^\beta u_\gamma\theta^\gamma + 2i(u^\beta{}_\alpha + 2u_\alpha u^\beta + 2\delta_\alpha{}^\beta u_\gamma u^\gamma)\theta, \tag{2.11}$$

$$\hat A_{\alpha\beta} = A_{\alpha\beta} + i(u_{\alpha\beta} + u_{\beta\alpha}) - 4iu_\alpha u_\beta + i(N_{\gamma\alpha\beta} + N_{\gamma\beta\alpha})u^\gamma, \tag{2.12}$$

$$\hat R_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} - (n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) - \left(u_\gamma{}^\gamma + u^\gamma{}_\gamma + 4(n+1)u_\gamma u^\gamma\right)h_{\alpha\overline{\beta}}. \tag{2.13}$$

Proof. The new Reeb vector field is $\hat T = e^{-2u}(T - 2iu^\alpha Z_\alpha + 2iu^{\overline{\alpha}}Z_{\overline{\alpha}})$ and the admissible coframe dual to $\{\, Z_\alpha \,\}$ is $\{\, \hat\theta^\alpha = \theta^\alpha + 2iu^\alpha\theta \,\}$. To establish (2.11) and (2.12), it is enough to check that

$$d\hat h_{\alpha\overline{\beta}} = \hat h_{\gamma\overline{\beta}}\hat\omega_\alpha{}^\gamma + \hat h_{\alpha\overline{\gamma}}\hat\omega_{\overline{\beta}}{}^{\overline{\gamma}}$$

and

$$d\hat\theta^\gamma = \hat\theta^\alpha \wedge \hat\omega_\alpha{}^\gamma - \hat h^{\gamma\overline{\beta}}\hat A_{\overline{\alpha}\overline{\beta}}\hat\theta^{\overline{\alpha}} \wedge \hat\theta - \tfrac{1}{2}N_{\overline{\alpha}\overline{\beta}}{}^\gamma\hat\theta^{\overline{\alpha}} \wedge \hat\theta^{\overline{\beta}}.$$

They are shown straightforward using (2.6).

We compute $\hat{\Pi}_\gamma{}^\gamma = d\hat{\omega}_\gamma{}^\gamma$ modulo $\hat{\theta}^\alpha \wedge \hat{\theta}^\beta$, $\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta}^{\overline{\beta}}$, $\hat{\theta}$, or equivalently, modulo $\theta^\alpha \wedge \theta^\beta$, $\theta^{\overline{\alpha}} \wedge \theta^{\overline{\beta}}$, θ. By the first identity of (2.6) we obtain that, modulo $\hat{\theta}^\alpha \wedge \hat{\theta}^\beta$, $\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta}^{\overline{\beta}}$, $\hat{\theta}$,

$$\begin{aligned}\hat{\Pi}_\gamma{}^\gamma &\equiv \Pi_\gamma{}^\gamma - \Big[(n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) + \Big(u_\gamma{}^\gamma + u_{\overline{\gamma}}{}^{\overline{\gamma}} + 4(n+1)u_\gamma u^\gamma\Big) h_{\alpha\overline{\beta}}\Big] \theta^\alpha \wedge \theta^{\overline{\beta}} \\ &\equiv \Big[R_{\alpha\overline{\beta}} - (n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) - \Big(u_\gamma{}^\gamma + u_{\overline{\gamma}}{}^{\overline{\gamma}} + 4(n+1)u_\gamma u^\gamma\Big) h_{\alpha\overline{\beta}}\Big] \hat{\theta}^\alpha \wedge \hat{\theta}^{\overline{\beta}}.\end{aligned}$$

This proves (2.13). □

Finally we sketch the concept of density bundles following [GoGr]. Let us assume that we have fixed a complex line bundle $E(1,0)$ over M together with a duality between $E(1,0)^{\otimes(n+2)}$ and the canonical bundle K. Such a choice may not exist globally, but locally it does; when we use density bundles we restrict our scope to the local theory. Then $E(w,0)$ is the w^{th} tensor power of $E(1,0)$, and we set

$$E(w,w') = E(w,0) \otimes E(0,w'), \qquad w,\, w' \in \mathbb{Z},$$

where $E(0,w') := \overline{E(w',0)}$. We call $E(w,w')$ the *density bundle of biweight* (w,w'). Since there is a specified isomorphism $E(-n-2,0) \cong K$, we can define a connection ∇ on $E(w,w')$ so that it is compatible with the Tanaka–Webster connection on K. The space of local sections of $E(w,w')$ is denoted by $\mathcal{E}(w,w')$. (However we use the density bundles of integral biweights only, one can also consider those of complex biweights (w,w') with $w - w' \in \mathbb{Z}$. See [GoGr] or [ČGo].)

F. Farris [Fa] observed that, if ζ is a locally defined nonvanishing section of K, there is a unique pseudohermitian structure θ satisfying (0.5). If we replace ζ with $\lambda\zeta$, $\lambda \in C^\infty(M, \mathbb{C}^\times)$, then θ is replaced by $|\lambda|^{2/(n+2)}\theta$. We set

$$|\zeta|^{2/(n+2)} = \zeta^{1/(n+2)} \otimes \overline{\zeta}^{1/(n+2)} \in \mathcal{E}(-1,-1),$$

which is independent of the choice of the $(n+2)^{\text{nd}}$ root of ζ and is in one-to-one correspondence with θ, and define $|\zeta|^{-2/(n+2)} \in \mathcal{E}(1,1)$ as its dual. Then we obtain a CR-invariant section $\boldsymbol{\theta} := \theta \otimes |\zeta|^{-2/(n+2)}$ of $T^*M \otimes E(1,1)$.

The Levi form h is a section of the bundle $(T^{1,0})^* \otimes (T^{0,1})^*$, which is also denoted by $E_{\alpha\overline{\beta}}$ using abstract indices α and $\overline{\beta}$. Since $h_{\alpha\overline{\beta}}$ and θ have the same scaling factor, $\boldsymbol{h}_{\alpha\overline{\beta}} := h_{\alpha\overline{\beta}} \otimes |\zeta|^{-2/(n+2)} \in \mathcal{E}_{\alpha\overline{\beta}}(1,1)$ is a CR-invariant section of $E_{\alpha\overline{\beta}}(1,1) := E_{\alpha\overline{\beta}} \otimes E(1,1)$. Its dual is $\boldsymbol{h}^{\alpha\overline{\beta}} \in \mathcal{E}^{\alpha\overline{\beta}}(-1,-1)$. Indices of density-weighted tensors are lowered and raised by $\boldsymbol{h}_{\alpha\overline{\beta}}$ and $\boldsymbol{h}^{\alpha\overline{\beta}}$.

One can show that $\nabla\boldsymbol{\theta}$ and $\nabla\boldsymbol{h}$ are both zero. To see this it is enough to show that $\nabla|\zeta|^2 = 0$, which follows from $\nabla h = 0$. For details see the proof of Proposition 2.1 in [GoGr].

The density-weighted versions of the Nijenhuis tensor, the Tanaka–Webster torsion tensor and the curvature tensor are defined by

$$\begin{gathered}\boldsymbol{N}_{\alpha\beta}{}^{\overline{\gamma}} := N_{\alpha\beta}{}^{\overline{\gamma}} \in \mathcal{E}_{\alpha\beta}{}^{\overline{\gamma}}, \qquad \boldsymbol{A}_{\alpha\beta} := A_{\alpha\beta} \in \mathcal{E}_{\alpha\beta}, \\ \boldsymbol{R}_{\alpha\overline{\beta}\sigma\overline{\tau}} := R_{\alpha\overline{\beta}\sigma\overline{\tau}} \otimes |\zeta|^{-2/(n+2)} \in \mathcal{E}_{\alpha\overline{\beta}\sigma\overline{\tau}}(1,1).\end{gathered}$$

When dealing with density-weighted tensors, we let ∇_α, $\nabla_{\overline{\alpha}}$ and $\nabla_{\mathbf{0}}$ denote the components of ∇ relative to θ^α, $\theta^{\overline{\alpha}}$ and $\boldsymbol{\theta}$. Since the transformation law (2.11) of the Tanaka–Webster connection forms does not contain the Nijenhuis tensor, equation (2.7) and Proposition 2.3 in [GoGr] also hold in the partially integrable case. Using them we can derive the transformation law of any covariant derivative of any density-weighted tensor.

3. Ricci tensor and some low-order terms

Let M be a nondegenerate partially integrable almost CR manifold with a fixed pseudohermitian structure θ and X an open neighborhood of $M = M \times \{ 0 \}$ in $M \times [0, \infty)_\rho$. We take a local frame

$$\{ \rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}} \} \tag{3.1}$$

of ${}^\Theta TX$, where T is the Reeb vector field associated to θ and $\{ Z_\alpha \}$ is a local frame of $T^{1,0}$, both extended constantly in the ρ-direction. The corresponding indices are ∞, 0, 1, $\dots$, n, $\overline{1}$, $\dots$, $\overline{n}$. The local frame (3.1) is denoted by $\{ \xi_I \}$ if needed.

Rule for the index notation. The following rule is observed for the index notation in the sequel, except in the proof of Proposition 5.5:

- α, β, γ, σ, τ run $\{ 1, \dots, n \}$ and $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, $\overline{\sigma}$, $\overline{\tau}$ run $\{ \overline{1}, \dots, \overline{n} \}$;
- i, j, k run $\{ 0, 1, \dots, n, \overline{1}, \dots, \overline{n} \}$;
- I, J, K, L run $\{ \infty, 0, 1, \dots, n, \overline{1}, \dots, \overline{n} \}$.

Lowercase Greek indices and their complex conjugates are raised and lowered by the Levi form unless otherwise stated.

We consider a normal-form ACH metric on X, i.e., a Θ-metric g satisfying

$$\begin{aligned} &g_{\infty\infty} = 4, \qquad g_{\infty 0} = 0, \qquad g_{\infty\alpha} = 0, \\ &g_{00} = 1 + O(\rho), \qquad g_{0\alpha} = O(\rho), \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + O(\rho), \qquad g_{\alpha\beta} = O(\rho), \end{aligned} \tag{3.2}$$

where $h_{\alpha\overline{\beta}}$ is the Levi form. Note that $\{ \xi_I \}$ is a distinguished local frame for g, θ. We shall compute the Ricci tensor of g and the Einstein tensor $\mathrm{Ein} := \mathrm{Ric} + \frac{1}{2}(n+2)g$. Our goal in this section is the following proposition. By abuse of notation, in what follows we use the same symbol for a tensor on M and its constant extension in the ρ-direction.

Proposition 3.1. *The Einstein tensor* Ein_{IJ} *of a normal-form ACH metric* g *is* $O(\rho^3)$ *if and only if*

$$\begin{aligned} &g_{00} = 1 + O(\rho^3), \qquad g_{0\alpha} = O(\rho^3), \\ &g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \rho^2 \Phi_{\alpha\overline{\beta}} + O(\rho^3), \qquad g_{\alpha\beta} = \rho^2 \Phi_{\alpha\beta} + O(\rho^3), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \Phi_{\alpha\overline{\beta}} &= -\frac{2}{n+2}\left(R_{\alpha\overline{\beta}} - 2N_{\alpha\sigma\tau}N_{\overline{\beta}}{}^{\tau\sigma} - \frac{1}{2(n+1)}(R - 2N_{\gamma\sigma\tau}N^{\gamma\tau\sigma})h_{\alpha\overline{\beta}} \right), \\ \Phi_{\alpha\beta} &= -2iA_{\alpha\beta} - \frac{2}{n}(N_{\gamma\alpha\beta,}{}^{\gamma} + N_{\gamma\beta\alpha,}{}^{\gamma}). \end{aligned} \tag{3.4}$$

The functions φ_{ij} are defined by

$$g_{00} = 1 + \varphi_{00}, \qquad g_{0\alpha} = \varphi_{0\alpha}, \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}}, \qquad g_{\alpha\beta} = \varphi_{\alpha\beta}. \tag{3.5}$$

The totality of (φ_{ij}) is seen as a symmetric 2-tensor on M with coefficients in $C^\infty(X)$ using the frame $\{ T, Z_\alpha, Z_{\overline{\alpha}} \}$. Hence the action of the Tanaka–Webster connection operator ∇ on (φ_{ij}) is naturally defined.

We define a connection $\overline{\nabla}$ on TX, which is "the trivial extension of ∇," by setting $\overline{\nabla}_Z W = \nabla_Z W$ for vector fields Z, W on M and

$$\overline{\nabla}\partial_\rho = 0, \qquad \overline{\nabla}_{\partial_\rho} T = \overline{\nabla}_{\partial_\rho} Z_\alpha = 0.$$

The connection forms of $\overline{\nabla}$ with respect to the frame $\{ \xi_I \}$ are given by

$$\overline{\omega}_\infty{}^\infty = \frac{d\rho}{\rho}, \qquad \overline{\omega}_0{}^0 = 2\frac{d\rho}{\rho}, \qquad \overline{\omega}_\alpha{}^\beta = \omega_\alpha{}^\beta + \delta_\alpha{}^\beta \frac{d\rho}{\rho}, \tag{3.6}$$

where $\omega_\alpha{}^\beta$ are the connection forms of ∇ with respect to $\{ Z_\alpha \}$. The torsion $\overline{\Theta}$ of $\overline{\nabla}$ is

$$\begin{aligned} &\overline{\Theta}_{IJ}{}^\infty = \overline{\Theta}_{\infty\infty}{}^0 = \overline{\Theta}_{0I}{}^0 = \overline{\Theta}_{\infty I}{}^\gamma = \overline{\Theta}_{00}{}^\gamma = \overline{\Theta}_{0\alpha}{}^\beta = \overline{\Theta}_{\alpha\overline{\beta}}{}^\gamma = \overline{\Theta}_{\alpha\beta}{}^\gamma = 0, \\ &\overline{\Theta}_{\alpha\overline{\beta}}{}^0 = ih_{\alpha\overline{\beta}}, \qquad \overline{\Theta}_{0\alpha}{}^{\overline{\beta}} = \rho^2 A_\alpha{}^{\overline{\beta}}, \qquad \overline{\Theta}_{\alpha\beta}{}^{\overline{\gamma}} = -\rho N_{\alpha\beta}{}^{\overline{\gamma}}; \end{aligned} \tag{3.7}$$

the Ricci tensor of $\overline{\nabla}$, defined by $\overline{R}_{IJ} := \overline{R}_I{}^K{}_{KJ}$, is given by

$$\begin{aligned} &\overline{R}_{\infty I} = \overline{R}_{I\infty} = \overline{R}_{0I} = \overline{R}_{I0} = 0, \\ &\overline{R}_{\alpha\overline{\beta}} = \rho^2 (R_{\alpha\overline{\beta}} - N_{\alpha\sigma\tau} N_{\overline{\beta}}{}^{\tau\sigma}), \qquad \overline{R}_{\alpha\beta} = \rho^2 \left(i(n-1)A_{\alpha\beta} + N_{\gamma\beta\alpha,}{}^\gamma \right). \end{aligned} \tag{3.8}$$

We sometimes reinterpret a tensor on X as a set of tensors on M with coefficient in $C^\infty(X)$. For example, a symmetric 2-tensor S_{IJ} is also regarded as the composed object of a scalar-valued function $S_{\infty\infty}$, a 1-tensor $S_{\infty i}$ and a 2-tensor S_{ij}, with coefficients in $C^\infty(X)$. Thus ∇ can be applied to $S_{IJ} = (S_{\infty\infty}, S_{\infty i}, S_{ij})$. Let $\#(I_1, \dots, I_N) := N +$ (the number of 0's in $I_1, \dots, I_N$). Then, from (3.6) we have the following formulae:

$$\overline{\nabla}_\infty S_{IJ} = (\rho\partial_\rho - \#(I,J))S_{IJ}, \qquad \overline{\nabla}_0 S_{IJ} = \rho^2 \nabla_0 S_{IJ}, \qquad \overline{\nabla}_\alpha S_{IJ} = \rho \nabla_\alpha S_{IJ}; \tag{3.9}$$

on the left-hand sides of the equalities $\{ \rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}} \}$ is used for covariant differentiation, while on the right-hand sides $\{ T, Z_\alpha, Z_{\overline{\alpha}} \}$ is used.

We set $\nabla^g_{\xi_J} \xi_I = \overline{\nabla}_{\xi_J} \xi_I + D_I{}^K{}_J \xi_K$, where ∇^g is the Levi-Civita connection of g. Then the Ricci tensor of g is given by

$$\mathrm{Ric}_{IJ} = \overline{R}_{IJ} + \overline{\nabla}_K D_I{}^K{}_J - \overline{\nabla}_J D_I{}^K{}_K - D_I{}^L{}_K D_J{}^K{}_L + D_I{}^L{}_J D_L{}^K{}_K. \tag{3.10}$$

Thus the calculation of the Ricci tensor essentially reduces to that of $D_I{}^K{}_J$. We can compute $D_{IKJ} := g_{KL} D_I{}^L{}_J$ by the formula

$$D_{IKJ} = \tfrac{1}{2}(\overline{\nabla}_I g_{JK} + \overline{\nabla}_J g_{IK} - \overline{\nabla}_K g_{IJ} + \overline{\Theta}_{IKJ} + \overline{\Theta}_{JKI} + \overline{\Theta}_{IJK}),$$

where $\overline{\Theta}_{IJK} := g_{KL} \overline{\Theta}_{IJ}{}^L$. The result is given in Table 3.1.

To prove Proposition 3.1, it is enough to calculate everything modulo $O(\rho^3)$. However, for later use, we shall carry out more precise computation. What we allow ourselves to neglect are

(N1) any term at least quadratic in $\varphi_{ij,k\dots}$ with $O(1)$ coefficients,

Type	Value
$D_{\infty\infty\infty}$	-4
$D_{\infty 0\infty}$	0
$D_{\infty\alpha\infty}$	0
$D_{\infty\infty 0}$	0
$D_{\infty 00}$	$-2+\frac{1}{2}(\rho\partial_\rho-4)\varphi_{00}$
$D_{\infty\alpha 0}$	$\frac{1}{2}(\rho\partial_\rho-3)\varphi_{0\alpha}$
$D_{\infty\infty\alpha}$	0
$D_{\infty 0\alpha}$	$\frac{1}{2}(\rho\partial_\rho-3)\varphi_{0\alpha}$
$D_{\infty\overline{\beta}\alpha}$	$-h_{\alpha\overline{\beta}}+\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}$
$D_{\infty\beta\alpha}$	$\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\beta}$
$D_{0\infty 0}$	$2-\frac{1}{2}(\rho\partial_\rho-4)\varphi_{00}$
D_{000}	$\frac{1}{2}\rho^2\nabla_0\varphi_{00}$
$D_{0\alpha 0}$	$-\frac{1}{2}\rho\nabla_\alpha\varphi_{00}+\rho^2(\nabla_0\varphi_{0\alpha}+A_\alpha{}^{\overline{\beta}}\varphi_{0\overline{\beta}})$
$D_{0\infty\alpha}$	$-\frac{1}{2}(\rho\partial_\rho-3)\varphi_{0\alpha}$
$D_{00\alpha}$	$\frac{1}{2}\rho\nabla_\alpha\varphi_{00}$
$D_{0\overline{\beta}\alpha}$	$\frac{i}{2}h_{\alpha\overline{\beta}}+\frac{i}{2}h_{\alpha\overline{\beta}}\varphi_{00}+\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}}-\nabla_{\overline{\beta}}\varphi_{0\alpha})+\frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\overline{\beta}}+A_\alpha{}^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}+A_{\overline{\beta}}{}^{\gamma}\varphi_{\alpha\gamma})$
$D_{0\beta\alpha}$	$\rho^2A_{\alpha\beta}+\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\beta}-\nabla_\beta\varphi_{0\alpha}-N_{\alpha\beta}{}^{\overline{\gamma}}\varphi_{0\overline{\gamma}})+\frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\beta}+A_\alpha{}^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}}+A_\beta{}^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}})$
$D_{\alpha\infty 0}$	$-\frac{1}{2}(\rho\partial_\rho-3)\varphi_{0\alpha}$
$D_{\alpha 00}$	$\frac{1}{2}\rho\nabla_\alpha\varphi_{00}-\rho^2A_\alpha{}^{\overline{\beta}}\varphi_{0\overline{\beta}}$
$D_{\alpha\overline{\beta}0}$	$\frac{i}{2}h_{\alpha\overline{\beta}}+\frac{i}{2}h_{\alpha\overline{\beta}}\varphi_{00}+\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}}-\nabla_{\overline{\beta}}\varphi_{0\alpha})+\frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\overline{\beta}}-A_\alpha{}^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}+A_{\overline{\beta}}{}^{\gamma}\varphi_{\alpha\gamma})$
$D_{\alpha\beta 0}$	$\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\beta}-\nabla_\beta\varphi_{0\alpha}-N_{\alpha\beta}{}^{\overline{\gamma}}\varphi_{0\overline{\gamma}})+\frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\beta}-A_\alpha{}^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}}+A_\beta{}^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}})$
$D_{\alpha\infty\overline{\beta}}$	$h_{\alpha\overline{\beta}}-\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}$
$D_{\alpha 0\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}+\frac{i}{2}h_{\alpha\overline{\beta}}\varphi_{00}+\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}}+\nabla_{\overline{\beta}}\varphi_{0\alpha})-\frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\overline{\beta}}+A_\alpha{}^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}+A_{\overline{\beta}}{}^{\gamma}\varphi_{\alpha\gamma})$
$D_{\alpha\overline{\gamma}\overline{\beta}}$	$\frac{i}{2}(h_{\alpha\overline{\gamma}}\varphi_{0\overline{\beta}}+h_{\alpha\overline{\beta}}\varphi_{0\overline{\gamma}})+\frac{1}{2}\rho(\nabla_\alpha\varphi_{\overline{\beta}\overline{\gamma}}+\nabla_{\overline{\beta}}\varphi_{\alpha\overline{\gamma}}-\nabla_{\overline{\gamma}}\varphi_{\alpha\overline{\beta}}-N_{\overline{\beta}\overline{\gamma}}{}^{\sigma}\varphi_{\alpha\sigma})$
$D_{\alpha\gamma\overline{\beta}}$	$-\frac{i}{2}(h_{\gamma\overline{\beta}}\varphi_{0\alpha}-h_{\alpha\overline{\beta}}\varphi_{0\gamma})+\frac{1}{2}\rho(\nabla_\alpha\varphi_{\gamma\overline{\beta}}+\nabla_{\overline{\beta}}\varphi_{\alpha\gamma}-\nabla_\gamma\varphi_{\alpha\overline{\beta}}-N_{\alpha\gamma}{}^{\overline{\sigma}}\varphi_{\overline{\beta}\overline{\sigma}})$
$D_{\alpha\infty\beta}$	$-\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\beta}$
$D_{\alpha 0\beta}$	$-\rho^2A_{\alpha\beta}+\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\beta}+\nabla_\beta\varphi_{0\alpha}+N_{\alpha\beta}{}^{\overline{\gamma}}\varphi_{0\overline{\gamma}})-\frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\beta}+A_\alpha{}^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}}+A_\beta{}^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}})$
$D_{\alpha\overline{\gamma}\beta}$	$\frac{i}{2}(h_{\alpha\overline{\gamma}}\varphi_{0\beta}+h_{\beta\overline{\gamma}}\varphi_{0\alpha})+\frac{1}{2}\rho(\nabla_\alpha\varphi_{\beta\overline{\gamma}}+\nabla_\beta\varphi_{\alpha\overline{\gamma}}-\nabla_{\overline{\gamma}}\varphi_{\alpha\beta}-N_{\alpha\beta}{}^{\overline{\sigma}}\varphi_{\overline{\gamma}\overline{\sigma}})$
$D_{\alpha\gamma\beta}$	$-\rho N_{\alpha\gamma\beta}+\frac{1}{2}\rho(\nabla_\alpha\varphi_{\beta\gamma}+\nabla_\beta\varphi_{\alpha\gamma}-\nabla_\gamma\varphi_{\alpha\beta}-N_{\alpha\beta}{}^{\overline{\sigma}}\varphi_{\gamma\overline{\sigma}}-N_{\alpha\gamma}{}^{\overline{\sigma}}\varphi_{\beta\overline{\sigma}}-N_{\beta\gamma}{}^{\overline{\sigma}}\varphi_{\alpha\overline{\sigma}})$

TABLE 3.1. D_{IKJ} for a normal-form ACH metric g. $D_{0K\infty}$ and $D_{\alpha K\infty}$ are omitted; we have $D_{0K\infty}=D_{\infty K0}$ and $D_{\alpha K\infty}=D_{\infty K\alpha}$

(N2) any term linear in $\varphi_{00,k\dots}$, $\varphi_{0\alpha,k\dots}$, $\varphi_{0\overline{\alpha},k\dots}$ or $\varphi_{\alpha\overline{\beta},k\dots}$ with $O(\rho)$ coefficients which vanish in the case of the CR sphere with standard pseudohermitian structure, and

(N3) any term linear in $\varphi_{\alpha\beta,k\dots}$ or $\varphi_{\overline{\alpha}\overline{\beta},k\dots}$ with $O(\rho^2)$ coefficients which vanish in the case of the CR sphere with standard pseudohermitian structure.

