On Rayleigh-Type Formulas for a Non-local Boundary Value Problem Associated with an Integral Operator Commuting with the Laplacian

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Abstract

In this article, we provide two different ways of producing recursive formulas for the Rayleigh functions of eigenvalues of the one-dimensional Laplacian with non-local boundary conditions, which commutes with an integral operator having a harmonic kernel.

Keywords: Rayleigh functions, Laplacian eigenvalue problems, non-local boundary conditions, sum rules, power sums, Euler-Rayleigh method

1. Introduction

The purpose of this article is to prove recursion formulas for power sums for the eigenvalues of a problem which has arisen in recent work by one of us [1]. We concentrate on the one-dimensional case for the non-local boundary value problem (BVP) described in [1]. Our recursion formulas emulate those developed by various authors for Rayleigh functions, or power sums, involving roots of various transcendental equations. It was Euler who first found the first few closed expressions for what later came to be known as the Rayleigh function [2] (see also [3, Sec. 15.5], [4]):

$$\sigma_{2\ell}(\nu) = \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{2\ell}}, \quad \ell = 1, 2, \ldots,$$

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where $j_{\nu,n}$ denotes the n-th positive root of $z^{-\nu}J_\nu(z)$, and $J_\nu(z)$ is the Bessel function of the first kind of order ν. Euler's method was further developed by Lord Rayleigh [5] and Carlitz [6]. Both Euler and Rayleigh analyzed eigenvalues of oscillations of physical systems (a hanging chain for Euler and a circular membrane for Rayleigh), which aroused their interest in computing zeros of the Bessel functions. By exploiting a differential equation of Riccati-type satisfied by the function $z^{-\nu}J_\nu(z)$, Kishore [7, 8, 9] developed recursion formulas for $\sigma_{2\ell}(\nu)$, starting with the known expression, due to Euler and Rayleigh

$$\begin{aligned} \sigma_2(\nu) &= \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)} \\ \sigma_4(\nu) &= \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^4} = \frac{1}{16(\nu+1)^2(\nu+2)}. \end{aligned} \tag{1}$$

In his famous book [10], Rayleigh was further led, in the context of treating the transverse vibrations of a clamped beam, to finding summation formulas for the reciprocal 4th and 8th powers of the positive roots of the equation

$$\cos x \cosh x \pm 1 = 0. \tag{2}$$

The early history of the techniques of proving these power sum rules can be found in Watson's book [3] as well as [11, 12, 13]. The more recent articles [14, 15, 16] offer modern views, survey recent results, and apply the techniques to various transcendental functions.

Properly speaking, the technique of resolution of many of these problems goes back to Euler and his famous resolution of the "Basel" problem, named after the native Swiss city of Euler and the Bernoulli brothers. Euler successfully solved the problem first posed by Pietro Mengoli in 1644 [12, 13] and found a closed form for the expression $\sum_{n=1}^{\infty}\frac{1}{n^2}$. It is now folklore that the sum is $\frac{\pi^2}{6}$. Heuristically, Euler's argument of 1740 [17] (see also [12, 13]) amounted to writing $\frac{\sin x}{x}$ in two different ways: as a Maclaurin series and as the infinite product

$$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right),$$

since the roots of the transcendental equation $\sin x/x = 0$ are given by $x = \pm n\pi$, for $n = 1, 2, \ldots$ Expanding the product, and equating the coefficients of x^2 gives the above formula. For rigorous justifications of these formulas one should consult [18, Chap. 1]. Euler's technique is exactly what Rayleigh employed in the case of equation (2). Many nice examples illustrating this technique appear in the excellent paper of Speigel [19] where generalizations of Newton's known formulas for the symmetric sums of the roots of a polynomial can be found (see also the comments in [15]).

Radoux [20], Liron [21, 22, 23], and more recently Gupta-Muldoon [14] and Ismail-Muldoon [15] employed similar techniques to generate various recursion formulas in the same spirit. In the case of Radoux and Liron, one finds explicit and recursive formulas for sums of even powers of reciprocals for the roots of the equation $\tan x = x$, and $\cot x = x$. To illustrate the case of the equation, $\tan x = x$, with $x_1, x_2, \ldots$ denoting the strictly positive roots of the equation, they derived the sums of even powers of x_k's, i.e., $\sum_{k=1}^{\infty} x_k^{-2p}$, $p = 1, 2, \ldots$. For example, the cases $p = 1, 2$ lead to

$$\sum_{k=1}^{\infty} \frac{1}{x_k^2} = \frac{1}{10},$$

$$\sum_{k=1}^{\infty} \frac{1}{x_k^4} = \frac{1}{350}.$$

All of these are manifestations of convolution formulas relating the trace of the compact operator defined by the Green's function, and power sums of the eigenvalues as detailed in [24] and the classical book of Mikhlin [25]. The recent survey paper of Grieser [26] offers a view that relates these formulas to what is known for matrices.