Modulo terms of type (N1), g^{IJ} is given by

$$\begin{aligned} &g^{\infty\infty} \equiv \tfrac{1}{4}, \qquad g^{\infty 0} \equiv g^{\infty\alpha} = 0, \\ &g^{00} \equiv 1 - \varphi_{00}, \qquad g^{0\alpha} \equiv -\varphi_0{}^{\alpha}, \qquad g^{\alpha\overline{\beta}} \equiv h^{\alpha\overline{\beta}} - \varphi^{\alpha\overline{\beta}}, \qquad g^{\alpha\beta} \equiv -\varphi^{\alpha\beta}. \end{aligned} \tag{3.11}$$

By these formulae and Table 3.1, we compute $D_I{}^K{}_J$ modulo terms of type (N1)–(N3) using the equality $D_I{}^K{}_J = g^{KL} D_{ILJ}$. Table 3.2 is the result.

Finally we can show the following formulae for the Einstein tensor. We define the sublaplacian by $\Delta_b := -(\nabla^\alpha\nabla_\alpha + \nabla^{\overline{\alpha}}\nabla_{\overline{\alpha}})$.

Lemma 3.2. *The Einstein tensor of an ACH metric g is, modulo terms of type (N1)–(N3),*

$$\begin{aligned}
\mathrm{Ein}_{\infty\infty} &\equiv -\tfrac{1}{2}\rho\partial_\rho(\rho\partial_\rho - 4)\varphi_{00} - \rho\partial_\rho(\rho\partial_\rho - 2)\varphi_\alpha{}^\alpha, \\
\mathrm{Ein}_{\infty 0} &\equiv \tfrac{1}{2}\rho(\rho\partial_\rho + 1)(\nabla^\alpha\varphi_{0\alpha} + \nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}}) - \rho^2(\rho\partial_\rho + 1)\nabla_0\varphi_\alpha{}^\alpha, \\
\mathrm{Ein}_{\infty\alpha} &\equiv -\tfrac{i}{2}(\rho\partial_\rho + 1)\varphi_{0\alpha} - \tfrac{1}{2}\rho(\rho\partial_\rho - 1)\nabla_\alpha\varphi_{00} - \rho^2\partial_\rho\nabla_\alpha\varphi_\beta{}^\beta \\
&\quad + \tfrac{1}{2}\rho^2\partial_\rho(\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \nabla^\beta\varphi_{\alpha\beta}) + \tfrac{1}{2}\rho^2\partial_\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}} + \tfrac{1}{2}\rho^2(\rho\partial_\rho - 1)\nabla_0\varphi_{0\alpha}, \\
\mathrm{Ein}_{00} &\equiv -\tfrac{1}{8}\left((\rho\partial_\rho)^2 - (2n+4)\rho\partial_\rho - 4n\right)\varphi_{00} + \tfrac{1}{2}(\rho\partial_\rho - 2)\varphi_\alpha{}^\alpha \\
&\quad + i\rho(\nabla^\alpha\varphi_{0\alpha} - \nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}}) + \tfrac{1}{2}\rho^2\Delta_b\varphi_{00} + \rho^3(\nabla_0\nabla^\alpha\varphi_{0\alpha} + \nabla_0\nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}}) - \rho^4\nabla_0\nabla_0\varphi_\alpha{}^\alpha \\
\mathrm{Ein}_{0\alpha} &\equiv \rho^3 A_{\alpha\beta,}{}^\beta + \rho^3 N_\alpha{}^{\overline{\beta}\overline{\gamma}}A_{\overline{\beta}\overline{\gamma}} - \tfrac{1}{8}(\rho\partial_\rho + 1)(\rho\partial_\rho - 2n - 3)\varphi_{0\alpha} \\
&\quad + \tfrac{3i}{4}\rho\nabla_\alpha\varphi_{00} + \tfrac{i}{2}\rho\nabla_\alpha\varphi_\beta{}^\beta - i\rho\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \tfrac{1}{2}\rho^2\Delta_b\varphi_{0\alpha} - \tfrac{i}{2}\rho^2\nabla_0\varphi_{0\alpha} \\
&\quad + \tfrac{1}{2}\rho^2(\nabla_\alpha\nabla^\beta\varphi_{0\beta} + \nabla_\alpha\nabla^{\overline{\beta}}\varphi_{0\overline{\beta}}) - \rho^3\nabla_0\nabla_\alpha\varphi_\beta{}^\beta + \tfrac{1}{2}\rho^3(\nabla_0\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \nabla_0\nabla^\beta\varphi_{\alpha\beta}), \\
\mathrm{Ein}_{\alpha\overline{\beta}} &\equiv \rho^2 R_{\alpha\overline{\beta}} - 2\rho^2 N_\alpha{}^{\overline{\gamma}}{}_\rho N_{\overline{\beta}}{}^\rho{}_{\overline{\gamma}} - \tfrac{1}{8}\left((\rho\partial_\rho)^2 - (2n+2)\rho\partial_\rho - 8\right)\varphi_{\alpha\overline{\beta}} \\
&\quad + \tfrac{1}{8}h_{\alpha\overline{\beta}}(\rho\partial_\rho - 4)\varphi_{00} + \tfrac{1}{4}h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_\gamma{}^\gamma + i\rho(\nabla_\alpha\varphi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\varphi_{0\alpha}) \\
&\quad - \tfrac{i}{4}\rho^2 h_{\alpha\overline{\beta}}\nabla_0\varphi_{00} - \tfrac{i}{2}\rho^2 h_{\alpha\overline{\beta}}\nabla_0\varphi_\gamma{}^\gamma - \tfrac{1}{2}\rho^2\nabla_\alpha\nabla_{\overline{\beta}}\varphi_{00} - \rho^2\nabla_\alpha\nabla_{\overline{\beta}}\varphi_\gamma{}^\gamma \\
&\quad + \tfrac{1}{2}\rho^2(\Delta_b\varphi_{\alpha\overline{\beta}} + \nabla_\alpha\nabla^\gamma\varphi_{\overline{\beta}\gamma} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^\gamma\varphi_{\alpha\gamma}) \\
&\quad + \tfrac{1}{2}\rho^3(\nabla_0\nabla_\alpha\varphi_{0\overline{\beta}} + \nabla_0\nabla_{\overline{\beta}}\varphi_{0\alpha}) - \tfrac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{\alpha\overline{\beta}}, \\
\mathrm{Ein}_{\alpha\beta} &\equiv in\rho^2 A_{\alpha\beta} + \rho^2(N_{\gamma\alpha\beta,}{}^\gamma + N_{\gamma\beta\alpha,}{}^\gamma) - \rho^4 A_{\alpha\beta,0} - \tfrac{1}{8}\rho\partial_\rho(\rho\partial_\rho - 2n - 2)\varphi_{\alpha\beta} \\
&\quad - \tfrac{1}{2}\rho^2\nabla_\alpha\nabla_\beta\varphi_{00} - \rho^2\nabla_\alpha\nabla_\beta\varphi_\gamma{}^\gamma \\
&\quad + \tfrac{1}{2}\rho^2(\Delta_b\varphi_{\alpha\beta} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}} + \nabla_\alpha\nabla^\gamma\varphi_{\beta\gamma} + \nabla_\beta\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}} + \nabla_\beta\nabla^\gamma\varphi_{\alpha\gamma} + 2i\nabla_0\varphi_{\alpha\beta}) \\
&\quad + \tfrac{1}{2}\rho^3(\nabla_0\nabla_\alpha\varphi_{0\beta} + \nabla_0\nabla_\beta\varphi_{0\alpha}) - \tfrac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{\alpha\beta}.
\end{aligned}$$

Proof. Using Table 3.2 we compute, modulo terms of type (N1)–(N3),

$$\overline{\nabla}_K D_I{}^K{}_J, \qquad \overline{\nabla}_J D_I{}^K{}_K, \qquad D_I{}^L{}_K D_J{}^K{}_L \qquad \text{and} \qquad D_I{}^L{}_J D_L{}^K{}_K$$

to obtain Tables 3.3–3.6. Then, from (3.8) and (3.10), the lemma follows. ∎

Type	Value (modulo terms of type (N1)–(N3))
${D_\infty}^\infty{}_\infty$	-1
${D_\infty}^0{}_\infty$	0
${D_\infty}^\alpha{}_\infty$	0
${D_\infty}^\infty{}_0$	0
${D_\infty}^0{}_0$	$-2+\frac{1}{2}\rho\partial_\rho\varphi_{00}$
${D_\infty}^\alpha{}_0$	$\frac{1}{2}(\rho\partial_\rho+1){\varphi_0}^\alpha$
${D_\infty}^\infty{}_\alpha$	0
${D_\infty}^0{}_\alpha$	$\frac{1}{2}(\rho\partial_\rho-1)\varphi_{0\alpha}$
${D_\infty}^\beta{}_\alpha$	$-{\delta_\alpha}^\beta+\frac{1}{2}\rho\partial_\rho{\varphi_\alpha}^\beta$
${D_\infty}^{\overline{\beta}}{}_\alpha$	$\frac{1}{2}\rho\partial_\rho{\varphi_\alpha}^{\overline{\beta}}$
${D_0}^\infty{}_0$	$\frac{1}{2}-\frac{1}{8}(\rho\partial_\rho-4)\varphi_{00}$
${D_0}^0{}_0$	$\frac{1}{2}\rho^2\nabla_0\varphi_{00}$
${D_0}^\alpha{}_0$	$-\frac{1}{2}\rho\nabla^\alpha\varphi_{00}+\rho^2\nabla_0{\varphi_0}^\alpha$
${D_0}^\infty{}_\alpha$	$-\frac{1}{8}(\rho\partial_\rho-3)\varphi_{0\alpha}$
${D_0}^0{}_\alpha$	$-\frac{i}{2}\varphi_{0\alpha}+\frac{1}{2}\rho\nabla_\alpha\varphi_{00}$
${D_0}^\beta{}_\alpha$	$\frac{i}{2}{\delta_\alpha}^\beta+\frac{i}{2}{\delta_\alpha}^\beta\varphi_{00}-\frac{i}{2}{\varphi_\alpha}^\beta+\frac{1}{2}\rho(\nabla_\alpha{\varphi_0}^\beta-\nabla^\beta\varphi_{0\alpha})+\frac{1}{2}\rho^2\nabla_0{\varphi_\alpha}^\beta$
${D_0}^{\overline{\beta}}{}_\alpha$	$\rho^2{A_\alpha}^{\overline{\beta}}-\frac{i}{2}{\varphi_\alpha}^{\overline{\beta}}+\frac{1}{2}\rho(\nabla_\alpha{\varphi_0}^{\overline{\beta}}-\nabla^{\overline{\beta}}\varphi_{0\alpha})+\frac{1}{2}\rho^2\nabla_0{\varphi_\alpha}^{\overline{\beta}}$
${D_\alpha}^\infty{}_0$	$-\frac{1}{8}(\rho\partial_\rho-3)\varphi_{0\alpha}$
${D_\alpha}^0{}_0$	$-\frac{i}{2}\varphi_{0\alpha}+\frac{1}{2}\rho\nabla_\alpha\varphi_{00}$
${D_\alpha}^\beta{}_0$	$\frac{i}{2}{\delta_\alpha}^\beta+\frac{i}{2}{\delta_\alpha}^\beta\varphi_{00}-\frac{i}{2}{\varphi_\alpha}^\beta+\frac{1}{2}\rho(\nabla_\alpha{\varphi_0}^\beta-\nabla^\beta\varphi_{0\alpha})+\frac{1}{2}\rho^2\nabla_0{\varphi_\alpha}^\beta$
${D_\alpha}^{\overline{\beta}}{}_0$	$-\frac{i}{2}{\varphi_\alpha}^{\overline{\beta}}+\frac{1}{2}\rho(\nabla_\alpha{\varphi_0}^{\overline{\beta}}-\nabla^{\overline{\beta}}\varphi_{0\alpha})+\frac{1}{2}\rho^2\nabla_0{\varphi_\alpha}^{\overline{\beta}}$
${D_\alpha}^\infty{}_{\overline{\beta}}$	$\frac{1}{4}h_{\alpha\overline{\beta}}-\frac{1}{8}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}$
${D_\alpha}^0{}_{\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}+\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}}+\nabla_{\overline{\beta}}\varphi_{0\alpha})-\frac{1}{2}\rho^2\nabla_0\varphi_{\alpha\overline{\beta}}$
${D_\alpha}^\gamma{}_{\overline{\beta}}$	$\frac{i}{2}{\delta_\alpha}^\gamma\varphi_{0\overline{\beta}}+\frac{1}{2}\rho(\nabla_\alpha{\varphi_{\overline{\beta}}}^\gamma+\nabla_{\overline{\beta}}{\varphi_\alpha}^\gamma-\nabla^\gamma\varphi_{\alpha\overline{\beta}})-\frac{1}{2}\rho {N_{\overline{\beta}}}^{\gamma\sigma}\varphi_{\alpha\sigma}$
${D_\alpha}^{\overline{\gamma}}{}_{\overline{\beta}}$	$-\frac{i}{2}{\delta_{\overline{\beta}}}^{\overline{\gamma}}\varphi_{0\alpha}+\frac{1}{2}\rho(\nabla_\alpha{\varphi_{\overline{\beta}}}^{\overline{\gamma}}+\nabla_{\overline{\beta}}{\varphi_\alpha}^{\overline{\gamma}}-\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\beta}})-\frac{1}{2}\rho {N_\alpha}^{\overline{\gamma}\overline{\sigma}}\varphi_{\overline{\beta}\overline{\sigma}}$
${D_\alpha}^\infty{}_\beta$	$-\frac{1}{8}(\rho\partial_\rho-2)\varphi_{\alpha\beta}$
${D_\alpha}^0{}_\beta$	$-\rho^2A_{\alpha\beta}+\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\beta}+\nabla_\beta\varphi_{0\alpha})-\frac{1}{2}\rho {N_{\alpha\beta}}^{\overline{\gamma}}\varphi_{0\overline{\gamma}}-\frac{1}{2}\rho^2\nabla_0\varphi_{\alpha\beta}$
${D_\alpha}^\gamma{}_\beta$	$\frac{i}{2}({\delta_\alpha}^\gamma\varphi_{0\beta}+{\delta_\beta}^\gamma\varphi_{0\alpha})+\frac{1}{2}\rho(\nabla_\alpha{\varphi_\beta}^\gamma+\nabla_\beta{\varphi_\alpha}^\gamma-\nabla^\gamma\varphi_{\alpha\beta})-\frac{1}{2}\rho {N_{\alpha\beta}}^{\overline{\sigma}}{\varphi_{\overline{\sigma}}}^\gamma$
${D_\alpha}^{\overline{\gamma}}{}_\beta$	$-\rho {N_\alpha}^{\overline{\gamma}}{}_\beta+\frac{1}{2}\rho(\nabla_\alpha{\varphi_\beta}^{\overline{\gamma}}+\nabla_\beta{\varphi_\alpha}^{\overline{\gamma}}-\nabla^{\overline{\gamma}}\varphi_{\alpha\beta})$

TABLE 3.2. ${D_I}^K{}_J$ for a normal-form ACH metric g. ${D_0}^K{}_\infty$ and ${D_\alpha}^K{}_\infty$ are omitted; we have ${D_0}^K{}_\infty={D_\infty}^K{}_0$ and ${D_\alpha}^K{}_\infty={D_\infty}^K{}_\alpha$

Type	Value (modulo terms of type (N1)–(N3))
$\overline{\nabla}_K D_\infty{}^K{}_\infty$	1
$\overline{\nabla}_K D_\infty{}^K{}_0$	$\frac{1}{2}\rho^3\partial_\rho\nabla_0\varphi_{00} + \frac{1}{2}\rho(\rho\partial_\rho+1)(\nabla^\alpha\varphi_{0\alpha} + \nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}})$
$\overline{\nabla}_K D_\infty{}^K{}_\alpha$	$\frac{1}{2}\rho^2(\rho\partial_\rho-1)\nabla_0\varphi_{0\alpha} + \frac{1}{2}\rho^2\partial_\rho(\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \nabla^\beta\varphi_{\alpha\beta})$
$\overline{\nabla}_K D_0{}^K{}_0$	$-\frac{3}{2} - \frac{1}{8}(\rho\partial_\rho-3)(\rho\partial_\rho-4)\varphi_{00} + \frac{1}{2}\rho^2\Delta_b\varphi_{00} + \rho^3(\nabla_0\nabla^\alpha\varphi_{0\alpha} + \nabla_0\nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}})$ $+\frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{00}$
$\overline{\nabla}_K D_0{}^K{}_\alpha$	$-\frac{1}{8}(\rho\partial_\rho-2)(\rho\partial_\rho-3)\varphi_{0\alpha} + \frac{i}{2}\rho\nabla_\alpha\varphi_{00} - \frac{i}{2}\rho(\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \nabla^\beta\varphi_{\alpha\beta})$ $+\frac{1}{2}\rho^2\Delta_b\varphi_{0\alpha} + \frac{1}{2}\rho^2(\nabla_\alpha\nabla^\beta\varphi_{0\beta} + \nabla_\alpha\nabla^{\overline{\beta}}\varphi_{0\overline{\beta}})$ $+\rho^3 A_{\alpha\beta,}{}^\beta + \frac{1}{2}\rho^3\nabla_0\nabla_\alpha\varphi_{00} + \frac{1}{2}\rho^3(\nabla_0\nabla^\beta\varphi_{\alpha\beta} + \nabla_0\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}})$
$\overline{\nabla}_K D_\alpha{}^K{}_{\overline{\beta}}$	$-\frac{1}{4}h_{\alpha\overline{\beta}} - \frac{1}{8}(\rho\partial_\rho-1)(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}} + \frac{i}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\varphi_{0\alpha})$ $+\frac{1}{2}\rho^2(\Delta_b\varphi_{\alpha\overline{\beta}} + \nabla_\alpha\nabla^\gamma\varphi_{\overline{\beta}\gamma} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^\gamma\varphi_{\alpha\gamma})$ $+\frac{1}{2}\rho^3(\nabla_0\nabla_\alpha\varphi_{0\overline{\beta}} + \nabla_0\nabla_{\overline{\beta}}\varphi_{0\alpha}) - \frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{\alpha\overline{\beta}}$
$\overline{\nabla}_K D_\alpha{}^K{}_\beta$	$\rho^2 N_{\gamma\alpha\beta,}{}^\gamma - \rho^4 A_{\alpha\beta,0} - \frac{1}{8}(\rho\partial_\rho-1)(\rho\partial_\rho-2)\varphi_{\alpha\beta} + \frac{i}{2}\rho(\nabla_\alpha\varphi_{0\beta} + \nabla_\beta\varphi_{0\alpha})$ $+\frac{1}{2}\rho^2(\Delta_b\varphi_{\alpha\beta} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}} + \nabla_\alpha\nabla^\gamma\varphi_{\beta\overline{\gamma}} + \nabla_\beta\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}} + \nabla_\beta\nabla^\gamma\varphi_{\alpha\gamma}) + i\rho^2\nabla_0\varphi_{\alpha\beta}$ $+\frac{1}{2}\rho^3(\nabla_0\nabla_\alpha\varphi_{0\beta} + \nabla_0\nabla_\beta\varphi_{0\alpha}) - \frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{\alpha\beta}$

TABLE 3.3. $\overline{\nabla}_K D_I{}^K{}_J$ for a normal-form ACH metric g

Type	Value (modulo terms of type (N1)–(N3))
$\overline{\nabla}_\infty D_\infty{}^K{}_K$	$2n+3+\frac{1}{2}\rho\partial_\rho(\rho\partial_\rho-1)\varphi_{00} + \rho\partial_\rho(\rho\partial_\rho-1)\varphi_\alpha{}^\alpha$
$\overline{\nabla}_0 D_\infty{}^K{}_K$	$\frac{1}{2}\rho^3\partial_\rho\nabla_0\varphi_{00} + \rho^3\partial_\rho\nabla_0\varphi_\alpha{}^\alpha$
$\overline{\nabla}_\alpha D_\infty{}^K{}_K$	$\frac{1}{2}\rho^2\partial_\rho\nabla_\alpha\varphi_{00} + \rho^2\partial_\rho\nabla_\alpha\varphi_\beta{}^\beta$
$\overline{\nabla}_0 D_0{}^K{}_K$	$\frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{00} + \rho^4\nabla_0\nabla_0\varphi_\alpha{}^\alpha$
$\overline{\nabla}_\alpha D_0{}^K{}_K$	$\frac{1}{2}\rho^3\nabla_0\nabla_\alpha\varphi_{00} + \rho^3\nabla_0\nabla_\alpha\varphi_\beta{}^\beta$
$\overline{\nabla}_{\overline{\beta}} D_\alpha{}^K{}_K$	$\frac{1}{2}\rho^2\nabla_\alpha\nabla_{\overline{\beta}}\varphi_{00} + \rho^2\nabla_\alpha\nabla_{\overline{\beta}}\varphi_\gamma{}^\gamma + \frac{i}{2}\rho^2 h_{\alpha\overline{\beta}}\nabla_0\varphi_{00} + i\rho^2 h_{\alpha\overline{\beta}}\nabla_0\varphi_\gamma{}^\gamma$
$\overline{\nabla}_\beta D_\alpha{}^K{}_K$	$\frac{1}{2}\rho^2\nabla_\alpha\nabla_\beta\varphi_{00} + \rho^2\nabla_\alpha\nabla_\beta\varphi_\gamma{}^\gamma$

TABLE 3.4. $\overline{\nabla}_J D_I{}^K{}_K$ for a normal-form ACH metric g

Since by definition φ_{ij} is $O(\rho)$, from Lemma 3.2 we have

$$
\begin{aligned}
&\mathrm{Ein}_{\infty\infty} = \tfrac{3}{2}\varphi_{00} + \varphi_\alpha{}^\alpha + O(\rho^2), \qquad \mathrm{Ein}_{\infty 0} = O(\rho^2), \qquad \mathrm{Ein}_{\infty\alpha} = -i\varphi_{0\alpha} + O(\rho^2),\\
&\mathrm{Ein}_{00} = \tfrac{3}{8}(2n+1)\varphi_{00} - \tfrac{1}{2}\varphi_\alpha{}^\alpha + O(\rho^2), \qquad \mathrm{Ein}_{0\alpha} = \tfrac{1}{2}(n+1)\varphi_{0\alpha} + O(\rho^2),\\
&\mathrm{Ein}_{\alpha\overline{\beta}} = \tfrac{1}{8}(2n+9)\varphi_{\alpha\overline{\beta}} - \tfrac{3}{8}h_{\alpha\overline{\beta}}\varphi_{00} + \tfrac{1}{4}h_{\alpha\overline{\beta}}\varphi_\gamma{}^\gamma + O(\rho^2),\\
&\mathrm{Ein}_{\alpha\beta} = \tfrac{1}{8}(2n+1)\varphi_{\alpha\beta} + O(\rho^2).
\end{aligned} \tag{3.12}
$$

Type	Value (modulo terms of type (N1)–(N3))
$D_\infty{}^L{}_K D_\infty{}^K{}_L$	$2n+5-2\rho\partial_\rho\varphi_{00}-2\rho\partial_\rho\varphi_\alpha{}^\alpha$
$D_\infty{}^L{}_K D_0{}^K{}_L$	$-\rho^2\nabla_0\varphi_{00}-\rho^2\nabla_0\varphi_\alpha{}^\alpha$
$D_\infty{}^L{}_K D_\alpha{}^K{}_L$	$\frac{i}{2}(\rho\partial_\rho+1)\varphi_{0\beta}-\rho\nabla_\alpha\varphi_{00}-\rho\nabla_\alpha\varphi_\beta{}^\beta-\frac{1}{2}\rho(\rho\partial_\rho-2)N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}$
$D_0{}^L{}_K D_0{}^K{}_L$	$-\frac{1}{2}(n+4)+(\rho\partial_\rho-n-2)\varphi_{00}+\varphi_\alpha{}^\alpha-i\rho(\nabla^\alpha\varphi_{0\alpha}-\nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}})$
$D_0{}^L{}_K D_\alpha{}^K{}_L$	$-\rho^3N_\alpha{}^{\overline{\beta}\overline{\gamma}}A_{\overline{\beta}\overline{\gamma}}+\frac{1}{4}(3\rho\partial_\rho-2n-5)\varphi_{0\alpha}+\frac{i}{2}(\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}}-\nabla^\beta\varphi_{\alpha\beta})-\frac{i}{2}\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}$ $+\frac{i}{2}\rho^2\nabla_0\varphi_{0\alpha}$
$D_\alpha{}^L{}_K D_{\overline{\beta}}{}^K{}_L$	$\rho^2N_\alpha{}^{\overline{\gamma}}{}_\rho N_{\overline{\beta}}{}^\rho{}_{\overline{\gamma}}+\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}+\frac{1}{2}h_{\alpha\overline{\beta}}\varphi_{00}-\frac{i}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}}-\nabla_{\overline{\beta}}\varphi_{0\alpha})$
$D_\alpha{}^L{}_K D_\beta{}^K{}_L$	$\frac{1}{2}\rho\partial_\rho\varphi_{\alpha\beta}+i\rho^2A_{\alpha\beta}+\frac{i}{2}\rho(\nabla_\alpha\varphi_{0\beta}+\nabla_\beta\varphi_{0\alpha})$

TABLE 3.5. $D_I{}^L{}_K D_J{}^K{}_L$ for a normal-form ACH metric g

Type	Value (modulo terms of type (N1)–(N3))
$D_\infty{}^L{}_\infty D_L{}^K{}_K$	$2n+3-\frac{1}{2}\rho\partial_\rho\varphi_{00}-\rho\partial_\rho\varphi_\alpha{}^\alpha$
$D_\infty{}^L{}_0 D_L{}^K{}_K$	$-\rho^2\nabla_0\varphi_{00}-2\rho^2\nabla_0\varphi_\alpha{}^\alpha$
$D_\infty{}^L{}_\alpha D_L{}^K{}_K$	$-\frac{1}{2}\rho\nabla_\alpha\varphi_{00}-\rho\nabla_\alpha\varphi_\beta{}^\beta+\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}$
$D_0{}^L{}_0 D_L{}^K{}_K$	$-\frac{1}{2}(2n+3)+\frac{1}{4}\rho\partial_\rho\varphi_{00}+\frac{1}{2}\rho\partial_\rho\varphi_\alpha{}^\alpha+\frac{1}{8}(2n+3)(\rho\partial_\rho-4)\varphi_{00}$
$D_0{}^L{}_\alpha D_L{}^K{}_K$	$\frac{1}{8}(2n+3)(\rho\partial_\rho-3)\varphi_{0\alpha}+\frac{i}{4}\rho\nabla_\alpha\varphi_{00}+\frac{i}{2}\rho\nabla_\alpha\varphi_\beta{}^\beta-\frac{i}{2}\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}$
$D_\alpha{}^L{}_{\overline{\beta}} D_L{}^K{}_K$	$-\frac{1}{4}(2n+3)h_{\alpha\overline{\beta}}+\frac{1}{8}(2n+3)(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}+\frac{1}{8}h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_{00}+\frac{1}{4}h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_\gamma{}^\gamma$ $+\frac{i}{4}\rho^2h_{\alpha\overline{\beta}}\nabla_0\varphi_{00}+\frac{i}{2}\rho^2h_{\alpha\overline{\beta}}\nabla_0\varphi_\gamma{}^\gamma$
$D_\alpha{}^L{}_\beta D_L{}^K{}_K$	$\frac{1}{8}(2n+3)(\rho\partial_\rho-2)\varphi_{\alpha\beta}$

TABLE 3.6. $D_I{}^L{}_J D_L{}^K{}_K$ for a normal-form ACH metric g

These identities show that all φ_{ij} must be $O(\rho^2)$ in order Ein_{IJ} to be $O(\rho^2)$. If $\varphi_{ij}=O(\rho^2)$, then repeating this process we obtain

$$
\begin{gathered}
\mathrm{Ein}_{\infty\infty}=2\varphi_{00}+O(\rho^3),\qquad \mathrm{Ein}_{\infty0}=O(\rho^3),\qquad \mathrm{Ein}_{\infty\alpha}=-\tfrac{3i}{2}\varphi_{0\alpha}+O(\rho^3),\\
\mathrm{Ein}_{00}=\tfrac{1}{2}(2n+1)\varphi_{00}+O(\rho^3),\qquad \mathrm{Ein}_{0\alpha}=\tfrac{3}{8}(2n+1)\varphi_{0\alpha}+O(\rho^3),\\
\mathrm{Ein}_{\alpha\overline{\beta}}=\rho^2R_{\alpha\overline{\beta}}-2\rho^2N_\alpha{}^{\overline{\gamma}}{}_\rho N_{\overline{\beta}}{}^\rho{}_{\overline{\gamma}}+\tfrac{1}{2}(n+2)\varphi_{\alpha\overline{\beta}}-\tfrac{1}{4}h_{\alpha\overline{\beta}}\varphi_{00}+\tfrac{1}{2}h_{\alpha\overline{\beta}}\varphi_\gamma{}^\gamma+O(\rho^3),\\
\mathrm{Ein}_{\alpha\beta}=in\rho^2A_{\alpha\beta}+\rho^2(N_{\gamma\alpha\beta,}{}^\gamma+N_{\gamma\beta\alpha,}{}^\gamma)+\tfrac{1}{2}n\varphi_{\alpha\beta}+O(\rho^3).
\end{gathered}
\tag{3.13}
$$

These identities immediately show Proposition 3.1.

4. Higher-order perturbation

Taking over the setting from the last section, we next introduce a perturbation in g and see what happens to the Einstein tensor. Let $m\geq1$ be a fixed integer and ψ_{ij} a 2-tensor on M with coefficients in $C^\infty(X)$ such that

$$\psi_{00} = O(\rho^{m+2}), \qquad \psi_{0\alpha} = O(\rho^{\max\{\,m+1,3\,\}}),$$
$$\psi_{\alpha\overline{\beta}} = O(\rho^{m+2}), \qquad \psi_{\alpha\beta} = O(\rho^{\max\{\,m,3\,\}}).$$

Let g be a normal-form ACH metric satisfying (3.3) and consider another metric g' with the following components with respect to $\{\,\xi_I\,\} = \{\,\rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}}\,\}$:

$$g'_{ij} = g_{ij} + \psi_{ij}. \tag{4.1}$$

Note that g' also satisfies (3.3). We can read off from Lemma 3.2 the amount to which the Einstein tensor changes, which is denoted by $\delta\,\mathrm{Ein}_{IJ}$. For example we have

$$\begin{aligned}
\delta\,\mathrm{Ein}_{\infty\alpha} &= -\tfrac{i}{2}(\rho\partial_\rho+1)\psi_{0\alpha} + \tfrac{1}{2}\rho^2\partial_\rho\nabla^\beta\psi_{\alpha\beta} + \tfrac{1}{2}\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\rho\partial_\rho\psi_{\overline{\beta}\overline{\gamma}} + O(\rho^{m+2}),\\
\delta\,\mathrm{Ein}_{0\alpha} &= -\tfrac{1}{8}\left((\rho\partial_\rho)^2 - (2n+2)\rho\partial_\rho - 2n - 3\right)\psi_{0\alpha} + O(\rho^{m+2}),\\
\delta\,\mathrm{Ein}_{\alpha\beta} &= -\tfrac{1}{8}\rho\partial_\rho(\rho\partial_\rho - 2n - 2)\psi_{\alpha\beta} + O(\rho^{m+1}).
\end{aligned} \tag{4.2}$$

In the same way we can compute $\delta\,\mathrm{Ein}_{\infty\infty}$, $\delta\,\mathrm{Ein}_{\infty 0}$, $\delta\,\mathrm{Ein}_{00}$ and $\delta\,\mathrm{Ein}_{\alpha\overline{\beta}}$ modulo $O(\rho^{m+2})$. But we want them to be given modulo one order higher. In this section we shall prove the following.

Proposition 4.1. *The components $\delta\,\mathrm{Ein}_{\infty\infty}$, $\delta\,\mathrm{Ein}_{\infty 0}$, $\delta\,\mathrm{Ein}_{00}$, $\delta\,\mathrm{Ein}_{\alpha\overline{\beta}}$ of the difference $\delta\,\mathrm{Ein} = \mathrm{Ein}' - \mathrm{Ein}$ between the Einstein tensors of g and g' are given by, modulo $O(\rho^{m+3})$,*

$$\begin{aligned}
\delta\,\mathrm{Ein}_{\infty\infty} &\equiv -\tfrac{1}{2}\rho\partial_\rho(\rho\partial_\rho - 4)\psi_{00} - \rho\partial_\rho(\rho\partial_\rho - 2)\psi_\alpha{}^\alpha\\
&\quad + \tfrac{1}{2}\rho^2(\rho\partial_\rho)^2(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}),
\end{aligned} \tag{4.3a}$$

$$\delta\,\mathrm{Ein}_{\infty 0} \equiv \tfrac{1}{2}\rho(\rho\partial_\rho + 1)(\nabla^\alpha\psi_{0\alpha} + \nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}}) - \tfrac{1}{2}\rho^3\partial_\rho(A^{\alpha\beta}\psi_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}), \tag{4.3b}$$

$$\begin{aligned}
\delta\,\mathrm{Ein}_{00} &\equiv -\tfrac{1}{8}\left((\rho\partial_\rho)^2 - (2n+4)\rho\partial_\rho - 4n\right)\psi_{00} + \tfrac{1}{2}(\rho\partial_\rho - 2)\psi_\alpha{}^\alpha\\
&\quad + i\rho(\nabla^\alpha\psi_{0\alpha} - \nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}}) - \tfrac{1}{4}\rho^3\partial_\rho(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}),
\end{aligned} \tag{4.3c}$$

$$\begin{aligned}
\delta\,\mathrm{Ein}_\alpha{}^\alpha &\equiv \tfrac{1}{8}n(\rho\partial_\rho - 4)\psi_{00} - \tfrac{1}{8}\left((\rho\partial_\rho)^2 - (4n+2)\rho\partial_\rho - 8\right)\psi_\alpha{}^\alpha\\
&\quad - i\rho(\nabla^\alpha\psi_{0\alpha} - \nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}})\\
&\quad - \tfrac{1}{8}\rho^2\left((n-2)\rho\partial_\rho + (2n+4)\right)(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})\\
&\quad + \tfrac{1}{2}\rho^2(\nabla^\alpha\nabla^\beta\psi_{\alpha\beta} + \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}) + \tfrac{1}{2}\rho^2(N^{\gamma\alpha\beta}{}_{,\gamma}\psi_{\alpha\beta} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}{}_{,\overline{\gamma}}\psi_{\overline{\alpha}\overline{\beta}})\\
&\quad + \tfrac{1}{2}\rho^2(N^{\gamma\alpha\beta}\nabla_\gamma\psi_{\alpha\beta} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}\psi_{\overline{\alpha}\overline{\beta}}),
\end{aligned} \tag{4.3d}$$

$$\begin{aligned}
\mathrm{tf}(\delta\,\mathrm{Ein}_{\alpha\overline{\beta}}) &\equiv -\tfrac{1}{8}\left((\rho\partial_\rho)^2 - (2n+2)\rho\partial_\rho - 8\right)\mathrm{tf}(\psi_{\alpha\overline{\beta}})\\
&\quad + i\rho\,\mathrm{tf}(\nabla_\alpha\psi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\psi_{0\alpha}) + \rho^2\,\mathrm{tf}(\Psi_{\alpha\overline{\beta}}),
\end{aligned} \tag{4.3e}$$

where $\delta\,\mathrm{Ein}_\alpha{}^\alpha$ is the trace of $\delta\,\mathrm{Ein}_{\alpha\overline{\beta}}$ with respect to $h_{\alpha\overline{\beta}}$, tf *denotes the trace-free part, and*

$$\begin{aligned}
\Psi_{\alpha\overline{\beta}} &= \tfrac{1}{4}(\rho\partial_\rho - 2)(\Phi_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + \Phi_{\overline{\beta}}{}^\gamma\psi_{\alpha\gamma}) + \tfrac{1}{2}(\nabla^{\overline{\gamma}}\nabla_\alpha\psi_{\overline{\beta}\overline{\gamma}} + \nabla^\gamma\nabla_{\overline{\beta}}\psi_{\alpha\gamma})\\
&\quad - N_\alpha{}^{\overline{\gamma}\overline{\sigma}}{}_{,\overline{\gamma}}\psi_{\overline{\beta}\overline{\sigma}} - N_{\overline{\beta}}{}^{\gamma\sigma}{}_{,\gamma}\psi_{\alpha\sigma} + N_\alpha{}^{\overline{\gamma}\overline{\sigma}}(\nabla_{\overline{\beta}}\psi_{\overline{\gamma}\overline{\sigma}} - \nabla_{\overline{\sigma}}\psi_{\overline{\beta}\overline{\gamma}}) + N_{\overline{\beta}}{}^{\gamma\sigma}(\nabla_\alpha\psi_{\gamma\sigma} - \nabla_\sigma\psi_{\alpha\gamma}).
\end{aligned}$$

First, let

$$\nabla^{g'}_{\xi_J}\xi_I = \overline{\nabla}_{\xi_J}\xi_I + D'_I{}^K{}_J\xi_K$$

and $D'_{IKJ} := g'_{KL}D'_I{}^L{}_J$. Then $\delta D_{IKJ} = D'_{IKJ} - D_{IKJ}$ is given in Table 4.1, which is seen immediately from Table 3.1.

Type	Value (modulo $O(\rho^{m+3})$)
$\delta D_{\infty\infty\infty}$	0
$\delta D_{\infty 0\infty}$	0
$\delta D_{\infty\alpha\infty}$	0
$\delta D_{\infty\infty 0}$	0
$\delta D_{\infty 00}$	$\frac{1}{2}(\rho\partial_\rho - 4)\psi_{00}$
$\delta D_{\infty\alpha 0}$	$\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\delta D_{\infty\infty\alpha}$	0
$\delta D_{\infty 0\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\delta D_{\infty\overline{\beta}\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\overline{\beta}}$
$\delta D_{\infty\beta\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\beta}$
$\delta D_{0\infty 0}$	$-\frac{1}{2}(\rho\partial_\rho - 4)\psi_{00}$
δD_{000}	0
$\delta D_{0\alpha 0}$	0
$\delta D_{0\infty\alpha}$	$-\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\delta D_{00\alpha}$	0
$\delta D_{0\overline{\beta}\alpha}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{00} + \frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\psi_{0\alpha}) + \frac{1}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\psi_{\alpha\gamma})$
$\delta D_{0\beta\alpha}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\beta} - \nabla_\beta\psi_{0\alpha} - N_{\alpha\beta}{}^{\overline{\gamma}}\psi_{0\overline{\gamma}}) + \frac{1}{2}\rho^2\nabla_0\psi_{\alpha\beta}$
$\delta D_{\alpha\infty\overline{\beta}}$	$-\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\overline{\beta}}$
$\delta D_{\alpha 0\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{00} + \frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}} + \nabla_{\overline{\beta}}\psi_{0\alpha}) - \frac{1}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\psi_{\alpha\gamma})$
$\delta D_{\alpha\overline{\gamma}\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{0\overline{\gamma}} + \frac{i}{2}h_{\alpha\overline{\gamma}}\psi_{0\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha\psi_{\overline{\beta}\overline{\gamma}} - N_{\overline{\beta}\overline{\gamma}}{}^{\sigma}\psi_{\alpha\sigma})$
$\delta D_{\alpha\gamma\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{0\gamma} - \frac{i}{2}h_{\gamma\overline{\beta}}\psi_{0\alpha} + \frac{1}{2}\rho(\nabla_{\overline{\beta}}\psi_{\alpha\gamma} - N_{\alpha\gamma}{}^{\overline{\sigma}}\psi_{\overline{\beta}\overline{\sigma}})$
$\delta D_{\alpha\infty\beta}$	$-\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\beta}$
$\delta D_{\alpha 0\beta}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\beta} + \nabla_\beta\psi_{0\alpha} - N_{\alpha\beta}{}^{\overline{\gamma}}\psi_{0\overline{\gamma}}) - \frac{1}{2}\rho^2\nabla_0\psi_{\alpha\beta}$
$\delta D_{\alpha\overline{\gamma}\beta}$	$\frac{i}{2}h_{\alpha\overline{\gamma}}\psi_{0\beta} + \frac{i}{2}h_{\beta\overline{\gamma}}\psi_{0\alpha} - \frac{1}{2}\rho(\nabla_{\overline{\gamma}}\psi_{\alpha\beta} + N_{\alpha\beta}{}^{\overline{\sigma}}\psi_{\overline{\gamma}\overline{\sigma}})$
$\delta D_{\alpha\gamma\beta}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{\beta\gamma} + \nabla_\beta\psi_{\alpha\gamma} - \nabla_\gamma\psi_{\alpha\beta})$

TABLE 4.1. $\delta D_{IKJ} = D'_{IKJ} - D_{IKJ}$ for a perturbation (4.1) of a normal-form ACH metric

Next we compute $\delta D_I{}^K{}_J := D'_I{}^K{}_J - D_I{}^K{}_J$. To do this we need the knowledge of the following quantities: D_{IKJ} modulo $O(\rho^3)$, g^{IJ} modulo $O(\rho^3)$ and $\delta g^{IJ} := g'^{IJ} - g^{IJ}$ modulo $O(\rho^{m+3})$. They can be read off from Table 3.1, (3.3) and (3.11). Namely, D_{IKJ} mod $O(\rho^3)$ are given by

$$
\begin{array}{llll}
D_{\infty\infty\infty} \equiv -4, & D_{\infty 0\infty} \equiv 0, & D_{\infty\alpha\infty} \equiv 0, & \\
D_{\infty\infty 0} \equiv 0, & D_{\infty 00} \equiv -2, & D_{\infty\alpha 0} \equiv 0, & \\
D_{\infty\infty\alpha} \equiv 0, & D_{\infty 0\alpha} \equiv 0, & D_{\infty\overline{\beta}\alpha} \equiv -h_{\alpha\overline{\beta}}, & D_{\infty\beta\alpha} \equiv 0, \\
D_{0\infty 0} \equiv 2, & D_{000} \equiv 0, & D_{0\alpha 0} \equiv 0, & \\
D_{0\infty\alpha} \equiv 0, & D_{00\alpha} \equiv 0, & D_{0\overline{\beta}\alpha} \equiv \frac{i}{2} h_{\alpha\overline{\beta}}, & D_{0\beta\alpha} \equiv \rho^2 A_{\alpha\beta}, \\
D_{\alpha\infty 0} \equiv 0, & D_{\alpha 00} \equiv 0, & D_{\alpha\overline{\beta}0} \equiv \frac{i}{2} h_{\alpha\overline{\beta}}, & D_{\alpha\beta 0} \equiv 0, \\
D_{\alpha\infty\overline{\beta}} \equiv h_{\alpha\overline{\beta}}, & D_{\alpha 0\overline{\beta}} \equiv \frac{i}{2} h_{\alpha\overline{\beta}}, & D_{\alpha\overline{\gamma}\overline{\beta}} \equiv 0, & D_{\alpha\gamma\overline{\beta}} \equiv 0, \\
D_{\alpha\infty\beta} \equiv 0, & D_{\alpha 0\beta} \equiv -\rho^2 A_{\alpha\beta}, & D_{\alpha\overline{\gamma}\beta} \equiv 0, & D_{\alpha\gamma\beta} \equiv -\rho N_{\alpha\gamma\beta};
\end{array}
$$

g^{IJ} mod $O(\rho^3)$ are

$$
\begin{gathered}
g^{\infty\infty} \equiv \tfrac{1}{4}, \qquad g^{\infty 0} \equiv g^{\infty\alpha} \equiv 0, \qquad g^{00} \equiv 1, \qquad g^{0\alpha} \equiv 0, \\
g^{\alpha\overline{\beta}} \equiv h^{\alpha\overline{\beta}} - \rho^2 \Phi^{\alpha\overline{\beta}}, \qquad g^{\alpha\beta} \equiv -\rho^2 \Phi^{\alpha\beta};
\end{gathered}
$$

δg^{IJ} mod $O(\rho^{m+3})$ are

$$
\begin{aligned}
&\delta g^{\infty\infty} \equiv \delta g^{\infty 0} \equiv g^{\infty\alpha} \equiv 0, \qquad \delta g^{00} \equiv -\psi_{00}, \qquad \delta g^{0\alpha} \equiv -\psi_0{}^\alpha, \\
&\delta g^{\alpha\overline{\beta}} \equiv -\psi^{\alpha\overline{\beta}} + \rho^2 (\Phi^\alpha{}_{\overline{\gamma}} \psi^{\overline{\beta}\overline{\gamma}} + \Phi^{\overline{\beta}}{}_\gamma \psi^{\alpha\gamma}), \\
&\delta g^{\alpha\beta} \equiv -\psi^{\alpha\beta} + \rho^2 (\Phi^\alpha{}_\gamma \psi^{\beta\gamma} + \Phi^\beta{}_\gamma \psi^{\alpha\gamma}).
\end{aligned} \tag{4.4}
$$

Since Table 4.1 and (4.4) shows that δg^{IJ} and δD_{IKJ} are both $O(\rho^{\max\{m,3\}})$, we have $\delta D^{KL} \cdot \delta D_{ILJ} = O(\rho^{m+3})$ and hence

$$
\delta D_I{}^K{}_J \equiv g^{KL} \cdot \delta D_{ILJ} + \delta g^{KL} \cdot D_{ILJ} \mod O(\rho^{m+3}),
$$

where $\delta D_I{}^K{}_J := D'_I{}^K{}_J - D_I{}^K{}_J$. Thus we obtain Table 4.2.