Radoux [20] attributes the method of finding sums of reciprocals of powers of eigenvalues of certain operators to Sèrge Nicaisse, but as detailed in [24, 25, 26] this is truly classical.

In what follows, we provide two proofs of a recursive scheme to obtain explicit values of Rayleigh functions for a non-local BVP posed in [1]. Our main contribution is Theorems 2, 3, 4 for that non-local BVP. The proofs of the first two theorems are demonstrated directly using the properties of the eigenvalues of the non-local BVP without using the trace formulas unlike the way Goodwin proved for the regular BVPs [24]. The proof of Theorem 4 uses the generating functions as Radoux [20] and Liron [21] did for different BVPs (see also Ismail and Muldoon [15]).

2. A Non-local Boundary Value Problem

In this section we describe our eigenvalue problem. We first recall Corollary 6 from the article [1]:

Corollary 1. *The eigenfunctions of the integral operator* $\mathcal{K}$ *with the kernel* $K(x, y) = -|x-y|/2$ *for the unit interval* $\Omega = (0,1)$ *satisfy the following Laplacian eigenvalue problem:*

$$-\phi'' = \lambda\phi, \quad x \in (0,1);$$
$$\phi(0) + \phi(1) = -\phi'(0) = \phi'(1), \tag{3}$$

which can be solved explicitly as follows.

- $\lambda_0 \approx -5.756915$ *is the smallest (and the only negative) eigenvalue and is the solution of the following secular equation:*

  $$\coth \frac{\sqrt{-\lambda_0}}{2} = \frac{\sqrt{-\lambda_0}}{2}, \tag{4}$$

  The corresponding eigenfunction is:

  $$\phi_0(x) = C_0 \cosh \sqrt{-\lambda_0}\left(x - \frac{1}{2}\right),$$

  where $C_0 = \sqrt{2}\left(1 + \frac{\sinh\sqrt{-\lambda_0}}{\sqrt{-\lambda_0}}\right)^{-1/2} \approx 0.7812598$ *is a normalization constant to have* $\|\phi_0\|_{L^2(\Omega)} = 1$.

- $\lambda_{2m-1} = (2m-1)^2\pi^2$, $m = 1,2,\ldots$, *and the corresponding eigenfunction is:*

  $$\phi_{2m-1}(x) = \sqrt{2}\cos(2m-1)\pi x.$$

  These are canonical cosines with odd modes.

- λ_{2m}, $m = 1,2,\ldots$, *is the solution of the secular equation:*

  $$\cot \frac{\sqrt{\lambda_{2m}}}{2} = -\frac{\sqrt{\lambda_{2m}}}{2}, \tag{5}$$

  and the corresponding eigenfunction is:

  $$\phi_{2m}(x) = C_{2m} \cos \sqrt{\lambda_{2m}}\left(x - \frac{1}{2}\right),$$

  where $C_{2m} = \sqrt{2}\left\{1 + \frac{\sin\sqrt{\lambda_{2m}}}{\sqrt{\lambda_{2m}}}\right\}^{-1/2}$ *is a normalization constant.*

Remark 1. We refer the reader to [1] for the motivation of considering such an integral operator $\mathcal{K}$, the description of the higher dimensional versions, and a variety of applications. Here, however, we would like to point out our new interpretation of the above eigenvalue problem that was not explicitly stated in [1]. The above problem turns out to be equivalent to the following problem defined for the whole real axis and then restricting the solutions to the unit interval Ω.

$$-\psi'' = \begin{cases} \lambda\psi & \text{for } x \in \Omega; \\ 0 & \text{for } x \in \mathbb{R} \setminus \overline{\Omega}, \end{cases}$$

with the continuity conditions at the boundary points: $\psi(0-) = \psi(0+)$, $\psi'(0-) = \psi'(0+)$, $\psi(1-) = \psi(1+)$, $\psi'(1-) = \psi'(1+)$. Then, $\phi(x)$ in Corollary 1 is $\chi_\Omega(x)\psi(x)$.

Remark 2. The three cases of the eigenvalues in Corollary 1, i.e., λ_0; $\{\lambda_{2m-1}\}$; and $\{\lambda_{2m}\}$ can also be derived from a single equation:

$$\left(e^{\alpha/2} + e^{-\alpha/2}\right) \cdot \left(\frac{e^{\alpha/2} + e^{-\alpha/2}}{e^{\alpha/2} - e^{-\alpha/2}} - \frac{\alpha}{2}\right) = 0,$$

where $\lambda = -\alpha^2$, and $\alpha \in \mathbb{C}$. Searching zeros of the first factor for $\alpha \in \mathrm{i}\mathbb{R}$ leads to $\lambda_{2m-1} = (2m-1)^2\pi^2$ whereas doing so in the second factor for $\alpha \in \mathbb{R}$ leads to (4) and for $\alpha \in \mathrm{i}\mathbb{R}$ leads to (5).