On the other hand, Table 3.2 shows that, modulo $O(\rho^3)$,

$$
\begin{aligned}
&D_\infty{}^\infty{}_\infty \equiv -1, \quad D_\infty{}^0{}_\infty \equiv 0, \quad D_\infty{}^\alpha{}_\infty \equiv 0, \\
&D_\infty{}^\infty{}_0 \equiv 0, \quad D_\infty{}^0{}_0 \equiv -2, \quad D_\infty{}^\alpha{}_0 \equiv 0, \\
&D_\infty{}^\infty{}_\alpha \equiv 0, \quad D_\infty{}^0{}_\alpha \equiv 0, \quad D_\infty{}^\beta{}_\alpha \equiv -\delta_\alpha{}^\beta + \rho^2 \Phi_\alpha{}^\beta, \quad D_\infty{}^{\overline{\beta}}{}_\alpha \equiv \rho^2 \Phi_\alpha{}^{\overline{\beta}}, \\
&D_0{}^\infty{}_0 \equiv \tfrac{1}{2}, \quad D_0{}^0{}_0 \equiv 0, \quad D_0{}^\alpha{}_0 \equiv 0, \\
&D_0{}^\infty{}_\alpha \equiv 0, \quad D_0{}^0{}_\alpha \equiv 0, \quad D_0{}^\beta{}_\alpha \equiv \tfrac{i}{2} \delta_\alpha{}^\beta - \tfrac{i}{2} \rho^2 \Phi_\alpha{}^\beta, \quad D_0{}^{\overline{\beta}}{}_\alpha \equiv -\tfrac{i}{2} \rho^2 \Phi_\alpha{}^{\overline{\beta}} + \rho^2 A_\alpha{}^{\overline{\beta}}, \\
&D_\alpha{}^\infty{}_0 \equiv 0, \quad D_\alpha{}^0{}_0 \equiv 0, \quad D_\alpha{}^\beta{}_0 \equiv \tfrac{i}{2} \delta_\alpha{}^\beta - \tfrac{i}{2} \rho^2 \Phi_\alpha{}^\beta, \quad D_\alpha{}^{\overline{\beta}}{}_0 \equiv -\tfrac{i}{2} \rho^2 \Phi_\alpha{}^{\overline{\beta}}, \\
&D_\alpha{}^\infty{}_{\overline{\beta}} \equiv \tfrac{1}{4} h_{\alpha\overline{\beta}}, \quad D_\alpha{}^0{}_{\overline{\beta}} \equiv \tfrac{i}{2} h_{\alpha\overline{\beta}}, \quad D_\alpha{}^\gamma{}_{\overline{\beta}} \equiv 0, \quad D_\alpha{}^{\overline{\gamma}}{}_{\overline{\beta}} \equiv 0, \\
&D_\alpha{}^\infty{}_\beta \equiv 0, \quad D_\alpha{}^0{}_\beta \equiv -\rho^2 A_{\alpha\beta}, \quad D_\alpha{}^\gamma{}_\beta \equiv 0, \quad D_\alpha{}^{\overline{\gamma}}{}_\beta \equiv -\rho N_\alpha{}^{\overline{\gamma}}{}_\beta.
\end{aligned} \tag{4.5}
$$

Type	Value (modulo $O(\rho^{m+3})$)
$\delta D_{\infty}{}^{\infty}{}_{\infty}$	0
$\delta D_{\infty}{}^{0}{}_{\infty}$	0
$\delta D_{\infty}{}^{\alpha}{}_{\infty}$	0
$\delta D_{\infty}{}^{\infty}{}_{0}$	0
$\delta D_{\infty}{}^{0}{}_{0}$	$\frac{1}{2}\rho\partial_\rho\psi_{00}$
$\delta D_{\infty}{}^{\alpha}{}_{0}$	$\frac{1}{2}(\rho\partial_\rho+1)\psi_0{}^{\alpha}$
$\delta D_{\infty}{}^{\infty}{}_{\alpha}$	0
$\delta D_{\infty}{}^{0}{}_{\alpha}$	$\frac{1}{2}(\rho\partial_\rho-1)\psi_{0\alpha}$
$\delta D_{\infty}{}^{\beta}{}_{\alpha}$	$\frac{1}{2}\rho\partial_\rho\psi_\alpha{}^{\beta}-\frac{1}{2}\rho^3\partial_\rho\Phi^{\beta\gamma}\psi_{\alpha\gamma}-\rho^2\Phi_{\alpha\gamma}\psi^{\beta\gamma}$
$\delta D_{\infty}{}^{\overline{\beta}}{}_{\alpha}$	$\frac{1}{2}\rho\partial_\rho\psi_\alpha{}^{\overline{\beta}}-\frac{1}{2}\rho^3\partial_\rho\Phi^{\gamma\overline{\beta}}\psi_{\alpha\gamma}-\rho^2\Phi_{\alpha\overline{\gamma}}\psi^{\overline{\beta}\overline{\gamma}}$
$\delta D_{0}{}^{\infty}{}_{0}$	$-\frac{1}{8}(\rho\partial_\rho-4)\psi_{00}$
$\delta D_{0}{}^{0}{}_{0}$	0
$\delta D_{0}{}^{\alpha}{}_{0}$	0
$\delta D_{0}{}^{\infty}{}_{\alpha}$	$-\frac{1}{8}(\rho\partial_\rho-3)\psi_{0\alpha}$
$\delta D_{0}{}^{0}{}_{\alpha}$	$-\frac{i}{2}\psi_{0\alpha}$
$\delta D_{0}{}^{\beta}{}_{\alpha}$	$\frac{i}{2}\delta_\alpha{}^{\beta}\psi_{00}-\frac{i}{2}\psi_\alpha{}^{\beta}+\frac{1}{2}\rho(\nabla_\alpha\psi_0{}^{\beta}-\nabla^{\beta}\psi_{0\alpha})$ $-\frac{1}{2}\rho^2(A_{\alpha\gamma}\psi^{\beta\gamma}-A^{\beta\gamma}\psi_{\alpha\gamma})+\frac{i}{2}\rho^2(\Phi^{\beta\gamma}\psi_{\alpha\gamma}+\Phi_{\alpha\gamma}\psi^{\beta\gamma})$
$\delta D_{0}{}^{\overline{\beta}}{}_{\alpha}$	$-\frac{i}{2}\psi_\alpha{}^{\overline{\beta}}+\frac{1}{2}\rho(\nabla_\alpha\psi_0{}^{\overline{\beta}}-\nabla^{\overline{\beta}}\psi_{0\alpha}-N_\alpha{}^{\overline{\beta}\overline{\gamma}}\psi_{0\overline{\gamma}})+\frac{1}{2}\rho^2\nabla_0\psi_\alpha{}^{\overline{\beta}}+\frac{i}{2}\rho^2(\Phi^{\gamma\overline{\beta}}\psi_{\alpha\gamma}+\Phi_{\alpha\overline{\gamma}}\psi^{\overline{\beta}\overline{\gamma}})$
$\delta D_{\alpha}{}^{\infty}{}_{\overline{\beta}}$	$-\frac{1}{8}(\rho\partial_\rho-2)\psi_{\alpha\overline{\beta}}$
$\delta D_{\alpha}{}^{0}{}_{\overline{\beta}}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}}+\nabla_{\overline{\beta}}\psi_{0\alpha})-\frac{1}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}}+A_{\overline{\beta}}{}^{\gamma}\psi_{\alpha\gamma})$
$\delta D_{\alpha}{}^{\gamma}{}_{\overline{\beta}}$	$\frac{i}{2}\delta_\alpha{}^{\gamma}\psi_{0\overline{\beta}}+\frac{1}{2}\rho(\nabla_\alpha\psi_{\overline{\beta}}{}^{\gamma}-N_{\overline{\beta}}{}^{\gamma\sigma}\psi_{\alpha\sigma})$
$\delta D_{\alpha}{}^{\overline{\gamma}}{}_{\overline{\beta}}$	$-\frac{i}{2}\delta_{\overline{\beta}}{}^{\overline{\gamma}}\psi_{0\alpha}+\frac{1}{2}\rho(\nabla_{\overline{\beta}}\psi_\alpha{}^{\overline{\gamma}}-N_\alpha{}^{\overline{\gamma}\overline{\sigma}}\psi_{\overline{\beta}\overline{\sigma}})$
$\delta D_{\alpha}{}^{\infty}{}_{\beta}$	$-\frac{1}{8}(\rho\partial_\rho-2)\psi_{\alpha\beta}$
$\delta D_{\alpha}{}^{0}{}_{\beta}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\beta}+\nabla_\beta\psi_{0\alpha})-\frac{1}{2}\rho(N^{\overline{\gamma}}{}_{\alpha\beta}+N^{\overline{\gamma}}{}_{\beta\alpha})\psi_{0\overline{\gamma}}-\frac{1}{2}\rho^2\nabla_0\psi_{\alpha\beta}$
$\delta D_{\alpha}{}^{\gamma}{}_{\beta}$	$\frac{i}{2}\delta_\alpha{}^{\gamma}\psi_{0\beta}+\frac{i}{2}\delta_\beta{}^{\gamma}\psi_{0\alpha}-\frac{1}{2}\rho\nabla^{\gamma}\psi_{\alpha\beta}-\frac{1}{2}\rho(N^{\overline{\sigma}}{}_{\alpha\beta}+N^{\overline{\sigma}}{}_{\beta\alpha})\psi_{\overline{\sigma}}{}^{\gamma}$
$\delta D_{\alpha}{}^{\overline{\gamma}}{}_{\beta}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_\beta{}^{\overline{\gamma}}+\nabla_\beta\psi_\alpha{}^{\overline{\gamma}}-\nabla^{\overline{\gamma}}\psi_{\alpha\beta})$

TABLE 4.2. $\delta D_I{}^K{}_J = D'_I{}^K{}_J - D_I{}^K{}_J$ for a perturbation (4.1) of a normal-form ACH metric

Using Table 4.2 and (4.5), we compute

$$\overline{\nabla}_K(\delta D_I{}^K{}_J),\quad \overline{\nabla}_J(\delta D_I{}^K{}_K),\quad D_I{}^L{}_K\cdot\delta D_J{}^K{}_L,\quad D_I{}^K{}_L\cdot\delta D_K{}^L{}_L\quad\text{and}\quad D_K{}^L{}_L\cdot\delta D_I{}^K{}_J,$$

all modulo $O(\rho^{m+3})$. The result is Tables 4.3–4.7. From these tables and

$$\begin{aligned}\delta\,\mathrm{Ein}_{IJ} \equiv \tfrac{1}{2}(n+2)\delta g_{IJ} &+ \overline{\nabla}_K(\delta D_I{}^K{}_J) - \overline{\nabla}_J(\delta D_I{}^K{}_K)\\ &- D_I{}^L{}_K\cdot\delta D_J{}^K{}_L - D_J{}^L{}_K\cdot\delta D_I{}^K{}_L\\ &+ D_I{}^L{}_J\cdot\delta D_L{}^K{}_K + D_L{}^K{}_K\cdot\delta D_I{}^L{}_J \mod O(\rho^{m+3}),\end{aligned}$$

we can verify Proposition 4.1.

Type	Value (modulo $O(\rho^{m+3})$)
$\overline{\nabla}_K(\delta D_\infty{}^K{}_\infty)$	0
$\overline{\nabla}_K(\delta D_\infty{}^K{}_0)$	$\frac{1}{2}\rho(\rho\partial_\rho+1)(\nabla^\alpha\psi_{0\alpha}+\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}})$
$\overline{\nabla}_K(\delta D_0{}^K{}_0)$	$-\frac{1}{8}(\rho\partial_\rho-3)(\rho\partial_\rho-4)\psi_{00}$
$\overline{\nabla}_K(\delta D_\alpha{}^K{}_{\overline{\beta}})$	$-\frac{1}{8}(\rho\partial_\rho-1)(\rho\partial_\rho-2)\psi_{\alpha\overline{\beta}}+\frac{i}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}}-\nabla_{\overline{\beta}}\psi_{0\alpha})$ $+\frac{1}{2}\rho^2\nabla^{\overline{\gamma}}(\nabla_\alpha\psi_{\overline{\beta}\overline{\gamma}}-N_{\overline{\beta}\overline{\gamma}}{}^\sigma\psi_{\alpha\sigma})+\frac{1}{2}\rho^2\nabla^\gamma(\nabla_{\overline{\beta}}\psi_{\alpha\gamma}-N_{\alpha\gamma}{}^{\overline{\sigma}}\psi_{\overline{\beta}\overline{\sigma}})$

TABLE 4.3. $\overline{\nabla}_K(\delta D_I{}^K{}_J)$ for a perturbation (4.1) of a normal-form ACH metric

Type	Value (modulo $O(\rho^{m+3})$)
$\overline{\nabla}_\infty(\delta D_\infty{}^K{}_K)$	$\frac{1}{2}\rho\partial_\rho(\rho\partial_\rho-1)\psi_{00}+\rho\partial_\rho(\rho\partial_\rho-1)\psi_\alpha{}^\alpha$ $-\frac{1}{2}\rho^2(\rho\partial_\rho+1)(\rho\partial_\rho+2)(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$\overline{\nabla}_0(\delta D_\infty{}^K{}_K)$	0
$\overline{\nabla}_0(\delta D_0{}^K{}_K)$	0
$\overline{\nabla}_{\overline{\beta}}(\delta D_\alpha{}^K{}_K)$	0

TABLE 4.4. $\overline{\nabla}_J(\delta D_I{}^K{}_K)$ for a perturbation (4.1) of a normal-form ACH metric

Type	Value (modulo $O(\rho^{m+3})$)
$D_\infty{}^L{}_K\cdot\delta D_\infty{}^K{}_L$	$-\rho\partial_\rho\psi_{00}-\rho\partial_\rho\psi_\alpha{}^\alpha+\rho^2(\rho\partial_\rho+1)(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$D_\infty{}^L{}_K\cdot\delta D_0{}^K{}_L$	$-\frac{i}{2}\rho^2(\Phi^{\alpha\beta}\psi_{\alpha\beta}-\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$D_0{}^L{}_K\cdot\delta D_\infty{}^K{}_L$	$\frac{i}{2}\rho^2(\Phi^{\alpha\beta}\psi_{\alpha\beta}-\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})+\frac{1}{2}\rho^2(A^{\alpha\beta}\rho\partial_\rho\psi_{\alpha\beta}+A^{\overline{\alpha}\overline{\beta}}\rho\partial_\rho\psi_{\overline{\alpha}\overline{\beta}})$
$D_0{}^L{}_K\cdot\delta D_0{}^K{}_L$	$\frac{1}{2}(\rho\partial_\rho-n-2)\psi_{00}+\frac{1}{2}\psi_\alpha{}^\alpha-\frac{i}{2}\rho(\nabla^\alpha\psi_{0\alpha}-\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}})$ $-\frac{1}{4}\rho^2(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$D_\alpha{}^L{}_K\cdot\delta D_{\overline{\beta}}{}^K{}_L$	$\frac{1}{4}(\rho\partial_\rho-2)\psi_{\alpha\overline{\beta}}+\frac{1}{4}h_{\alpha\overline{\beta}}\psi_{00}+\frac{i}{2}\rho\nabla_{\overline{\beta}}\psi_{0\alpha}-\frac{1}{4}\rho^2(\rho\partial_\rho-2)\Phi_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}}$ $-\frac{i}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}}+A_{\overline{\beta}}{}^\gamma\psi_{\alpha\gamma})-\frac{1}{2}\rho^2N_\alpha{}^{\overline{\gamma}\overline{\sigma}}(\nabla_{\overline{\beta}}\psi_{\overline{\gamma}\overline{\sigma}}+\nabla_{\overline{\gamma}}\psi_{\overline{\beta}\overline{\sigma}}-\nabla_{\overline{\sigma}}\psi_{\overline{\beta}\overline{\gamma}})$

TABLE 4.5. $D_I{}^L{}_K\cdot\delta D_J{}^K{}_L$ for a perturbation (4.1) of a normal-form ACH metric

Type	Value (modulo $O(\rho^{m+3})$)
$D_\infty{}^K{}_\infty \cdot \delta D_K{}^L{}_L$	$-\frac{1}{2}\rho\partial_\rho\psi_{00} - \rho\partial_\rho\psi_\alpha{}^\alpha + \frac{1}{2}\rho^2(\rho\partial_\rho + 2)(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha\beta}}\psi_{\overline{\alpha\beta}})$
$D_\infty{}^K{}_0 \cdot \delta D_K{}^L{}_L$	0
$D_0{}^K{}_0 \cdot \delta D_K{}^L{}_L$	$\frac{1}{4}\rho\partial_\rho\psi_{00} + \frac{1}{2}\rho\partial_\rho\psi_\alpha{}^\alpha - \frac{1}{4}\rho^2(\rho\partial_\rho + 2)(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha\beta}}\psi_{\overline{\alpha\beta}})$
$D_\alpha{}^K{}_{\overline\beta} \cdot \delta D_K{}^L{}_L$	$\frac{1}{8}h_{\alpha\overline\beta}\rho\partial_\rho\psi_{00} + \frac{1}{4}h_{\alpha\overline\beta}\rho\partial_\rho\psi_\gamma{}^\gamma - \frac{1}{8}\rho^2 h_{\alpha\overline\beta}(\rho\partial_\rho + 2)(\Phi^{\sigma\tau}\psi_{\sigma\tau} + \Phi^{\overline{\sigma\tau}}\psi_{\overline{\sigma\tau}})$

TABLE 4.6. $D_I{}^K{}_L \cdot \delta D_K{}^L{}_L$ for a perturbation (4.1) of a normal-form ACH metric

Type	Value (modulo $O(\rho^{m+3})$)
$D_K{}^L{}_L \cdot \delta D_\infty{}^K{}_\infty$	0
$D_K{}^L{}_L \cdot \delta D_\infty{}^K{}_0$	0
$D_K{}^L{}_L \cdot \delta D_0{}^K{}_0$	$\frac{1}{8}(2n+3)(\rho\partial_\rho - 4)\psi_{00}$
$D_K{}^L{}_L \cdot \delta D_\alpha{}^K{}_{\overline\beta}$	$\frac{1}{8}(2n+3)(\rho\partial_\rho - 2)\psi_{\alpha\overline\beta}$

TABLE 4.7. $D_K{}^L{}_L \cdot \delta D_I{}^K{}_J$ for a perturbation (4.1) of a normal-form ACH metric

Proposition 4.2. *Let g be a normal-form ACH metric satisfying* (3.3) *and g' given by* (4.1)*. Then,*

$$\begin{aligned}\delta\,\mathrm{Ein}_{\infty\infty} &= -\tfrac{1}{2}(m+2)(m-2)\psi_{00} - m(m+2)\psi_\alpha{}^\alpha \\ &\quad + \tfrac{1}{2}m^2\rho^2(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha\beta}}\psi_{\overline{\alpha\beta}}) + O(\rho^{m+3}),\end{aligned} \tag{4.6a}$$

$$\begin{aligned}\delta\,\mathrm{Ein}_{\infty 0} &= \tfrac{1}{2}(m+2)\rho(\nabla^\alpha\psi_{0\alpha} + \nabla^{\overline\alpha}\psi_{0\overline\alpha}) - \tfrac{1}{2}m\rho^2(A^{\alpha\beta}\psi_{\alpha\beta} + A^{\overline{\alpha\beta}}\psi_{\overline{\alpha\beta}}) \\ &\quad + O(\rho^{m+3}),\end{aligned} \tag{4.6b}$$

$$\delta\,\mathrm{Ein}_{\infty\alpha} = -\tfrac{i}{2}(m+2)\psi_{0\alpha} + \tfrac{1}{2}m\rho\nabla^\beta\psi_{\alpha\beta} + \tfrac{1}{2}m\rho N_\alpha{}^{\overline{\beta\gamma}}\psi_{\overline{\beta\gamma}} + O(\rho^{m+2}), \tag{4.6c}$$

$$\begin{aligned}\delta\,\mathrm{Ein}_{00} &= -\tfrac{1}{8}(m^2 - 2nm - 8n - 4)\psi_{00} + \tfrac{1}{2}m\psi_\alpha{}^\alpha + i\rho(\nabla^\alpha\psi_{0\alpha} - \nabla^{\overline\alpha}\psi_{0\overline\alpha}) \\ &\quad - \tfrac{1}{4}\rho^2 m(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha\beta}}\psi_{\overline{\alpha\beta}}) + O(\rho^{m+3}),\end{aligned} \tag{4.6d}$$

$$\delta\,\mathrm{Ein}_{0\alpha} = -\tfrac{1}{8}(m+2)(m-2n-2)\psi_{0\alpha} + O(\rho^{m+2}), \tag{4.6e}$$

$$\begin{aligned}\delta\,\mathrm{Ein}_\alpha{}^\alpha &= \tfrac{1}{8}n(m-2)\psi_{00} - \tfrac{1}{8}\left(m^2 - (4n-2)m - 8n - 8\right)\psi_\alpha{}^\alpha \\ &\quad + (O(\rho^{m+2}) \textit{ terms depending on } \psi_{0\alpha} \textit{ and } \psi_{\alpha\beta}) + O(\rho^{m+3}),\end{aligned} \tag{4.6f}$$

$$\begin{aligned}\mathrm{tf}(\delta\,\mathrm{Ein}_{\alpha\overline\beta}) &= -\tfrac{1}{8}(m^2 - 2nm - 2n - 9)\,\mathrm{tf}(\psi_{\alpha\overline\beta}) \\ &\quad + (O(\rho^{m+2}) \textit{ terms depending on } \psi_{0\alpha} \textit{ and } \psi_{\alpha\beta}) + O(\rho^{m+3}),\end{aligned} \tag{4.6g}$$

$$\delta\,\mathrm{Ein}_{\alpha\beta} = -\tfrac{1}{8}m(m-2n-2)\psi_{\alpha\beta} + O(\rho^{m+1}). \tag{4.6h}$$

Proof. This follows from (4.2), (4.3) and the fact that the Euler vector field $\rho\partial_\rho$ acts on an $O(\rho^m)$ function as, modulo $O(\rho^{m+1})$, a scalar multiplication by m. $\square$

5. Approximate solution and obstruction tensor

By using the results in §3 and §4, in this section we construct a normal-form ACH metric whose Einstein tensor vanishes to as high order as possible. First we observe the contracted Bianchi identity satisfied by the Einstein tensor.

Lemma 5.1. *Let $m \ge 1$ be a positive integer. Suppose that g is a normal-form ACH metric satisfying*

$$\begin{aligned} \operatorname{Ein}_{\infty\infty} &= O(\rho^{m+2}), & \operatorname{Ein}_{\infty 0} &= O(\rho^{m+2}), & \operatorname{Ein}_{\infty\alpha} &= O(\rho^{\max\{m+1,3\}}),\\ \operatorname{Ein}_{00} &= O(\rho^{m+2}), & \operatorname{Ein}_{0\alpha} &= O(\rho^{\max\{m+1,3\}}), & &\\ \operatorname{Ein}_{\alpha\overline{\beta}} &= O(\rho^{m+2}), & \operatorname{Ein}_{\alpha\beta} &= O(\rho^{\max\{m,3\}}). & & \end{aligned}$$

Then we have

$$\begin{aligned} O(\rho^{m+3}) &= (m-4n-2)\operatorname{Ein}_{\infty\infty} - 4(m-2)\operatorname{Ein}_{00} - 8m \operatorname{Ein}_{\alpha}{}^{\alpha} \\ &\quad + 8\rho(\nabla^{\alpha}\operatorname{Ein}_{\infty\alpha} + \nabla^{\overline{\alpha}}\operatorname{Ein}_{\infty\overline{\alpha}}) + 4\rho^2(m-2)(\Phi^{\alpha\beta}\operatorname{Ein}_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\operatorname{Ein}_{\overline{\alpha}\overline{\beta}}), \end{aligned} \tag{5.1a}$$

$$\begin{aligned} O(\rho^{m+3}) &= (m-2n-2)\operatorname{Ein}_{\infty 0} + 4\rho(\nabla^{\alpha}\operatorname{Ein}_{0\alpha} + \nabla^{\overline{\alpha}}\operatorname{Ein}_{0\overline{\alpha}}) \\ &\quad + 4\rho^2(A^{\alpha\beta}\operatorname{Ein}_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\operatorname{Ein}_{\overline{\alpha}\overline{\beta}}), \end{aligned} \tag{5.1b}$$

$$O(\rho^{m+2}) = 2(m-2n-2)\operatorname{Ein}_{\infty\alpha} + 4\rho\nabla^{\beta}\operatorname{Ein}_{\alpha\beta} - 4i\operatorname{Ein}_{0\alpha} + 4\rho N_{\alpha}{}^{\overline{\beta}\overline{\gamma}}\operatorname{Ein}_{\overline{\beta}\overline{\gamma}}. \tag{5.1c}$$

Proof. We have the contracted Bianchi identity $g^{IJ}\nabla^g{}_K \operatorname{Ric}_{IJ} = 2g^{IJ}\nabla^g{}_I \operatorname{Ric}_{JK}$, where ∇^g is the Levi-Civita connection determined by g. Since ∇^g is a metric connection we further have

$$g^{IJ}\nabla^g{}_K \operatorname{Ein}_{IJ} = 2g^{IJ}\nabla^g{}_I \operatorname{Ein}_{JK}.$$

In terms of the extended Tanaka–Webster connection $\overline{\nabla}$ and the tensor D, we can rewrite this identity as

$$g^{IJ}(\overline{\nabla}_K \operatorname{Ein}_{IJ} - 2D_I{}^L{}_K \operatorname{Ein}_{JL}) = 2g^{IJ}(\overline{\nabla}_I \operatorname{Ein}_{JK} - D_J{}^L{}_I \operatorname{Ein}_{LK} - D_K{}^L{}_I \operatorname{Ein}_{JL}),$$

or equivalently,

$$0 = g^{IJ}(\overline{\nabla}_K \operatorname{Ein}_{IJ} - 2\overline{\nabla}_I \operatorname{Ein}_{JK} + 2D_I{}^L{}_J \operatorname{Ein}_{KL} - 2\overline{\Theta}_{IK}{}^L \operatorname{Ein}_{JL}),$$

where $\overline{\Theta}$ is the torsion form of $\overline{\nabla}$. Since $g^{0\alpha} = O(\rho^3)$ and $\operatorname{Ein}_{IJ} = O(\rho^m)$, we obtain

$$\begin{aligned} O(\rho^{m+3}) &= g^{\infty\infty}(\overline{\nabla}_K \operatorname{Ein}_{\infty\infty} - 2\overline{\nabla}_\infty \operatorname{Ein}_{\infty K} + 2D_\infty{}^L{}_\infty \operatorname{Ein}_{KL} - 2\overline{\Theta}_{\infty K}{}^L \operatorname{Ein}_{\infty L}) \\ &\quad + g^{00}(\overline{\nabla}_K \operatorname{Ein}_{00} - 2\overline{\nabla}_0 \operatorname{Ein}_{0K} + 2D_0{}^L{}_0 \operatorname{Ein}_{KL} - 2\overline{\Theta}_{0K}{}^L \operatorname{Ein}_{0L}) \\ &\quad + 2g^{\beta\overline{\gamma}}(\overline{\nabla}_K \operatorname{Ein}_{\beta\overline{\gamma}} - \overline{\nabla}_\beta \operatorname{Ein}_{\overline{\gamma}K} - \overline{\nabla}_{\overline{\gamma}} \operatorname{Ein}_{\beta K} + (D_\beta{}^L{}_{\overline{\gamma}} + D_{\overline{\gamma}}{}^L{}_\beta)\operatorname{Ein}_{KL} \\ &\qquad\qquad - \overline{\Theta}_{\beta K}{}^L \operatorname{Ein}_{\overline{\gamma}L} - \overline{\Theta}_{\overline{\gamma}K}{}^L \operatorname{Ein}_{\beta L}) \\ &\quad + g^{\beta\gamma}(\overline{\nabla}_K \operatorname{Ein}_{\beta\gamma} - 2\overline{\nabla}_\beta \operatorname{Ein}_{\gamma K} + 2D_\beta{}^L{}_\gamma \operatorname{Ein}_{KL} - 2\overline{\Theta}_{\beta K}{}^L \operatorname{Ein}_{\gamma L}) \\ &\quad + g^{\overline{\beta}\overline{\gamma}}(\overline{\nabla}_K \operatorname{Ein}_{\overline{\beta}\overline{\gamma}} - 2\overline{\nabla}_{\overline{\beta}} \operatorname{Ein}_{\overline{\gamma}K} + 2D_{\overline{\beta}}{}^L{}_{\overline{\gamma}} \operatorname{Ein}_{KL} - 2\overline{\Theta}_{\overline{\beta}K}{}^L \operatorname{Ein}_{\overline{\gamma}L}). \end{aligned}$$

Substituting $K=\infty$, $K=0$ and $K=\alpha$ into this formula, in view of (3.7), (3.9) and (4.5) we find that

$$\begin{aligned} O(\rho^{m+3}) &= (\rho\partial_\rho - 4n - 4)\operatorname{Ein}_{\infty\infty} - 4(\rho\partial_\rho - 4)\operatorname{Ein}_{00} - 8(\rho\partial_\rho - 2)\operatorname{Ein}_\alpha{}^\alpha \\ &\quad + 8\rho(\nabla^\alpha \operatorname{Ein}_{\infty\alpha} + \nabla^{\overline{\alpha}} \operatorname{Ein}_{\infty\overline{\alpha}}) \\ &\quad + 4\rho^2(\rho\partial_\rho - 2)(\Phi^{\alpha\beta}\operatorname{Ein}_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\operatorname{Ein}_{\overline{\alpha}\overline{\beta}}), \end{aligned} \tag{5.2a}$$

$$\begin{aligned} O(\rho^{m+3}) &= (\rho\partial_\rho - 2n - 4)\operatorname{Ein}_{\infty0} + 4\rho(\nabla^\alpha\operatorname{Ein}_{0\alpha} + \nabla^{\overline{\alpha}}\operatorname{Ein}_{0\overline{\alpha}}) \\ &\quad + 4\rho^2(A^{\alpha\beta}\operatorname{Ein}_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\operatorname{Ein}_{\overline{\alpha}\overline{\beta}}), \end{aligned} \tag{5.2b}$$

$$O(\rho^{m+2}) = 2(\rho\partial_\rho - 2n - 3)\operatorname{Ein}_{\infty\alpha} + 4\rho\nabla^\beta\operatorname{Ein}_{\alpha\beta} - 4i\operatorname{Ein}_{0\alpha} + 4\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\operatorname{Ein}_{\overline{\beta}\overline{\gamma}}, \tag{5.2c}$$

which imply (5.1). □

Let

$$a(I,J) = \begin{cases} 3, & (I,J) = (\infty,\infty),\ (\infty,0),\ (0,0),\ (\alpha,\overline{\beta}), \\ 2, & (I,J) = (\infty,\alpha),\ (0,\alpha), \\ 1, & (I,J) = (\alpha,\beta). \end{cases}$$

The next theorem proves Theorem 0.1.

Theorem 5.2. *Let $(M, T^{1,0})$ be a nondegenerate partially integrable almost CR manifold, θ any pseudohermitian structure and X an open neighborhood of $M = M\times\{0\}$ in $M\times[0,\infty)$. Then there exists a normal-form ACH metric g on X which satisfies*

$$\operatorname{Ein}_{IJ} = O(\rho^{2n+1+a(I,J)}) \tag{5.3}$$

with respect to the frame (3.1) *of ${}^\Theta TX$. For such a metric, each g_{ij} is uniquely determined modulo $O(\rho^{2n+1+a(i,j)})$.*

Proof. By Proposition 3.1 we already have a normal-form ACH metric $g^{(0)}$ satisfying $\operatorname{Ein}_{IJ} = O(\rho^3)$ for every I, J, with $O(\rho^3)$ ambiguity in each component $g^{(0)}_{ij}$. We shall inductively show that there exists a normal-form ACH metric $g^{(m)}$ satisfying

$$\operatorname{Ein}_{IJ} = O(\rho^{\max\{\,m+a(I,J),3\,\}}). \tag{5.4}$$

for each m, $m = 1, \ldots, 2n+1$, and for such $g^{(m)}$ its components $g^{(m)}_{ij}$ are unique modulo $O(\rho^{\max\{\,m+a(i,j),3\,\}})$.

Suppose we have a normal-form ACH metric $g^{(m-1)}$ that satisfies (5.4) for $m-1$ as well as (3.3). Consider a new ACH metric $g^{(m)}$ given by $g^{(m)}_{ij} = g^{(m-1)}_{ij} + \psi_{ij}$, where ψ_{ij} is such that $\psi_{ij} = O(\rho^{\max\{\,m-1+a(i,j),3\,\}})$. Then the difference $\delta\operatorname{Ein} = \operatorname{Ein}' - \operatorname{Ein}$ between the Einstein tensors is given in Proposition 4.2. In view of (4.6e) and (4.6h) we can determine $\psi_{0\alpha}$ mod $O(\rho^{m+2})$ and $\psi_{\alpha\beta}$ mod $O(\rho^{\max\{\,m+1,3\,\}})$ so that $\operatorname{Ein}^{(m)}_{0\alpha} = O(\rho^{m+2})$ and $\operatorname{Ein}^{(m)}_{\alpha\beta} = O(\rho^{\max\{\,m+1,3\,\}})$ hold, because the exponents $-\frac{1}{8}(m+2)(m-2n-2)$ and $-\frac{1}{8}m(m-2n-2)$ are nonzero for $m = 1, \ldots, 2n+1$. After that, by a similar reasoning using (4.6g), we can determine $\operatorname{tf}(\psi_{\alpha\overline{\beta}})$ mod $O(\rho^{m+3})$ so that $\operatorname{tf}(\operatorname{Ein}^{(m)}_{\alpha\overline{\beta}}) = O(\rho^{m+3})$ hold. Next we see (4.6d) and (4.6f) as a system of linear equations for ψ_{00} and $\psi_\alpha{}^\alpha$. The determinant of the coefficients is

$$\begin{vmatrix} -\frac{1}{8}(m^2-2nm-8n-4) & \frac{1}{2}m \\ \frac{1}{8}n(m-2) & -\frac{1}{8}\left(m^2-(4n-2)m-8n-8\right) \end{vmatrix} = \frac{1}{64}(m+2)(m+4)(m-2n-2)(m-4n-2), \tag{5.5}$$

which shows that this system is nondegenerate for $m = 1, \ldots, 2n+1$. Hence we can determine ψ_{00} and $\psi_\alpha{}^\alpha$, both modulo $O(\rho^{m+3})$, so that $\mathrm{Ein}^{(m)}_{00} = O(\rho^{m+3})$ and $\mathrm{Ein}^{(m)}{}_\alpha{}^\alpha = O(\rho^{m+3})$ hold. Thus we have attained $\mathrm{Ein}_{ij} = O(\rho^{\max\{\, m+a(i,j),3\,\}})$, and if $g^{(m-1)}_{ij}$ are unique up to $O(\rho^{\max\{\, m-1+a(i,j),3\,\}})$, the desired uniqueness result holds for $g^{(m)}_{ij}$.

Finally we check that $g^{(m)}$ is determined in such a way that it satisfies (5.4) for $I=\infty$, too. This is done by using Lemma 5.1. In fact, for $g^{(m)}$, $\mathrm{Ein}^{(m)}_{\infty 0} = O(\rho^{m+3})$ and $\mathrm{Ein}^{(m)}_{\infty\alpha} = O(\rho^{m+2})$ should hold, because in (5.1b) and (5.1c) the terms on the right-hand sides are, except the first terms in each identity, already $O(\rho^{m+3})$ and $O(\rho^{m+2})$, respectively, and the coefficients of the first terms are both nonzero. Similarly (5.1a) shows that $\mathrm{Ein}^{(m)}_{\infty\infty} = O(\rho^{m+3})$. Hence the induction is complete. □

In spite of the success of the inductive determination of g_{ij} up to the stage in the theorem above, the next step cannot be executed, as (4.6e) and (4.6h) indicate; the freedom of the choice of g satisfying (5.3) does not affect the ρ^{2n+2}-term coefficient of $\mathrm{Ein}_{\alpha\beta}$ and the ρ^{2n+3}-term coefficient of $\mathrm{Ein}_{0\alpha}$. So we define

$$\mathcal{O}_{\alpha\beta} := \left(\rho^{-2n-2}\,\mathrm{Ein}_{\alpha\beta}\right)\big|_{\rho=0} \tag{5.6}$$

and call it the *obstruction tensor* associated with $(M, T^{1,0}, \theta)$. In fact, the condition $\mathrm{Ein}_{\alpha\overline{\beta}} = O(\rho^{2n+4})$ on the metric from which $\mathcal{O}_{\alpha\beta}$ is computed can be weakened to $\mathrm{Ein}_{\alpha\overline{\beta}} = O(\rho^{2n+3})$, for the $O(\rho^{2n+3})$ ambiguity in $\mathrm{tf}(g_{\alpha\overline{\beta}})$ emerging from that does not have any effect on ρ^{2n+2}-term coefficient of $\mathrm{Ein}_{\alpha\beta}$ as (4.6h) shows. This fact further implies that we can use any approximately Einstein ACH metric g that Theorem 0.1 claims its existence, because if ρ is a model boundary defining function for g and θ, then there is a boundary-fixing $[\Theta]$-preserving diffeomorphism Φ such that Φ^*g is a normal-form ACH metric for which the second coordinate function is equal to $\Phi^*\rho$, and its Einstein tensor vanishes to the same order as that of g does.

The ρ^{2n+3}-term coefficient of $\mathrm{Ein}_{0\alpha}$ is not a new obstruction, since by (5.1c) we have

$$\left(\rho^{-2n-3}\,\mathrm{Ein}_{0\alpha}\right)\big|_M = -i\nabla^\beta\mathcal{O}_{\alpha\beta} - iN_\alpha{}^{\overline{\beta}\overline{\gamma}}\mathcal{O}_{\overline{\beta}\overline{\gamma}}. \tag{5.7}$$

Proposition 5.3. *Let θ and $\hat{\theta} = e^{2u}\theta$, $u \in C^\infty(M)$, be two pseudohermitian structures on M. Then*

$$\hat{\mathcal{O}}_{\alpha\beta} = e^{-2nu}\mathcal{O}_{\alpha\beta}, \tag{5.8}$$

where $\mathcal{O}_{\alpha\beta}$ is the obstruction tensor for $(M, T^{1,0}, \theta)$ and $\hat{\mathcal{O}}_{\alpha\beta}$ is that for $(M, T^{1,0}, \hat{\theta})$.

Proof. Let $(X, [\Theta])$ be a manifold-with-boundary with a conformal Θ-structure such that $\partial X = M$ and $\iota^*[\Theta]$ is the conformal class of the pseudohermitian structures on M, and take any ACH metric g satisfying the condition in Theorem 0.1. If ρ is a model boundary defining function for θ and $\hat{\rho} = e^\psi\rho$, $\psi \in C^\infty(X)$, is one for $\hat{\theta}$, then we have $\psi|_M = u$ by the condition $\iota^*(\hat\rho^4 g) = \hat\theta^2$. Hence, if $\{ Z_\alpha \}$ is any extension of a local frame of $T^{1,0}$, we have $\hat{\mathcal{O}}_{\alpha\beta} = \left(\hat\rho^{-2n-2}\operatorname{Ein}(\hat\rho Z_\alpha, \hat\rho Z_\beta)\right)\big|_M = e^{-2nu}\left(\rho^{-2n-2}\operatorname{Ein}(\rho Z_\alpha, \rho Z_\beta)\right)\big|_M = e^{-2nu}\mathcal{O}_{\alpha\beta}$. □

The proposition above implies that the density-weighted version of the obstruction tensor

$$\boldsymbol{\mathcal{O}}_{\alpha\beta} := \mathcal{O}_{\alpha\beta} \otimes |\zeta|^{2n/(n+2)} \in \mathcal{E}_{(\alpha\beta)}(-n,-n)$$

is a CR-invariant tensor, where $\mathcal{E}_{(\alpha\beta)}$ denotes the space of local sections of $\operatorname{Sym}^2(T^{1,0})^*$.

Next we recall (5.7). Let us also look at a similar result

$$\left(\rho^{-2n-3}(\nabla^\alpha \operatorname{Ein}_{0\alpha} + \nabla^{\overline{\alpha}} \operatorname{Ein}_{0\overline{\alpha}})\right)\big|_M = -A^{\alpha\beta}\mathcal{O}_{\alpha\beta} - A^{\overline{\alpha}\overline{\beta}}\mathcal{O}_{\overline{\alpha}\overline{\beta}},$$

which follows from (5.1b). Combining these identities we obtain

$$P^{\alpha\beta}\mathcal{O}_{\alpha\beta} - P^{\overline{\alpha}\overline{\beta}}\mathcal{O}_{\overline{\alpha}\overline{\beta}} = 0, \tag{5.9}$$

where

$$P^{\alpha\beta} = \nabla^\alpha\nabla^\beta - iA^{\alpha\beta} - N^{\gamma\alpha\beta}\nabla_\gamma - N^{\gamma\alpha\beta}{}_{,\gamma}. \tag{5.10}$$

Replacing N, A with $\boldsymbol{N}$, $\boldsymbol{A}$ and taking contractions with respect not to h but to $\boldsymbol{h}$, we may also interpret $P^{\alpha\beta}$ as a differential operator $\mathcal{E}_{(\alpha\beta)}(-n,-n) \to \mathcal{E}(-n-2,-n-2)$ between density-weighted bundles. Then we have $P^{\alpha\beta}\boldsymbol{\mathcal{O}}_{\alpha\beta} - P^{\overline{\alpha}\overline{\beta}}\boldsymbol{\mathcal{O}}_{\overline{\alpha}\overline{\beta}} = 0$. Furthermore, in this setting, the operator $P^{\alpha\beta}$ belongs to a one-parameter family of CR-invariant differential operators, as we shall describe in the following proposition.

Proposition 5.4. *Let $(M, T^{1,0})$ be a nondegenerate partially integrable almost CR manifold. Let*

$$P_t^{\alpha\beta} \colon \mathcal{E}_{(\alpha\beta)}(-n,-n) \to \mathcal{E}(-n-2,-n-2), \quad t \in \mathbb{C}$$

be a one-parameter family of differential operators defined by, in terms of any pseudohermitian structure θ,

$$P_t^{\alpha\beta} = \nabla^\alpha\nabla^\beta - i\boldsymbol{A}^{\alpha\beta} - (1+tn)\boldsymbol{N}^{\gamma\alpha\beta}\nabla_\gamma - (1+t(n+1))\,\boldsymbol{N}^{\gamma\alpha\beta}{}_{,\gamma}. \tag{5.11}$$

Then this is well-defined, i.e., the right-hand side of (5.11) is independent of θ.

Proof. This can be checked by using equation (2.7) and Proposition 2.3 of [GoGr], as we have discussed in §2. The details are left to the reader. □

The next proposition finishes the proof of Proposition 0.2.

Proposition 5.5. *The obstruction tensor $\mathcal{O}_{\alpha\beta}$ for a nondegenerate (integrable) CR manifold vanishes.*

Proof. Since $\mathcal{O}_{\alpha\beta}$ is a certain polynomial of derivatives of pseudohermitian torsion and curvature, using the formal embedding we can reduce the problem to the case of a (small piece of) nondegenerate real hypersurface $M \subset \mathbb{C}^{n+1}$. In this proof we use indices j, k for components with respect to the complex coordinates $(z^1, \dots, z^{n+1})$.

Let r be Fefferman's approximate solution of the complex Monge–Ampère equation [Fe], i.e., a smooth defining function of M such that $J(r) = 1 + O(r^{n+2})$, where

$$J(r) := (-1)^{n+1}\det\begin{pmatrix} r & \partial r/\partial\overline{z}^k \\ \partial r/\partial z^j & \partial^2 r/\partial z^j\partial\overline{z}^k \end{pmatrix}.$$

We set $\tilde{\theta} := \frac{i}{2}(\partial r - \overline{\partial} r)$ and $\theta := \iota^* \tilde{\theta}$, where $\iota \colon M \hookrightarrow \mathbb{C}^{n+1}$ is the inclusion. Let g_0 be the Kähler metric on $\Omega = \{ r > 0 \}$ associated with Kähler form $i\,\partial\overline{\partial}(\log(1/r))$ as in Example 1.1. Then it is easily verified that $\det((g_0)_{j\overline{k}}) = r^{-(n+2)} J(r)$, and the usual formula for the Ricci tensor of Kähler metric shows that

$$\operatorname{Ric}(g_0)_{j\overline{k}} = -\tfrac{1}{2}(n+2)(g_0)_{j\overline{k}} + \frac{\partial^2}{\partial z^j \partial \overline{z}^k} \log J(r).$$

Observe that, if we set $\log J(r) = r^{n+2} f$,

$$\begin{aligned}\partial\overline{\partial} \log J(r) &= (n+2)(n+1) r^n f \partial r \wedge \overline{\partial} r + (n+2) r^{n+1} (f \partial\overline{\partial} r + \partial f \wedge \overline{\partial} r + \partial r \wedge \overline{\partial} f) \\ &\quad + r^{n+2} \partial\overline{\partial} f.\end{aligned} \tag{5.12}$$

Let ξ be the unique $(1,0)$ vector field satisfying

$$\xi \rfloor \partial\overline{\partial} r = 0 \mod \overline{\partial} r, \qquad \partial r(\xi) = 1$$

and $N := \operatorname{Re} \xi$, $\tilde{T} := 2 \operatorname{Im} \xi$. We set $\xi \rfloor \partial\overline{\partial} r = \tau \overline{\partial} r$, or $\tau = \partial\overline{\partial} r(\xi, \overline{\xi})$. Then, since τ is a real-valued function, $\tilde{T} \rfloor \partial\overline{\partial} r = -i(\xi - \overline{\xi}) \rfloor \partial\overline{\partial} r = -i(\tau \overline{\partial} r + \tau \partial r) = -i\tau dr$. Therefore $T \rfloor d\theta = T \rfloor \iota^*(-i\partial\overline{\partial} r) = \iota^*(-\tau dr) = 0$, where T is the restriction of $\tilde{T}$ to M. This shows that T is the Reeb vector field on M associated with θ.

Let $\xi_1, \ldots, \xi_n$ be $(1,0)$ vector fields spanning $\ker \partial r \subset T^{1,0}\mathbb{C}^{n+1}$ near M. Restricted on M, they form a local frame $\{ Z_1, \ldots, Z_n \}$ of $T^{1,0}M$.

We identify a (one-sided) neighborhood of M in $\overline{\Omega}$ with $M \times [0, \epsilon)$ by

$$M \times [0, \epsilon) \to \overline{\Omega}, \qquad (p, s) \mapsto \operatorname{Fl}_s(p),$$

where Fl_s is the flow generated by N. In view of the fact that s is equal to the pullback of r, we write r instead of s in the sequel. The constant extensions of T and Z_α in the r-direction are also denoted by the same symbols. Then obviously $T = \tilde{T} + O(r)$, $Z_\alpha = \xi_\alpha + O(r)$. By (5.12) we have

$$\begin{aligned}
\operatorname{Ein}(g_0)(N, N) &= \operatorname{Ein}(g_0)\left(\tfrac{1}{2}(\xi + \overline{\xi}), \tfrac{1}{2}(\xi + \overline{\xi})\right) = \tfrac{1}{2} \operatorname{Ein}(g_0)(\xi, \overline{\xi}) = O(r^n), \\
\operatorname{Ein}(g_0)(\tilde{T}, \tilde{T}) &= \operatorname{Ein}(g_0)(-i(\xi - \overline{\xi}), -i(\xi - \overline{\xi})) = 2 \operatorname{Ein}(g_0)(\xi, \overline{\xi}) = O(r^n), \\
\operatorname{Ein}(g_0)(N, \tilde{T}) &= 0, \qquad \operatorname{Ein}(g_0)(N, \xi_\alpha) = O(r^{n+1}), \qquad \operatorname{Ein}(g_0)(\tilde{T}, \xi_\alpha) = O(r^{n+1}), \\
\operatorname{Ein}(g_0)(\xi_\alpha, \xi_{\overline{\beta}}) &= O(r^{n+1}), \qquad \operatorname{Ein}(g_0)(\xi_\alpha, \xi_\beta) = 0.
\end{aligned}$$

Hence, with respect to the local frame $\{ \partial_r = N, T, Z_\alpha, Z_{\overline{\alpha}} \}$ of $T_{\mathbb{C}}(M \times [0, \epsilon))$, we have

$$\begin{aligned}
&\operatorname{Ein}(g_0)_{\infty\infty} = O(r^n), \qquad \operatorname{Ein}(g_0)_{\infty 0} = O(r^{n+1}), \qquad \operatorname{Ein}(g_0)_{\infty\alpha} = O(r^{n+1}), \\
&\operatorname{Ein}(g_0)_{00} = O(r^n), \qquad \operatorname{Ein}(g_0)_{0\alpha} = O(r^{n+1}), \\
&\operatorname{Ein}(g_0)_{\alpha\overline{\beta}} = O(r^{n+1}), \qquad \operatorname{Ein}(g_0)_{\alpha\beta} = O(r^{n+2}).
\end{aligned}$$

Therefore the Einstein tensor of the induced ACH metric g on the square root of $M \times [0, \epsilon)$ in the sense of [EMM] satisfies, with respect to the frame $\{ \rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}} \}$,

$$\begin{aligned}
&\operatorname{Ein}_{\infty\infty} = O(\rho^{2n+4}), \qquad \operatorname{Ein}_{\infty 0} = O(\rho^{2n+6}), \qquad \operatorname{Ein}_{\infty\alpha} = O(\rho^{2n+5}), \\
&\operatorname{Ein}_{00} = O(\rho^{2n+4}), \qquad \operatorname{Ein}_{0\alpha} = O(\rho^{2n+5}), \\
&\operatorname{Ein}_{\alpha\overline{\beta}} = O(\rho^{2n+4}), \qquad \operatorname{Ein}_{\alpha\beta} = O(\rho^{2n+6}).
\end{aligned}$$

Hence g satisfies the condition (5.3). Moreover, since $\mathrm{Ein}_{\alpha\beta} = O(\rho^{2n+3})$, it follows that $\mathcal{O}_{\alpha\beta} = 0$. $\square$

6. On first variation

In this section, we calculate the first-order term of the obstruction tensor with respect to a variation from the standard CR sphere. First we introduce a tensor that describes a modification of partially integrable almost CR structures.

Proposition 6.1. *Let $(M, T^{1,0})$ be a nondegenerate partially integrable almost CR manifold and $\{ Z_\alpha \}$ a local frame of the bundle $T^{1,0}$. Let $\mu_\alpha{}^{\overline{\beta}} \in \mathcal{E}_\alpha{}^{\overline{\beta}}$ and set*

$$\hat{Z}_\alpha := Z_\alpha + \mu_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}};$$

$\{ \hat{Z}_\alpha \}$ defines a new almost CR structure on M without changing the contact distribution H. Then this is partially integrable if and only if

$$\mu_{\alpha\beta} = \mu_{\beta\alpha},$$

where the upper index is lowered by the Levi form of $(M, T^{1,0})$ associated to any pseudohermitian structure.

Proof. The new almost CR structure is partially integrable if and only if

$$\theta([\hat{Z}_\alpha, \hat{Z}_\beta]) = \theta([Z_\alpha + \mu_\alpha{}^{\overline{\sigma}} Z_{\overline{\sigma}}, Z_\beta + \mu_\beta{}^{\overline{\tau}} Z_{\overline{\tau}}]) = 0,$$

where θ is any pseudohermitian structure for $(M, T^{1,0})$. Since $\theta([Z_\alpha, Z_\beta]) = \theta([Z_{\overline{\sigma}}, Z_{\overline{\tau}}]) = 0$, this is equivalent to

$$\theta([Z_{\overline{\sigma}}, Z_\beta])\mu_\alpha{}^{\overline{\sigma}} + \theta([Z_\alpha, Z_{\overline{\tau}}])\mu_\beta{}^{\overline{\tau}} = 0,$$

or $\mu_{\alpha\beta} - \mu_{\beta\alpha} = 0$. $\square$

Let $M = S^{2n+1}$ be the $(2n+1)$-dimensional sphere and θ the standard contact form. Then the obstruction tensor $\mathcal{O}_{\alpha\beta}$ with respect to θ is a function of partially integrable almost CR structures on $\ker\theta$. For the standard CR structure we have $\mathcal{O}_{\alpha\beta} = 0$. We shall compute the derivative of $\mathcal{O}_{\alpha\beta}$ at the standard CR structure in the direction of $\mu_{\alpha\beta}$, where the second index of $\mu_{\alpha\beta}$ is understood to be lowered by the Levi form of the standard CR sphere associated to θ. In this section, the differentials of various quantities will be indicated by the bullet $\bullet$.

Proposition 6.2. *Consider $h_{\alpha\overline{\beta}}$, $N_{\alpha\beta\gamma}$, $A_{\alpha\beta}$ and $R_{\alpha\overline{\beta}}$ associated to the standard contact form θ on the sphere. Then, their differentials at the standard CR structure are as follows:*

$$h^\bullet_{\alpha\overline{\beta}} = 0, \qquad N^\bullet_{\alpha\beta\gamma} = \nabla_\alpha \mu_{\beta\gamma} - \nabla_\beta \mu_{\alpha\gamma},$$
$$A^\bullet_{\alpha\beta} = -\nabla_0 \mu_{\alpha\beta}, \qquad R^\bullet_{\alpha\overline{\beta}} = -\nabla_\alpha \nabla^{\overline{\sigma}} \mu_{\overline{\beta}\overline{\sigma}} - \nabla_{\overline{\beta}} \nabla^\tau \mu_{\alpha\tau}.$$

Proof. Since the both sides of all four equalities are tensorial, we may take any frame to derive them. Let $\{ Z_\alpha \}$ be a local frame of $T^{1,0}$ of the standard CR sphere such that

$$[Z_\alpha, Z_{\overline{\beta}}] = -i h_{\alpha\overline{\beta}} T, \qquad [Z_\alpha, Z_\beta] = [Z_\alpha, T] = 0$$

and

$$h_{\alpha\overline{\beta}} = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise,} \end{cases}$$

where T is the Reeb vector field associated with θ. Then the differentials of the Lie brackets are given by

$$[\hat{Z}_\alpha, \hat{Z}_{\overline{\beta}}]^\bullet = (\nabla_\alpha \mu_{\overline{\beta}}{}^\sigma) Z_\sigma - (\nabla_{\overline{\beta}} \mu_\alpha{}^{\overline{\tau}}) Z_{\overline{\tau}},$$
$$[\hat{Z}_\alpha, \hat{Z}_\beta]^\bullet = (\nabla_\alpha \mu_\beta{}^{\overline{\gamma}} - \nabla_{\overline{\beta}} \mu_\alpha{}^{\overline{\gamma}}) Z_{\overline{\gamma}},$$
$$[\hat{Z}_\alpha, T]^\bullet = -(\nabla_0 \mu_\alpha{}^{\overline{\gamma}}) Z_{\overline{\gamma}}.$$

They immediately show that $h^\bullet_{\alpha\overline{\beta}} = 0$ and $N^\bullet{}_{\alpha\beta}{}^{\overline{\gamma}} = \nabla_\alpha \mu_\beta{}^{\overline{\gamma}} - \nabla_\beta \mu_\alpha{}^{\overline{\gamma}}$. The first structure equation (2.3) implies

$$A^\bullet{}_\alpha{}^{\overline{\beta}} = \theta^{\overline{\beta}}([\hat{Z}_\alpha, T]^\bullet) = -\nabla_0 \mu_\alpha{}^{\overline{\beta}}.$$

Similarly we have

$$\omega^\bullet{}_\alpha{}^\beta(Z_{\overline{\gamma}}) = -\nabla_\alpha \mu_{\overline{\gamma}}{}^\beta, \qquad \omega^\bullet{}_\alpha{}^\beta(T) = 0,$$

and this together with $\omega^\bullet_{\alpha\overline{\beta}} + \omega^\bullet_{\overline{\beta}\alpha} = (dh_{\alpha\overline{\beta}})^\bullet = 0$ implies $\omega^\bullet{}_\alpha{}^\beta(Z_\gamma) = \nabla^\beta \mu_{\alpha\gamma}$. From (2.7) we have

$$R^\bullet_{\alpha\overline{\beta}} = Z_\alpha \omega^\bullet{}_\gamma{}^\gamma(Z_{\overline{\beta}}) - Z_{\overline{\beta}} \omega^\bullet{}_\gamma{}^\gamma(Z_\alpha) - \omega^\bullet{}_\gamma{}^\gamma([Z_\alpha, Z_{\overline{\beta}}]) = -\nabla_\alpha \nabla^{\overline{\gamma}} \mu_{\overline{\beta}\overline{\gamma}} - \nabla_{\overline{\beta}} \nabla^\gamma \mu_{\alpha\gamma}.$$

This completes the proof. □

Let g be a normal-form ACH metric for θ satisfying the condition in Theorem 5.2. Let

$$g_{00} = 1 + \varphi_{00}, \qquad g_{0\alpha} = \varphi_{0\alpha}, \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}}, \qquad g_{\alpha\beta} = \varphi_{\alpha\beta}.$$

Then, as seen in Theorem 5.2,

$$\varphi[m]_{ij} := \frac{1}{m!} \left(\partial_\rho^m \varphi_{ij}\right)\Big|_{\rho=0}, \qquad m \le 2n + 1 + a(i,j)$$

are uniquely determined. For the standard CR structure they completely vanish. We shall observe the differentials $\varphi[m]^\bullet_{ij}$ of $\varphi[m]_{ij}$. For notational convenience, we set $\varphi[m]_{ij} := 0$ for $m \le 0$ and

$$\chi_k(m) := \begin{cases} 1, & m = k, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 6.3. *The differentials $\varphi[m]^\bullet_{ij}$ of $\varphi[m]_{ij}$ at the standard CR structure satisfy*

$$\begin{aligned}
0 &= -\tfrac{1}{8}\left(m^2 - (2n+4)m - 4n\right)\varphi[m]^\bullet_{00} + \tfrac{1}{2}(m-2)\varphi[m]^\bullet{}_\alpha{}^\alpha \\
&\quad + i(\nabla^\alpha \varphi[m-1]^\bullet_{0\alpha} - \nabla^{\overline{\alpha}} \varphi[m-1]^\bullet_{0\overline{\alpha}}) + \tfrac{1}{2}\Delta_b \varphi[m-2]^\bullet_{00} \\
&\quad + (\nabla_0 \nabla^\alpha \varphi[m-3]^\bullet_{0\alpha} + \nabla_0 \nabla^{\overline{\alpha}} \varphi[m-3]^\bullet_{0\overline{\alpha}}) - \nabla_0 \nabla_0 \varphi[m-4]^\bullet{}_\alpha{}^\alpha, \\
0 &= -\chi_3(m) \nabla_0 \nabla^\beta \mu_{\alpha\beta} - \tfrac{1}{8}(m+1)(m-2n-3)\varphi[m]^\bullet_{0\alpha} \\
&\quad + \tfrac{3i}{4}\nabla_\alpha \varphi[m-1]^\bullet_{00} + \tfrac{i}{2}\nabla_\alpha \varphi[m-1]^\bullet{}_\beta{}^\beta - i\nabla^{\overline{\beta}} \varphi[m-1]^\bullet_{\alpha\overline{\beta}} \\
&\quad + \tfrac{1}{2}\Delta_b \varphi[m-2]^\bullet_{0\alpha} - \tfrac{i}{2}\nabla_0 \varphi[m-2]^\bullet_{0\alpha} + \tfrac{1}{2}(\nabla_\alpha \nabla^\beta \varphi[m-2]^\bullet_{0\beta} + \nabla_\alpha \nabla^{\overline{\beta}} \varphi[m-2]^\bullet_{0\overline{\beta}}) \\
&\quad - \nabla_0 \nabla_\alpha \varphi[m-3]^\bullet{}_\beta{}^\beta + \tfrac{1}{2}(\nabla_0 \nabla^{\overline{\beta}} \varphi[m-3]^\bullet_{\alpha\overline{\beta}} + \nabla_0 \nabla^\beta \varphi[m-3]^\bullet_{\alpha\beta}),
\end{aligned}$$

$$
\begin{aligned}
0 = &-\chi_2(m)(\nabla_\alpha\nabla^{\overline{\gamma}}\mu_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^{\gamma}\mu_{\alpha\gamma}) - \tfrac{1}{8}\left(m^2-(2n+2)m-8\right)\varphi[m]^\bullet_{\alpha\overline{\beta}} \\
&+ \tfrac{1}{8}h_{\alpha\overline{\beta}}(m-4)\varphi[m]^\bullet_{00} + \tfrac{1}{4}h_{\alpha\overline{\beta}}m\varphi[m]^\bullet{}_{\gamma}{}^{\gamma} \\
&+ i(\nabla_\alpha\varphi[m-1]^\bullet_{0\overline{\beta}} - \nabla_{\overline{\beta}}\varphi[m-1]^\bullet_{0\alpha}) - \tfrac{i}{4}h_{\alpha\overline{\beta}}\nabla_0\varphi[m-2]^\bullet_{00} - \tfrac{i}{2}h_{\alpha\overline{\beta}}\nabla_0\varphi[m-2]^\bullet{}_{\gamma}{}^{\gamma} \\
&- \tfrac{1}{2}\nabla_\alpha\nabla_{\overline{\beta}}\varphi[m-2]^\bullet_{00} - \nabla_\alpha\nabla_{\overline{\beta}}\varphi[m-2]^\bullet{}_{\gamma}{}^{\gamma} + \tfrac{1}{2}\Delta_b\varphi[m-2]^\bullet_{\alpha\overline{\beta}} \\
&+ \tfrac{1}{2}(\nabla_\alpha\nabla^{\gamma}\varphi[m-2]^\bullet_{\overline{\beta}\gamma} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi[m-2]^\bullet_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^{\overline{\gamma}}\varphi[m-2]^\bullet_{\alpha\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^{\gamma}\varphi[m-2]^\bullet_{\alpha\gamma}) \\
&+ \tfrac{1}{2}(\nabla_0\nabla_\alpha\varphi[m-3]^\bullet_{0\overline{\beta}} + \nabla_0\nabla_{\overline{\beta}}\varphi[m-3]^\bullet_{0\alpha}) - \tfrac{1}{2}\nabla_0\nabla_0\varphi[m-4]^\bullet_{\alpha\overline{\beta}}, \\
0 = &-\chi_2(m)(\Delta_b\mu_{\alpha\beta} + \nabla_\alpha\nabla^{\gamma}\mu_{\beta\gamma} + \nabla_\beta\nabla^{\gamma}\mu_{\alpha\gamma} + 2i\nabla_0\mu_{\alpha\beta}) + \chi_4(m)\nabla_0\nabla_0\mu_{\alpha\beta} \\
&- \tfrac{1}{8}m(m-2n-2)\varphi[m]^\bullet_{\alpha\beta} - \tfrac{1}{2}\nabla_\alpha\nabla_\beta\varphi[m-2]^\bullet_{00} - \nabla_\alpha\nabla_\beta\varphi[m-2]^\bullet{}_{\gamma}{}^{\gamma} + \tfrac{1}{2}\Delta_b\varphi[m-2]^\bullet_{\alpha\beta} \\
&+ \tfrac{1}{2}(\nabla_\alpha\nabla^{\overline{\gamma}}\varphi[m-2]^\bullet_{\beta\overline{\gamma}} + \nabla_\alpha\nabla^{\gamma}\varphi[m-2]^\bullet_{\beta\gamma} + \nabla_\beta\nabla^{\overline{\gamma}}\varphi[m-2]^\bullet_{\alpha\overline{\gamma}} + \nabla_\beta\nabla^{\gamma}\varphi[m-2]^\bullet_{\alpha\gamma}) \\
&+ i\nabla_0\varphi[m-2]^\bullet_{\alpha\beta} + \tfrac{1}{2}(\nabla_0\nabla_\alpha\varphi[m-3]^\bullet_{0\beta} + \nabla_0\nabla_\beta\varphi[m-3]^\bullet_{0\alpha}) - \tfrac{1}{2}\nabla_0\nabla_0\varphi[m-4]^\bullet_{\alpha\beta},
\end{aligned}
$$

where in each equality m takes any nonnegative integer and ∇ denotes the Tanaka–Webster connection for the standard CR sphere with θ.

Proof. This follows from Lemma 3.2, because terms of type (N1)–(N3), which are neglected in the formulae recorded in that lemma, are at least quadratic in $\mu_{\alpha\beta}$. By setting $\mathrm{Ein}_{IJ} = O(\rho^{2n+1+a(I,J)})$, the Taylor expansions of the last four equalities in Lemma 3.2 give the claimed formulae, thanks to Proposition 6.2. □

In principle we can calculate all $\varphi[m]^\bullet_{ij}$ using the recurrence formulae above. It is easy to see that $\varphi[m]^\bullet_{00} = \varphi[m]^\bullet_{\alpha\overline{\beta}} = \varphi[m]^\bullet_{\alpha\beta} = 0$ for m odd and $\varphi[m]^\bullet_{0\alpha} = 0$ for m even, and each nonzero $\varphi[m]^\bullet_{ij}$ is a linear combination over $\mathbb{C}$ of covariant derivatives of $\mu_{\alpha\beta}$ which are given in Table 6.1. As a result the differential $\mathcal{O}^\bullet_{\alpha\beta}$ of the obstruction tensor is a linear combination of

$$
\begin{aligned}
&\Delta_b^k\nabla_0^{n+1-k}\mu_{\alpha\beta}, \qquad \Delta_b^k\nabla_0^{n-k}\nabla_{(\alpha}\nabla^{\sigma}\mu_{\beta)\sigma}, \\
&\Delta_b^k\nabla_0^{n-1-k}\nabla_\alpha\nabla_\beta\nabla^\sigma\nabla^\tau\mu_{\sigma\tau} \quad \text{and} \quad \Delta_b^k\nabla_0^{n-1-k}\nabla_\alpha\nabla_\beta\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma}\overline{\tau}},
\end{aligned}
$$

which are linearly independent if $n \geq 2$.

Proposition 6.4. *Let $n \geq 2$ and*

$$
\begin{aligned}
\mathcal{O}^\bullet_{\alpha\beta} = &\sum_{k=0}^{n+1} a_k\Delta_b^k\nabla_0^{n+1-k}\mu_{\alpha\beta} + \sum_{k=0}^{n} b_k\Delta_b^k\nabla_0^{n-k}\nabla_{(\alpha}\nabla^\sigma\mu_{\beta)\sigma} \\
&+ \sum_{k=0}^{n-1} c_k\Delta_b^k\nabla_0^{n-1-k}\nabla_\alpha\nabla_\beta\nabla^\sigma\nabla^\tau\mu_{\sigma\tau} + \sum_{k=0}^{n-1} d_k\Delta_b^k\nabla_0^{n-1-k}\nabla_\alpha\nabla_\beta\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma}\overline{\tau}}.
\end{aligned}
$$

Then $a_{n+1} = (-1)^n/(n!)^2$.

Proof. The last equality in Lemma 6.3 and Table 6.1 show

$$
0 \equiv -\chi_2(2l)\Delta_b\mu_{\alpha\beta} - \tfrac{1}{2}l(l-n-1)\varphi[2l]^\bullet_{\alpha\beta} + \tfrac{1}{2}\Delta_b\varphi[2l-2]^\bullet_{\alpha\beta}
$$

modulo $\Delta_b^k\nabla_0^{l-k}\mu_{\alpha\beta}$, $k < l$, and

$$
\Delta_b^k\nabla_0^{l-1-k}\nabla_{(\alpha}\nabla^\sigma\mu_{\beta)\sigma}, \qquad \Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_\beta\nabla^\sigma\nabla^\tau\mu_{\sigma\tau}, \qquad \Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_\beta\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma}\overline{\tau}}.
$$

Type	Linear combination of
$\varphi[2l]^{\bullet}_{00}$	$\Delta_b^k \nabla_0^{l-1-k} \nabla^\alpha \nabla^\beta \mu_{\alpha\beta}$, $\Delta_b^k \nabla_0^{l-1-k} \nabla^{\overline{\alpha}} \nabla^{\overline{\beta}} \mu_{\overline{\alpha}\overline{\beta}}$
$\varphi[2l+1]^{\bullet}_{0\alpha}$	$\Delta_b^k \nabla_0^{l-k} \nabla^\beta \mu_{\alpha\beta}$, $\Delta_b^k \nabla_0^{l-1-k} \nabla_\alpha \nabla^\sigma \nabla^\tau \mu_{\sigma\tau}$, $\Delta_b^k \nabla_0^{l-1-k} \nabla_\alpha \nabla^{\overline{\sigma}} \nabla^{\overline{\tau}} \mu_{\overline{\sigma}\overline{\tau}}$
$\varphi[2l]^{\bullet}_{\alpha\overline{\beta}}$	$\Delta_b^k \nabla_0^{l-1-k} \nabla_\alpha \nabla^{\overline{\sigma}} \mu_{\overline{\beta}\overline{\sigma}}$, $\Delta_b^k \nabla_0^{l-1-k} \nabla_{\overline{\beta}} \nabla^\sigma \mu_{\alpha\sigma}$, $\Delta_b^k \nabla_0^{l-2-k} \nabla_\alpha \nabla_{\overline{\beta}} \nabla^\sigma \nabla^\tau \mu_{\sigma\tau}$, $\Delta_b^k \nabla_0^{l-2-k} \nabla_{\overline{\beta}} \nabla_\alpha \nabla^{\overline{\sigma}} \nabla^{\overline{\tau}} \mu_{\overline{\sigma}\overline{\tau}}$, $h_{\alpha\overline{\beta}} \Delta_b^k \nabla_0^{l-1-k} \nabla^\sigma \nabla^\tau \mu_{\sigma\tau}$, $h_{\alpha\overline{\beta}} \Delta_b^k \nabla_0^{l-1-k} \nabla^{\overline{\sigma}} \nabla^{\overline{\tau}} \mu_{\overline{\sigma}\overline{\tau}}$
$\varphi[2l]^{\bullet}_{\alpha\beta}$	$\Delta_b^k \nabla_0^{l-k} \mu_{\alpha\beta}$, $\Delta_b^k \nabla_0^{l-1-k} \nabla_{(\alpha} \nabla^\sigma \mu_{\beta)\sigma}$, $\Delta_b^k \nabla_0^{l-2-k} \nabla_\alpha \nabla_\beta \nabla^\sigma \nabla^\tau \mu_{\sigma\tau}$, $\Delta_b^k \nabla_0^{l-2-k} \nabla_\alpha \nabla_\beta \nabla^{\overline{\sigma}} \nabla^{\overline{\tau}} \mu_{\overline{\sigma}\overline{\tau}}$

Table 6.1. Terms appearing in the differentials $\varphi[m]^{\bullet}_{ij}$ of the coefficients of the approximate normal-form ACH-Einstein metric at the standard CR structure on the sphere

Hence we have $\varphi[2]^{\bullet}_{\alpha\beta} \equiv (2/n)\Delta_b \mu_{\alpha\beta}$ and

$$\varphi[2l]^{\bullet}_{\alpha\beta} \equiv -\frac{1}{l(n+1-l)} \Delta_b \varphi[2l-2]^{\bullet}_{\alpha\beta}.$$

This immediately shows that

$$\varphi[2l]^{\bullet}_{\alpha\beta} \equiv \frac{2}{n} \cdot \frac{-1}{2(n-1)} \cdot \frac{-1}{3(n-2)} \cdot \cdots \cdot \frac{-1}{l(n+1-l)} \Delta_b^l \mu_{\alpha\beta}, \qquad l = 1, 2, \ldots, n.$$

Then we use the last equality in Lemma 3.2 to see

$$\mathcal{O}^{\bullet}_{\alpha\beta} \equiv -\frac{1}{2} \Delta_b \varphi[2n]^{\bullet}_{\alpha\beta} \equiv -\frac{1}{2} \cdot \frac{2}{n} \cdot \frac{-1}{2(n-1)} \cdot \frac{-1}{3(n-2)} \cdot \cdots \cdot \frac{-1}{n \cdot 1} \Delta_b^{n+1} \mu_{\alpha\beta} \equiv \frac{(-1)^n}{(n!)^2} \Delta_b^{n+1} \mu_{\alpha\beta},$$

which implies the claim. □

Corollary 6.5. *Let $n \geq 2$. Then there is a partially integrable almost CR structure on the $(2n+1)$-dimensional sphere, arbitrarily close to the standard one, for which the obstruction tensor does not vanish.*

7. Formal solution involving logarithmic singularity

Let X be a manifold-with-boundary and ρ a boundary defining function. We say that a function $f \in C^0(X) \cap C^\infty(\mathring{X})$ belongs to $\mathcal{A}(X)$, or simply $\mathcal{A}$, if it admits an asymptotic expansion of the form (0.6). By this we mean that for any $m \geq 0$,

$$r_N := f - \sum_{q=0}^{N} f^{(q)} (\log \rho)^q \in C^m(X) \quad \text{and} \quad r_N = O(\rho^m)$$

holds for sufficiently large N. The Taylor expansions of $f^{(q)}$ at ∂X are uniquely determined; we write $f \in \mathcal{A}^m$ if $f^{(q)} \in O(\rho^m)$, $q \geq 0$, and $\mathcal{A}^\infty := \cap_{m=0}^\infty \mathcal{A}^m$. One can show that $\mathcal{A}$ is closed under multiplication, and that if $f \in \mathcal{A}$, $f|_{\partial X} \neq 0$ then $f^{-1} \in \mathcal{A}$. Furthermore, $\mathcal{A}$ is closed under the action of a totally characteristic linear differential operator, i.e., a noncommutative polynomial of C^∞ vector fields tangent to the boundary.

As in §§3–5, again in this section X is an open neighborhood of M in $M \times [0, \infty)$, where $(M, T^{1,0})$ a nondegenerate partially integrable almost CR manifold. We fix a pseudohermitian structure θ and consider (nonsmooth) Θ-metrics of the form (1.9) with $g_{ij} \in \mathcal{A}$ satisfying (1.10), which we call *singular normal-form ACH metrics.*

All the calculations regarding the Ricci tensor go in the same way as in §3 and §4 except that, while on the space of smooth $O(\rho^m)$ functions $\rho\partial_\rho$ behaves as a mere "m times" operator modulo $O(\rho^{m+1})$, it is no longer the case when $O(\rho^m)$ and $O(\rho^{m+1})$ are replaced by $\mathcal{A}^m$ and $\mathcal{A}^{m+1}$. Nevertheless, since $\mathcal{A}$ is closed under the action of totally characteristic operators, the Ricci tensors for singular normal-form ACH metrics have expansions of the form (0.6) with respect to the frame $\{ \rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}} \}$.

Proposition 7.1. *There exists a singular normal-form ACH metric g satisfying*

$$\mathrm{Ein}_{IJ} = \mathcal{A}^{2n+1+a(I,J)}. \tag{7.1}$$

The components g_{ij} are uniquely determined, and do not contain logarithmic terms, modulo $\mathcal{A}^{2n+1+a(i,j)}$.

Proof. This is proved by following the argument in §3, §4 and the first half of §5 again. We shall include here a detailed account of the following fact only, which is a version of Proposition 3.1: $\mathrm{Ein}_{IJ} = \mathcal{A}^3$ if and only if

$$g_{00} = 1 + \mathcal{A}^3, \qquad g_{0\alpha} = \mathcal{A}^3, \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \rho^2\Phi_{\alpha\overline{\beta}} + \mathcal{A}^3, \qquad g_{\alpha\beta} = \rho^2\Phi_{\alpha\beta} + \mathcal{A}^3,$$

where $\Phi_{\alpha\overline{\beta}}$ and $\Phi_{\alpha\beta}$ are defined by (3.4). Then the rest of the proof goes similarly.

If we define φ_{ij} by (3.5), then Lemma 3.2 is again valid. Take $N \geq 1$ large enough so that φ_{ij} and Ein_{IJ} for given g are of the form

$$\varphi_{ij} = \sum_{q=0}^{N} \varphi_{ij}^{(q)} (\log\rho)^q + \mathcal{A}^3, \qquad \varphi_{ij}^{(q)} \in C^\infty(X),$$

and

$$\mathrm{Ein}_{IJ} = \sum_{q=0}^{N} \mathrm{Ein}_{IJ}^{(q)} (\log\rho)^q + \mathcal{A}^3, \qquad \mathrm{Ein}_{IJ}^{(q)} \in C^\infty(X).$$

Then by Lemma 3.2 we have the same identities as (3.12) between $\mathrm{Ein}_{IJ}^{(N)}$ and $\varphi_{ij}^{(N)}$; namely, the following holds for $q = N$:

$$\begin{aligned}
&\mathrm{Ein}_{\infty\infty}^{(q)} = \tfrac{3}{2}\varphi_{00}^{(q)} + \varphi^{(q)}{}_{\alpha}{}^{\alpha} + O(\rho^2), \\
&\mathrm{Ein}_{\infty 0}^{(q)} = O(\rho^2), \qquad \mathrm{Ein}_{\infty\alpha}^{(q)} = -i\varphi_{0\alpha}^{(q)} + O(\rho^2), \\
&\mathrm{Ein}_{00}^{(q)} = \tfrac{3}{8}(2n+1)\varphi_{00}^{(q)} - \tfrac{1}{2}\varphi^{(q)}{}_{\alpha}{}^{\alpha} + O(\rho^2), \qquad \mathrm{Ein}_{0\alpha}^{(q)} = \tfrac{1}{2}(n+1)\varphi_{0\alpha}^{(q)} + O(\rho^2), \\
&\mathrm{Ein}_{\alpha\overline{\beta}}^{(q)} = \tfrac{1}{8}(2n+9)\varphi_{\alpha\overline{\beta}}^{(q)} - \tfrac{3}{8}h_{\alpha\overline{\beta}}\varphi_{00}^{(q)} + \tfrac{1}{4}h_{\alpha\overline{\beta}}\varphi^{(q)}{}_{\gamma}{}^{\gamma} + O(\rho^2), \\
&\mathrm{Ein}_{\alpha\beta}^{(q)} = \tfrac{1}{8}(2n+1)\varphi_{\alpha\beta}^{(q)} + O(\rho^2).
\end{aligned}$$

Hence $\varphi_{ij}^{(N)}$ must be $O(\rho^2)$ so as to make $\mathrm{Ein}_{IJ}^{(N)} = O(\rho^2)$. If $\varphi_{ij}^{(q)} = O(\rho^2)$, $q_0 + 1 \leq q \leq N$, then the identities above hold for $q = q_0$, which shows that $\mathrm{Ein}_{IJ}^{(q_0)} = O(\rho^2)$ is equivalent to $\varphi_{ij}^{(q_0)} = O(\rho^2)$. Hence we conclude that $\mathrm{Ein}_{IJ} = \mathcal{A}^2$ if and only if $\varphi_{ij} = \mathcal{A}^2$.

Next, again by Lemma 3.2 we see that the following is true for $q = N$:

$$\begin{aligned}
&\mathrm{Ein}^{(q)}_{\infty\infty} = 2\varphi^{(q)}_{00} + O(\rho^3), \qquad \mathrm{Ein}^{(q)}_{\infty 0} = O(\rho^3), \qquad \mathrm{Ein}^{(q)}_{\infty\alpha} = -\tfrac{3i}{2}\varphi^{(q)}_{0\alpha} + O(\rho^3),\\
&\mathrm{Ein}^{(q)}_{00} = \tfrac{1}{2}(2n+1)\varphi^{(q)}_{00} + O(\rho^3), \qquad \mathrm{Ein}^{(q)}_{0\alpha} = \tfrac{3}{8}(2n+1)\varphi^{(q)}_{0\alpha} + O(\rho^3),\\
&\mathrm{Ein}^{(q)}_{\alpha\overline{\beta}} = \tfrac{1}{2}(n+2)\varphi^{(q)}_{\alpha\overline{\beta}} - \tfrac{1}{4}h_{\alpha\overline{\beta}}\varphi^{(q)}_{00} + \tfrac{1}{2}h_{\alpha\overline{\beta}}\varphi^{(q)}{}_{\gamma}{}^{\gamma} + O(\rho^3),\\
&\mathrm{Ein}^{(q)}_{\alpha\beta} = \tfrac{1}{2}n\varphi^{(q)}_{\alpha\beta} + O(\rho^3).
\end{aligned}$$

An inductive argument shows that $\mathrm{Ein}^{(q)}_{IJ} = O(\rho^3)$, $1 \le q \le N$, if and only if $\varphi^{(q)}_{ij} = O(\rho^3)$, $1 \le q \le N$. Finally, the same identities as (3.13) hold for $\mathrm{Ein}^{(0)}_{IJ}$ and $\varphi^{(0)}_{ij}$, which imply that $\varphi^{(0)}_{ij}$ must satisfy $\varphi^{(0)}_{00} = O(\rho^3)$, $\varphi^{(0)}_{0\alpha} = O(\rho^3)$, $\varphi^{(0)}_{\alpha\overline{\beta}} = \rho^2\Phi_{\alpha\overline{\beta}} + O(\rho^3)$ and $\varphi^{(0)}_{ij} = \rho^2\Phi_{\alpha\beta} + O(\rho^3)$ as desired. □

Let g_0 be such a normal-form ACH metric, and for specificity, let its components $(g_0)_{ij}$ be polynomials of order $2n + a(i,j)$ in ρ, which are uniquely determined by the condition (7.1). We set

$$(\mathrm{Ein}_0)_{IJ} = \rho^{2n+1+a(I,J)}E_{IJ} + O(\rho^{2n+2+a(I,J)}), \tag{7.2}$$

where E_{IJ} is constant in the ρ-direction. The tensor E_{IJ} is also seen as a composition $(E_{\infty\infty}, E_{\infty i}, E_{ij})$ of a function and tensors on M, which are universally-defined polynomials of pseudohermitian invariants of (M,θ). We already know that $E_{\alpha\beta} = \mathcal{O}_{\alpha\beta}$ and $E_{0\alpha} = -i\nabla^{\beta}\mathcal{O}_{\alpha\beta} - iN_{\alpha}{}^{\overline{\beta}\overline{\gamma}}\mathcal{O}_{\overline{\beta}\overline{\gamma}}$. Set

$$u := -\frac{1}{n+1}(iE_{\infty 0} + \nabla^{\alpha}E_{\infty\alpha} - \nabla^{\overline{\alpha}}E_{\infty\overline{\alpha}}).$$

Theorem 7.2. *Let κ be a smooth function and $\lambda_{\alpha\beta}$ a tensor satisfying*

$$P^{\alpha\beta}\lambda_{\alpha\beta} + P^{\overline{\alpha}\overline{\beta}}\lambda_{\overline{\alpha}\overline{\beta}} = u. \tag{7.3}$$

Then there is a singular normal-form ACH metric g satisfying $\mathrm{Ein}_{IJ} = \mathcal{A}^{\infty}$ and

$$\frac{1}{(2n+4)!}\left(\partial_{\rho}^{2n+4}g^{(0)}_{00}\right)\Big|_M = \kappa, \qquad \frac{1}{(2n+2)!}\left(\partial_{\rho}^{2n+2}g^{(0)}_{\alpha\beta}\right)\Big|_M = \lambda_{\alpha\beta}, \tag{7.4}$$

where $g_{ij} \sim \sum_{q=0}^{\infty} g^{(q)}_{ij}(\log\rho)^q$ is the asymptotic expansion of g_{ij}. The components g_{ij} are uniquely determined modulo $\mathcal{A}^{\infty}$ by the condition above.

As is clear from the proof below, Theorem 7.2 also holds in the following formal sense. Let $p \in M$, κ a smooth function and $\lambda_{\alpha\beta}$ a tensor satisfying (7.3) to the infinite order at p. Then there exists a singular normal-form ACH metric g satisfying (7.4) and $\mathrm{Ein}_{IJ} = \mathcal{A}^{\infty}$ to the infinite order at p, and the Taylor expansions of $g^{(q)}_{ij}$ at p are unique. On the other hand, we can find a formal power series solution to (7.3) by the Cauchy–Kovalevskaya theorem. Hence, by Borel's Lemma, we have $\lambda_{\alpha\beta}$ solving (7.3) to the infinite order at p and prove the first statement of Theorem 0.3. We do not know whether (7.3) is solvable in the category of smooth tensors.

The first step to prove Theorem 7.2 is the following.

Lemma 7.3. *There exists a singular normal-form ACH metric g satisfying*

$$\begin{aligned} &\mathrm{Ein}_{\infty\infty} = \mathcal{A}^{2n+4}, \quad \mathrm{Ein}_{\infty 0} = \mathcal{A}^{2n+4}, \quad \mathrm{Ein}_{\infty\alpha} = \mathcal{A}^{2n+3}, \\ &\mathrm{Ein}_{00} = \mathcal{A}^{2n+4}, \quad \mathrm{Ein}_{0\alpha} = \mathcal{A}^{2n+4}, \quad \mathrm{Ein}_{\alpha\overline{\beta}} = \mathcal{A}^{2n+4} \quad \mathrm{Ein}_{\alpha\beta} = \mathcal{A}^{2n+3} \end{aligned}$$

and $\mathrm{Ein}_{\infty 0}$ *mod* $\mathcal{A}^{2n+5}$, $\mathrm{Ein}_{\infty\alpha}$ *mod* $\mathcal{A}^{2n+4}$, Ein_{00} *mod* $\mathcal{A}^{2n+5}$, $\mathrm{Ein}_{\alpha\overline{\beta}}$ *mod* $\mathcal{A}^{2n+5}$ *do not contain logarithmic terms. Such a metric g is of the form*

$$\begin{aligned} g_{00} &= (g_0)_{00} + \psi_{00}^{(0)} + \psi_{00}^{(1)} \log\rho + \psi_{00}^{(2)} (\log\rho)^2 + \mathcal{A}^{2n+5}, \\ g_{0\alpha} &= (g_0)_{0\alpha} + \psi_{0\alpha}^{(0)} + \psi_{0\alpha}^{(1)} \log\rho + \mathcal{A}^{2n+4}, \\ g_{\alpha\overline{\beta}} &= (g_0)_{\alpha\overline{\beta}} + \psi_{\alpha\overline{\beta}}^{(0)} + \psi_{\alpha\overline{\beta}}^{(1)} \log\rho + \psi_{\alpha\overline{\beta}}^{(2)} (\log\rho)^2 + \mathcal{A}^{2n+5}, \\ g_{\alpha\beta} &= (g_0)_{\alpha\beta} + \psi_{\alpha\beta}^{(0)} + \psi_{\alpha\beta}^{(1)} \log\rho + \mathcal{A}^{2n+3}, \end{aligned}$$

where $\psi_{ij}^{(q)} = O(\rho^{2n+1+a(i,j)})$*. Furthermore, among* $\psi_{ij}^{(q)}$*,*

$$\psi_{00}^{(2)}, \qquad \psi_{\alpha\overline{\beta}}^{(2)}, \qquad \psi_{0\alpha}^{(1)}, \qquad \mathrm{tf}(\psi_{\alpha\overline{\beta}}^{(1)}), \qquad \psi_{\alpha\beta}^{(1)} \qquad \text{and} \qquad \tfrac{1}{2} n \psi_{00}^{(1)} + (n+1) {\psi^{(1)}}_{\alpha}{}^{\alpha}$$

are determined modulo $O(\rho^{2n+2+a(i,j)})$*. In particular, if* $\mathcal{O}_{\alpha\beta} = 0$ *then they are zero modulo* $O(\rho^{2n+2+a(i,j)})$*.*

Proof. We set

$$g_{ij} = (g_0)_{ij} + \sum_{q=0}^{N} \psi_{ij}^{(q)} (\log\rho)^q, \qquad \psi_{ij}^{(q)} = O(\rho^{2n+1+a(i,j)}). \tag{7.5}$$

By (4.2) and (4.3), which are also valid here if $O(\rho^{m'})$ is replaced by $\mathcal{A}^{m'}$, the difference $\delta\,\mathrm{Ein}_{IJ}$ between the Einstein tensor of g_0 and that of g is of the form

$$\delta\,\mathrm{Ein}_{IJ} = \sum_{q=0}^{N} \delta\,\mathrm{Ein}_{IJ}^{(q)} (\log\rho)^q + \mathcal{A}^{2n+2+a(I,J)}.$$

We may assume $N \geq 3$. Then, by (4.2) we have $\delta\,\mathrm{Ein}_{0\alpha}^{(N)} = O(\rho^{2n+4})$ and $\delta\,\mathrm{Ein}_{\alpha\beta}^{(q-1)} = O(\rho^{2n+3})$, which imply that $\mathrm{Ein}_{0\alpha}^{(N)} = O(\rho^{2n+4})$, $\mathrm{Ein}_{\alpha\beta}^{(q-1)} = O(\rho^{2n+3})$ already hold, and

$$\begin{aligned} \delta\,\mathrm{Ein}_{0\alpha}^{(q-1)} &= -\tfrac{1}{4} q(n+2) \psi_{0\alpha}^{(q)} + O(\rho^{2n+4}), \\ \delta\,\mathrm{Ein}_{\alpha\beta}^{(q-1)} &= -\tfrac{1}{4} q(n+1) \psi_{\alpha\beta}^{(q)} + O(\rho^{2n+3}) \end{aligned} \tag{7.6}$$

for $q = N$. This shows that $\mathrm{Ein}_{0\alpha}^{(N-1)} = O(\rho^{2n+4})$, $\mathrm{Ein}_{\alpha\beta}^{(N-1)} = O(\rho^{2n+3})$ if and only if $\psi_{0\alpha}^{(N)} = O(\rho^{2n+4})$, $\psi_{\alpha\beta}^{(N)} = O(\rho^{2n+3})$, for Ein_0 contains no logarithmic terms. Since (7.6) holds for $q = q_0$ if $\psi_{0\alpha}^{(q)} = O(\rho^{2n+4})$ and $\psi_{\alpha\beta}^{(q)} = O(\rho^{2n+3})$ for $q_0 + 1 \leq q \leq N$, inductively we verify that $\mathrm{Ein}_{0\alpha} = \mathcal{A}^{2n+4}$, $\mathrm{Ein}_{\alpha\beta} = \mathcal{A}^{2n+3}$ if and only if $\psi_{0\alpha}^{(q)} = O(\rho^{2n+4})$, $\psi_{\alpha\beta}^{(q)} = O(\rho^{2n+3})$, $2 \leq q \leq N$ and

$$\psi_{0\alpha}^{(1)} = \frac{4}{n+2} \rho^{2n+3} E_{0\alpha} + O(\rho^{2n+4}), \qquad \psi_{\alpha\beta}^{(1)} = \frac{4}{n+1} \rho^{2n+2} E_{\alpha\beta} + O(\rho^{2n+3}). \tag{7.7}$$

Next, from (4.3c)–(4.3e) we have $n\delta\operatorname{Ein}^{(N)}_{00} - 2\delta\operatorname{Ein}^{(N)}{}_{\alpha}{}^{\alpha} = O(\rho^{2n+5})$ and

$$\begin{aligned}\delta\operatorname{Ein}^{(q)}_{00} &= \tfrac{1}{2}n\psi^{(q)}_{00} + (n+1)\psi^{(q)}{}_{\alpha}{}^{\alpha} + O(\rho^{2n+5}),\\ \operatorname{tf}(\delta\operatorname{Ein}^{(q)}_{\alpha\overline{\beta}}) &= -\tfrac{1}{2}n\operatorname{tf}(\psi^{(q)}_{\alpha\overline{\beta}}) + O(\rho^{2n+5}),\\ n\delta\operatorname{Ein}^{(q-1)}_{00} - 2\delta\operatorname{Ein}^{(q-1)}{}_{\alpha}{}^{\alpha} &= -\tfrac{1}{4}q(n+3)(n\psi^{(q)}_{00} - 2\psi^{(q)}{}_{\alpha}{}^{\alpha}) + O(\rho^{2n+5})\end{aligned}$$

for $q = N$. Hence both $\psi^{(N)}_{00}$ and $\psi^{(N)}_{\alpha\overline{\beta}}$ must be $O(\rho^{2n+5})$. Inductively the same must hold for all $\psi^{(q)}_{00}$, $\psi^{(q)}_{\alpha\overline{\beta}}$, $3 \le q \le N$, and $\frac{1}{2}n\psi^{(2)}_{00} + (n+1)\psi^{(2)}{}_{\alpha}{}^{\alpha} = O(\rho^{2n+5})$ and $\operatorname{tf}(\psi^{(2)}_{\alpha\overline{\beta}}) = O(\rho^{2n+5})$, in order for us to have $\operatorname{Ein}^{(q)}_{00} = O(\rho^{2n+5})$, $\operatorname{Ein}^{(q)}_{\alpha\overline{\beta}} = O(\rho^{2n+5})$, $2 \le q \le N$.

Again by (4.3c)–(4.3e), modulo $O(\rho^{2n+4})$ terms which linearly depend on $\psi^{(2)}_{00}$, $\psi^{(1)}_{0\alpha}$ and $\psi^{(1)}_{\alpha\beta}$,

$$\begin{aligned}\delta\operatorname{Ein}^{(1)}_{00} &\equiv \tfrac{1}{2}n\psi^{(1)}_{00} + (n+1)\psi^{(1)}{}_{\alpha}{}^{\alpha} + O(\rho^{2n+5}),\\ \operatorname{tf}(\delta\operatorname{Ein}^{(1)}_{\alpha\overline{\beta}}) &\equiv -\tfrac{1}{2}n\operatorname{tf}(\psi^{(1)}_{\alpha\overline{\beta}}) + O(\rho^{2n+5}),\\ n\delta\operatorname{Ein}^{(1)}_{00} - 2\delta\operatorname{Ein}^{(1)}{}_{\alpha}{}^{\alpha} &\equiv -\tfrac{1}{2}(n+3)(n\psi^{(2)}_{00} - 2\psi^{(2)}{}_{\alpha}{}^{\alpha}) + O(\rho^{2n+5}).\end{aligned}$$

Therefore $\psi^{(2)}_{00}$, $\psi^{(2)}{}_{\alpha}{}^{\alpha}$, $\operatorname{tf}(\psi^{(1)}_{\alpha\overline{\beta}})$ and $\frac{1}{2}n\psi^{(1)}_{00} + (n+1)\psi^{(1)}{}_{\alpha}{}^{\alpha}$ are uniquely determined modulo $O(\rho^{2n+5})$ by the requirement $\operatorname{Ein}^{(1)}_{00} = O(\rho^{2n+5})$, $\operatorname{Ein}^{(1)}_{\alpha\overline{\beta}} = O(\rho^{2n+5})$.

For g_{ij} that we have constructed, (5.2b) and (5.2c), or (5.9), show that $\operatorname{Ein}_{\infty 0}$ and $\operatorname{Ein}_{\infty\alpha}$ do not contain logarithmic terms modulo $\mathcal{A}^{2n+5}$ and $\mathcal{A}^{2n+4}$, respectively. If $\mathcal{O}_{\alpha\beta} = 0$, (7.7) implies that $\psi^{(1)}_{0\alpha}$ and $\psi^{(1)}_{\alpha\beta}$ are zero, and hence $\psi^{(2)}_{00}$, $\psi^{(2)}{}_{\alpha}{}^{\alpha}$, $\operatorname{tf}(\psi^{(1)}_{\alpha\overline{\beta}})$ and $\frac{1}{2}n\psi^{(1)}_{00} + (n+1)\psi^{(1)}{}_{\alpha}{}^{\alpha}$ are also zero. □

The rest of the proof of Theorem 7.2 consists of two parts, in the first of which we finish constructing a singular ACH metric satisfying $\operatorname{Ein}_{IJ} = \mathcal{A}^{2n+2+a(I,J)}$, and in the second we go through the inductive argument to achieve $\operatorname{Ein}_{IJ} = \mathcal{A}^{\infty}$.

Proof of Theorem 7.2. Let g be a singular normal-form ACH metric we have obtained in Lemma 7.3. By (4.3b), (4.2) and (7.7) we have

$$\begin{aligned}\delta\operatorname{Ein}^{(0)}_{\infty 0} &= (n+2)\rho(\nabla^{\alpha}\psi^{(0)}_{0\alpha} + \nabla^{\overline{\alpha}}\psi^{(0)}_{0\overline{\alpha}}) - (n+1)\rho^{2}(A^{\alpha\beta}\psi^{(0)}_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}})\\ &\quad + \rho^{2n+4}\left[\tfrac{2}{n+2}(\nabla^{\alpha}E_{0\alpha} + \nabla^{\overline{\alpha}}E_{0\overline{\alpha}}) - \tfrac{2}{n+1}(A^{\alpha\beta}E_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}E_{\overline{\alpha}\overline{\beta}})\right] + O(\rho^{2n+5}),\\ \delta\operatorname{Ein}^{(0)}_{\infty\alpha} &= -i(n+2)\psi^{(0)}_{0\alpha} + (n+1)\rho\nabla^{\beta}\psi^{(0)}_{\alpha\beta} + (n+1)\rho N_{\alpha}{}^{\overline{\beta}\overline{\gamma}}\psi^{(0)}_{\overline{\beta}\overline{\gamma}}\\ &\quad - \rho^{2n+3}\left[\tfrac{2i}{n+2}E_{0\alpha} - \tfrac{2}{n+1}(\nabla^{\beta}E_{\alpha\beta} + N_{\alpha}{}^{\overline{\beta}\overline{\gamma}}E_{\overline{\beta}\overline{\gamma}})\right] + O(\rho^{2n+4}).\end{aligned}$$

If we set $\psi^{(0)}_{0\alpha} = \rho^{2n+3}\nu_{\alpha} + O(\rho^{2n+4})$ and $\psi^{(0)}_{\alpha\beta} = \rho^{2n+2}\mu_{\alpha\beta} + O(\rho^{2n+3})$, then to attain $\operatorname{Ein}_{\infty 0} = O(\rho^{2n+5})$ and $\operatorname{Ein}_{\infty\alpha} = O(\rho^{2n+4})$ is equivalent to solve the following system of

PDEs:

$$
(7.8) \qquad \begin{cases} (n+2)(\nabla^\alpha \nu_\alpha + \nabla^{\overline{\alpha}} \nu_{\overline{\alpha}}) - (n+1)(A^{\alpha\beta}\mu_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}}) \\ \quad = -E_{\infty 0} - \frac{2}{n+2}(\nabla^\alpha E_{0\alpha} + \nabla^{\overline{\alpha}} E_{0\overline{\alpha}}) + \frac{2}{n+1}(A^{\alpha\beta}E_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}E_{\overline{\alpha}\overline{\beta}}), \\ -i(n+2)\nu_\alpha + (n+1)\nabla^\beta \mu_{\alpha\beta} + (n+1){N_\alpha}^{\overline{\beta}\overline{\gamma}}\mu_{\overline{\beta}\overline{\gamma}} \\ \quad = -E_{\infty\alpha} + \frac{2i}{n+2}E_{0\alpha} - \frac{2}{n+1}(\nabla^\beta E_{\alpha\beta} + {N_\alpha}^{\overline{\beta}\overline{\gamma}}E_{\overline{\beta}\overline{\gamma}}). \end{cases}
$$

If we substitute the second equation into the first one and use $E_{\alpha\beta} = \mathcal{O}_{\alpha\beta}$ and (5.9), the system is reduced to $P^{\alpha\beta}\mu_{\alpha\beta} - P^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}} = u$. Hence, by setting $\mu_{\alpha\beta} = \lambda_{\alpha\beta}$ and determining ν_α by (7.8) we achieve $\mathrm{Ein}_{\infty 0} = O(\rho^{2n+5})$ and $\mathrm{Ein}_{\infty\alpha} = O(\rho^{2n+4})$.

Having fixed $\psi^{(0)}_{0\alpha}$ and $\psi^{(0)}_{\alpha\beta}$, now we may determine $\psi^{(1)}_{00}$, ${\psi^{(1)}_\alpha}^\alpha$, $\mathrm{tf}(\psi^{(0)}_{\alpha\overline{\beta}})$ and $\frac{1}{2}n\psi^{(0)}_{00} + (n+1){\psi^{(0)}_\alpha}^\alpha$ modulo $O(\rho^{2n+5})$ so that $\mathrm{Ein}^{(0)}_{00}$, $\mathrm{Ein}^{(0)}_{\alpha\overline{\beta}}$ are $O(\rho^{2n+5})$ by observing (4.3c)–(4.3e). It automatically holds that $\mathrm{Ein}_{\infty\infty} = \mathcal{A}^{2n+5}$ by (5.2a). Although $\frac{1}{2}n\psi^{(0)}_{00} + (n+1){\psi^{(0)}_\alpha}^\alpha$ is fixed, $\psi^{(0)}_{00}$ (or ${\psi^{(0)}_\alpha}^\alpha$) remains to be free, so we prescribe it by $\psi^{(0)}_{00} = \rho^{2n+4}\kappa + O(\rho^{2n+5})$.

We have shown that there is a singular normal-form ACH metric satisfying $\mathrm{Ein}_{IJ} = \mathcal{A}^{2n+2+a(I,J)}$. If we impose the condition (7.4) then g_{ij} are unique modulo $\mathcal{A}^{2n+2+a(i,j)}$. Let $m \geq 2n+3$ and suppose that g is a singular normal-form ACH metric satisfying $\mathrm{Ein}_{IJ} = \mathcal{A}^{m-1+a(I,J)}$. We set

$$
{g'}_{ij} = g_{ij} + \sum_{q=0}^{N} \psi^{(q)}_{ij} (\log\rho)^q,
$$

where $\psi^{(q)}_{ij} = O(\rho^{m-1+a(i,j)})$, and prove that $\psi^{(q)}_{ij} \bmod O(\rho^{m+a(i,j)})$ may be uniquely determined so that $\mathrm{Ein}_{IJ} = \mathcal{A}^{m+a(I,J)}$ holds.

By replacing N with larger one if necessary, the difference $\delta\,\mathrm{Ein} = \mathrm{Ein}' - \mathrm{Ein}$ between the Einstein tensors is of the form

$$
\delta\,\mathrm{Ein}_{IJ} = \sum_{q=0}^{N} \delta\,\mathrm{Ein}^{(q)}_{IJ} (\log\rho)^q + \mathcal{A}^{m+a(I,J)}.
$$

Then by (4.2) and (4.3) we have, modulo terms linearly depending on $\psi^{(q+2)}_{ij}$ or $\psi^{(q+1)}_{ij}$,

$$
(7.9a) \qquad \begin{aligned} \delta\,\mathrm{Ein}^{(q)}_{00} &\equiv -\tfrac{1}{8}(m^2 - 2nm - 8n - 4)\psi^{(q)}_{00} + \tfrac{1}{2}m{\psi^{(q)}_\alpha}^\alpha \\ &\quad + (O(\rho^{m+2}) \text{ terms depending on } \psi^{(q)}_{0\alpha} \text{ and } \psi^{(q)}_{\alpha\beta}) + O(\rho^{m+3}) \end{aligned}
$$

$$
(7.9b) \qquad \delta\,\mathrm{Ein}^{(q)}_{0\alpha} \equiv -\tfrac{1}{8}(m+2)(m-2n-2)\psi^{(q)}_{0\alpha} + O(\rho^{m+2}),
$$

$$
(7.9c) \qquad \begin{aligned} \delta\,{\mathrm{Ein}^{(q)}_\alpha}^\alpha &\equiv \tfrac{1}{8}n(m-2)\psi^{(q)}_{00} - \tfrac{1}{8}\left(m^2 - (4n-2)m - 8n - 8\right){\psi^{(q)}_\alpha}^\alpha \\ &\quad + (O(\rho^{m+2}) \text{ terms depending on } \psi^{(q)}_{0\alpha} \text{ and } \psi^{(q)}_{\alpha\beta}) + O(\rho^{m+3}), \end{aligned}
$$

$$
(7.9d) \qquad \begin{aligned} \mathrm{tf}(\delta\,\mathrm{Ein}^{(q)}_{\alpha\overline{\beta}}) &\equiv -\tfrac{1}{8}(m^2 - 2nm - 2n - 9)\,\mathrm{tf}(\psi^{(q)}_{\alpha\overline{\beta}}) \\ &\quad + (O(\rho^{m+2}) \text{ terms depending on } \psi^{(q)}_{0\alpha} \text{ and } \psi^{(q)}_{\alpha\beta}) + O(\rho^{m+3}), \end{aligned}
$$

$$
(7.9e) \qquad \delta\,\mathrm{Ein}^{(q)}_{\alpha\beta} \equiv -\tfrac{1}{8}m(m-2n-2)\psi^{(q)}_{\alpha\beta} + O(\rho^{m+1}).
$$

By (5.5), if $m \neq 4n+2$, we may determine $\psi_{ij}^{(N)}$, $\psi_{ij}^{(N-1)}$, $\dots$, $\psi_{ij}^{(0)}$ inductively so that ${\mathrm{Ein}'}_{ij} = \mathcal{A}^{m+a(i,j)}$ hold. Then by (5.2) it automatically holds that ${\mathrm{Ein}'}_{\infty\infty} = \mathcal{A}^{m+3}$, ${\mathrm{Ein}'}_{\infty 0} = \mathcal{A}^{m+3}$ and ${\mathrm{Ein}'}_{\infty\alpha} = \mathcal{A}^{m+2}$. If $m = 4n+2$, instead of (7.9c) we use

$$\begin{aligned}\delta\,\mathrm{Ein}^{(q)}_{\infty\infty} \equiv{}& -8n(n+1)\psi^{(q)}_{00} - 8(n+1)(2n+1){\psi^{(q)}}_{\alpha}{}^{\alpha}\\ &+ (O(\rho^{4n+4}) \text{ terms depending on } \psi^{(q)}_{\alpha\beta}) + O(\rho^{4n+5}),\end{aligned}$$

which holds modulo $\psi_{ij}^{(q+2)}$ and $\psi_{ij}^{(q+1)}$. We may determine $\psi_{ij}^{(N)}$, $\psi_{ij}^{(N-1)}$, $\dots$, $\psi_{ij}^{(0)}$ inductively so that ${\mathrm{Ein}'}_{\infty\infty} = \mathcal{A}^{4n+5}$, ${\mathrm{Ein}'}_{00} = \mathcal{A}^{4n+5}$, ${\mathrm{Ein}'}_{0\alpha} = \mathcal{A}^{4n+4}$, $\mathrm{tf}({\mathrm{Ein}'}_{\alpha\overline{\beta}}) = \mathcal{A}^{4n+5}$ and ${\mathrm{Ein}'}_{\alpha\beta} = \mathcal{A}^{4n+4}$. By (5.2), we obtain ${\mathrm{Ein}'}_{\alpha}{}^{\alpha} = \mathcal{A}^{4n+3}$, ${\mathrm{Ein}'}_{\infty 0} = \mathcal{A}^{4n+5}$ and ${\mathrm{Ein}'}_{\infty\alpha} = \mathcal{A}^{4n+4}$. Hence the induction works and we obtain the theorem. □

Finally we shall discuss constructing a completely log-free solution when the $\mathcal{O}_{\alpha\beta} = 0$. We set

$$\begin{aligned}v :={}& -E_{00} + \tfrac{2}{n}E_{\alpha}{}^{\alpha} - \tfrac{1}{n}(\nabla^{\alpha}E_{\infty\alpha} + \nabla^{\overline{\alpha}}E_{\infty\overline{\alpha}}) + \tfrac{2i}{n(n+2)}(\nabla^{\alpha}E_{0\alpha} - \nabla^{\overline{\alpha}}E_{0\overline{\alpha}})\\ &- \tfrac{2}{n(n+1)}(\nabla^{\alpha}\nabla^{\beta}E_{\alpha\beta} + \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}E_{\overline{\alpha}\overline{\beta}} + N^{\gamma\alpha\beta}\nabla_{\gamma}E_{\alpha\beta} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}E_{\overline{\alpha}\overline{\beta}}\\ &\qquad + {N^{\gamma\alpha\beta}}_{,\gamma}E_{\alpha\beta} + {N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}}_{,\overline{\gamma}}E_{\overline{\alpha}\overline{\beta}}).\end{aligned}$$

Theorem 7.4. *Suppose that $\mathcal{O}_{\alpha\beta} = 0$. Let κ be a smooth function and $\lambda_{\alpha\beta}$ a tensor satisfying*

$$\begin{cases} P^{\alpha\beta}\lambda_{\alpha\beta} - P^{\overline{\alpha}\overline{\beta}}\lambda_{\overline{\alpha}\overline{\beta}} = u,\\ P^{\alpha\beta}_{-2/n}\lambda_{\alpha\beta} + P^{\overline{\alpha}\overline{\beta}}_{-2/n}\lambda_{\overline{\alpha}\overline{\beta}} = v.\end{cases} \tag{7.10}$$

Then there is a normal-form ACH metric g, which is free of logarithmic terms, satisfying $\mathrm{Ein}_{IJ} = \mathcal{A}^{\infty}$ and

$$\frac{1}{(2n+4)!}\left(\partial_{\rho}^{2n+4}g_{00}\right)\big|_{M} = \kappa, \qquad \frac{1}{(2n+2)!}\left(\partial_{\rho}^{2n+2}g_{\alpha\beta}\right)\big|_{M} = \lambda_{\alpha\beta}. \tag{7.11}$$

The components g_{ij} are unique.

Again this theorem also holds in the formal sense. Since the principal parts of $P^{\alpha\beta}$ and $P^{\alpha\beta}_{-2/n}$ agree, the Cauchy–Kovalevskaya theorem guarantees that the system (7.10) is formally solvable at any given point. Thus we show the second statement of Theorem 0.3.

Proof. If $\mathcal{O}_{\alpha\beta} = 0$, then a (potentially) singular normal-form ACH metric g satisfying the conditions in the statement of Lemma 7.3 is of the form

$$\begin{aligned} g_{00} &= (g_0)_{00} + \psi^{(0)}_{00} + \psi^{(1)}_{00}\log\rho + \mathcal{A}^{2n+5},\\ g_{0\alpha} &= (g_0)_{0\alpha} + \psi^{(0)}_{0\alpha} + \mathcal{A}^{2n+4},\\ g_{\alpha\overline{\beta}} &= (g_0)_{\alpha\overline{\beta}} + \psi^{(0)}_{\alpha\overline{\beta}} + \tfrac{1}{n}h_{\alpha\overline{\beta}}{\psi^{(1)}}_{\gamma}{}^{\gamma}\log\rho + \mathcal{A}^{2n+5},\\ g_{\alpha\beta} &= (g_0)_{\alpha\beta} + \psi^{(0)}_{\alpha\beta} + \mathcal{A}^{2n+3}.\end{aligned}$$

Here $\frac{1}{2}n\psi^{(1)}_{00} + (n+1){\psi^{(1)}}_{\alpha}{}^{\alpha} = O(\rho^{2n+5})$ should hold. After prescribing $\psi^{(0)}_{\alpha\beta}$, the potential log-term coefficients $\psi^{(1)}_{00}$ and ${\psi^{(1)}}_{\alpha}{}^{\alpha}$ are determined by requiring $n\,\mathrm{Ein}^{(0)}_{00} - 2\,{\mathrm{Ein}^{(0)}}_{\alpha}{}^{\alpha} =$

$O(\rho^{2n+5})$. So let us look at the dependence of $n{\mathrm{Ein}'}^{(0)}_{00} - 2{{\mathrm{Ein}'}^{(0)}}_{\alpha}{}^{\alpha}$ on $\psi^{(0)}_{\alpha\beta}$. Using (4.3c) and (4.3d) again, we obtain

$$\begin{aligned} n\delta\,\mathrm{Ein}^{(0)}_{00} - 2\delta\,{\mathrm{Ein}^{(0)}}_{\alpha}{}^{\alpha} &= -\tfrac{1}{2}(n+2)(n\psi^{(1)}_{00} - 2{\psi^{(1)}}_{\alpha}{}^{\alpha}) \\ &\quad + i(n+2)\rho(\nabla^{\alpha}\psi^{(0)}_{0\alpha} - \nabla^{\overline{\alpha}}\psi^{(0)}_{0\overline{\alpha}}) - \tfrac{1}{2}n\rho^2(\Phi^{\alpha\beta}\psi^{(0)}_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}}) \\ &\quad - \rho^2(\nabla^{\alpha}\nabla^{\beta}\psi^{(0)}_{\alpha\beta} + \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}} + N^{\gamma\alpha\beta}\nabla_{\gamma}\psi^{(0)}_{\alpha\beta} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}} \\ &\qquad + {N^{\gamma\alpha\beta}}_{,\gamma}\psi^{(0)}_{\alpha\beta} + {N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}}_{,\overline{\gamma}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}}) + O(\rho^{2n+5}). \end{aligned}$$

Hence if we can set $\psi^{(0)}_{\alpha\beta}$ and $\psi^{(0)}_{0\alpha}$ appropriately, then $\psi^{(1)}_{00} - 2{\psi^{(1)}}_{\alpha}{}^{\alpha}$ must be $O(\rho^{2n+5})$ and so are $\psi^{(1)}_{00}$ and ${\psi^{(1)}}_{\alpha}{}^{\alpha}$. Let $\psi^{(0)}_{0\alpha} = \rho^{2n+3}\nu_{\alpha} + O(\rho^{2n+4})$ and $\psi^{(0)}_{\alpha\beta} = \rho^{2n+2}\mu_{\alpha\beta} + O(\rho^{2n+3})$. Combined with (7.8), the equations to be solved are

$$\begin{cases} (n+2)(\nabla^{\alpha}\nu_{\alpha} + \nabla^{\overline{\alpha}}\nu_{\overline{\alpha}}) - (n+1)(A^{\alpha\beta}\mu_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}}) \\ \quad = -E_{\infty 0} - \frac{2}{n+2}(\nabla^{\alpha}E_{0\alpha} + \nabla^{\overline{\alpha}}E_{0\overline{\alpha}}) + \frac{2}{n+1}(A^{\alpha\beta}E_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}E_{\overline{\alpha}\overline{\beta}}), \\ -i(n+2)\nu_{\alpha} + (n+1)\nabla^{\beta}\mu_{\alpha\beta} + (n+1){N_{\alpha}}^{\overline{\beta}\overline{\gamma}}\mu_{\overline{\beta}\overline{\gamma}} \\ \quad = -E_{\infty\alpha} + \frac{2i}{n+2}E_{0\alpha} - \frac{2}{n+1}(\nabla^{\beta}E_{\alpha\beta} + {N_{\alpha}}^{\overline{\beta}\overline{\gamma}}E_{\overline{\beta}\overline{\gamma}}), \\ i(n+2)(\nabla^{\alpha}\nu_{\alpha} - \nabla^{\overline{\alpha}}\nu_{\overline{\alpha}}) - \frac{1}{2}n(\Phi^{\alpha\beta}\mu_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}}) \\ -\nabla^{\alpha}\nabla^{\beta}\mu_{\alpha\beta} - \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}} - N^{\gamma\alpha\beta}\nabla_{\gamma}\mu_{\alpha\beta} - N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}\mu_{\overline{\alpha}\overline{\beta}} - {N^{\gamma\alpha\beta}}_{,\gamma}\mu_{\alpha\beta} - {N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}}_{,\overline{\gamma}}\mu_{\overline{\alpha}\overline{\beta}} \\ \quad = -nE_{00} + 2{E_{\alpha}}^{\alpha}. \end{cases}$$

By substituting the second equation into the other two and using (3.4), the system is reduced to

$$\begin{cases} P^{\alpha\beta}\mu_{\alpha\beta} - P^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}} = u, \\ P^{\alpha\beta}_{-2/n}\mu_{\alpha\beta} + P^{\overline{\alpha}\overline{\beta}}_{-2/n}\mu_{\overline{\alpha}\overline{\beta}} = v. \end{cases}$$

So we set $\mu_{\alpha\beta} = \lambda_{\alpha\beta}$ and determine ν_{α} by the equations above. Then $\psi^{(1)}_{00} = O(\rho^{2n+5})$, ${\psi^{(1)}}_{\alpha}{}^{\alpha} = O(\rho^{2n+5})$ solve $\mathrm{Ein}_{\infty 0} = \mathcal{A}^{2n+5}$, $\mathrm{Ein}_{\infty\alpha} = \mathcal{A}^{2n+4}$ and $n\,\mathrm{Ein}_{00} - 2\,{\mathrm{Ein}_{\alpha}}^{\alpha} = \mathcal{A}^{2n+5}$. As before, $\mathrm{tf}(\psi^{(0)}_{\alpha\overline{\beta}}) \bmod O(\rho^{2n+5})$ and $\frac{1}{2}n\psi^{(0)}_{00} + (n+1){\psi^{(0)}}_{\alpha}{}^{\alpha} \bmod O(\rho^{2n+5})$ are uniquely determined so that $\mathrm{Ein}_{00} = \mathcal{A}^{2n+5}$, $\mathrm{Ein}_{\alpha\overline{\beta}} = \mathcal{A}^{2n+5}$. We set $\psi^{(0)}_{00} = \rho^{2n+4}\kappa + O(\rho^{2n+5})$. By (5.2a) we have $\mathrm{Ein}_{\infty\infty} = \mathcal{A}^{2n+5}$.

Now we have uniquely constructed a normal-form ACH metric g, which is log-free, satisfying $\mathrm{Ein}_{IJ} = \mathcal{A}^{2n+2+a(I,J)}$ and (7.11). After that we once again follow the latter half of the proof of Theorem 7.2 to determine all the higher-order terms of g_{ij}. No logarithmic terms occur in this process. □

References

[BaD] E. Barletta and S. Dragomir, *Differential equations on contact Riemannian manifolds,* Ann. Scoula Norm. Sup. Pisa Cl. Sci. (4) **30** (2001), 63–95.

[BlD] D. E. Blair and S. Dragomir, *Pseudohermitian geometry on contact Riemannian manifolds,* Rend. Mat. Appl. (7) **22** (2002), 275–341.

[Bi] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques,* Astérisque **265** (2000), vi+109.

[BiH] O. Biquard and M. Herzlich, *A Burns-Epstein invariant for ACHE 4-manifolds,* Duke Math. J. **126** (2005), 53–100.

[ČGo] A. Čap and A. R. Gover, *CR-tractors and the Fefferman space,* Indiana Univ. Math. J. **57** (2008), 2519–2570.

[ČSc] A. Čap and H. Schichl, *Parabolic geometries and canonical Cartan connections,* Hokkaido Math. J. **29** (2000), 453–505.

[CY] S. Y. Cheng and S. T. Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation,* Comm. Pure Appl. Math. **33** (1980), 507–544.

[EMM] C. L. Epstein, R. B. Melrose and G. A. Mendoza, *Resolvent of the Laplacian on strictly pseudo-convex domains,* Acta Math. **167** (1991), 1–106.

[Fa] F. Farris, *An intrinsic construction of Fefferman's CR metric,* Pacific J. Math. **123** (1986), 33–45.

[Fe] C. L. Fefferman, *Monge–Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains,* Ann. of Math. (2) **103** (1976), 395–416; erratum **104** (1976), 393–394.

[FeGr1] C. Fefferman and C. R. Graham, *Conformal invariants,* in *The mathematical heritage of Élie Cartan (Lyon, 1984),* Astérisque, 1985, hors série, 95–116.

[FeGr2] C. Fefferman and C. R. Graham, *The ambient metric,* `arXiv:math/0710.0919v2`.

[GoGr] A. R. Gover and C. R. Graham, *CR invariant powers of the sub-Laplacian,* J. Reine Angew. Math. **583** (2005), 1–27.

[Gr] C. R. Graham, *Higher asymptotics of the complex Monge–Ampère equation,* Compositio Math. **64** (1987), 133–155.

[GrL] C. R. Graham and J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball,* Adv. Math. **87** (1991), 186–225.

[GuSá] C. Guillarmou and A. Sá Barreto, *Scattering and inverse scattering on ACH manifolds,* J. Reine Angew. Math. **622** (2008), 1–55.

[L1] J. M. Lee, *The Fefferman metric and pseudohermitian invariants,* Trans. Amer. Math. Soc. **296** (1986), 411–429.

[L2] J. M. Lee, *Pseudo-Einstein structures on CR manifolds,* Amer. J. Math. **110** (1988), 157–178.

[LM] J. Lee and R. Melrose, *Boundary behaviour of the complex Monge–Ampère equation,* Acta Math. **148** (1982), 159–192.

[Mi] R. I. Mizner, *Almost CR structures, f-structures, almost product structures and associated connections,* Rocky Mountain J. Math. **23** (1993), 1337–1359.

[Se] N. Seshadri, *Approximately Einstein ACH metrics,* Bull. Soc. Math. France **137** (2009), 63–91.

[Tnk] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds,* Lectures in Mathematics 9, Department of Mathematics, Kyoto Univ., Kinokuniya, Tokyo, 1975.

[Tno] S. Tanno, *Variational problems on contact Riemannian manifolds,* Trans. Amer. Math. Soc. **314** (1989), 349–379.

[W] S. M. Webster, *Pseudo-hermitian structures on a real hypersurface,* J. Differential Geom. **13** (1978), 25–41.

Graduate School of Mathematical Sciences, The University of Tokyo

E-mail address: `yoshim@ms.u-tokyo.ac.jp`