Remark 3. Both Radoux [20] and Liron [21] dealt with the secular equation $\tan\beta = \beta$. They explicitly mention that this equation came from the one-dimensional Laplacian eigenvalue problem by setting $\lambda = -\alpha^2$, $\alpha = \mathrm{i}\beta$, $\beta \in \mathbb{R}$ with the following *Robin* boundary condition:

$$\phi(0) = 0, \quad \phi'(1) = \phi(1).$$

Note that $\phi'(0) = \phi(0)$, $\phi(1) = 0$ lead to $\tan\beta = -\beta$.

On the other hand, Radoux also dealt with the other secular equation $\cot\beta = \beta$ whereas Liron treated the case involving $\cot\beta = -\beta$. Neither of them explained why they wanted to treat these secular equations and neither of them explicitly listed the corresponding boundary condition unlike the case of $\tan\beta = \beta$. In fact, simple computations similar to those in [27, Sec. 4.3] suggest that $\cot\beta = \beta$ is associated with the Robin boundary conditions $(\phi'(0), \phi'(1)) = (\phi(0), 0)$ or $(0, -\phi(1))$ and $\cot\beta = -\beta$ is associated with $(\phi'(0), \phi'(1)) = (0, \phi(1))$ or $(-\phi(0), 0)$. But one also needs to consider the hyperbolic versions, i.e., $\coth\alpha = \alpha$, in order to fully solve the eigenvalue problems with the Robin boundary conditions

$(\phi'(0), \phi'(1)) = (0, \phi(1))$ or $(-\phi(0), 0)$. Note that these Robin boundary conditions are all decoupled, i.e., local. To the best of our knowledge, [1] is the first to explicitly describe the unusual non-local boundary condition (3).

Remark 4. One can exploit the well-known trace formula [24, 25, 26]

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n^p} = \int_0^1 K_p(x,x)\,\mathrm{d}x, \tag{6}$$

where $K_p(x, y)$ denotes the pth iterated kernel of $K(x, y)$, to determine the first few expressions for the Rayleigh function at hand. Indeed, one obtains at once

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \int_0^1 K(x,x)\,\mathrm{d}x = 0,$$

and

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} = \int_0^1 K_2(x,x)\,\mathrm{d}x = \int_0^1 \left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{4}\right) \mathrm{d}x = \frac{1}{24}.$$

However this task becomes tedious for $p \geq 3$, and we propose to obtain these power sums without recourse to iterated kernels, but by exploiting properties of the transcendental equations of which the eigenvalues are roots.

3. Sum of the Reciprocals of the Eigenvalues of Corollary 1

In light of Remark 4, we want to show the following directly.

Theorem 2. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be the eigenvalues of the boundary problem in Corollary 1, and let $K(x, y) = -|x-y|/2$. Then, they satisfy the following trace formula:*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \int_0^1 K(x,x)\,\mathrm{d}x = 0.$$

PROOF. Let us group the eigenvalues into the three groups as indicated in Corollary 1:

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \frac{1}{\lambda_0} + \sum_{m=1}^{\infty} \frac{1}{\lambda_{2m-1}} + \sum_{m=1}^{\infty} \frac{1}{\lambda_{2m}}. \tag{7}$$

Now, the second term of the sum is:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{1}{\lambda_{2m-1}} &= \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2\pi^2} \\ &= \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} \\ &= \frac{1}{\pi^2}\left(\sum_{m=1}^{\infty}\frac{1}{m^2}-\sum_{m=1}^{\infty}\frac{1}{(2m)^2}\right) \\ &= \frac{1}{\pi^2}\cdot\frac{3}{4}\sum_{m=1}^{\infty}\frac{1}{m^2} \\ &= \frac{1}{\pi^2}\cdot\frac{3}{4}\cdot\frac{\pi^2}{6}=\frac{1}{8},\end{aligned} \tag{8}$$

where we used the famous Basel problem identity $\sum_{m=1}^{\infty} 1/m^2 = \pi^2/6$ resolved by Euler [17] (see also [11, 12, 13, 26]).

As for the last term of (7),

$$\sum_{m=1}^{\infty}\frac{1}{\lambda_{2m}}=\frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{x_m^2}, \tag{9}$$

where $x_m := \sqrt{\lambda_{2m}}/2 > 0$ is the mth zero of the following transcendental equation; see (5):

$$\cot x = -x. \tag{10}$$

To proceed to compute (9) explicitly, let us analyze (10) more deeply. Following Radoux [20], let us first consider the following function and its Maclaurin series expansion:

$$\begin{aligned}(\cot x + x)\cdot\sin x &= \cos x + x\sin x \\ &= \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+x\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right) \\ &= 1+\frac{x^2}{2}-\left(\frac{1}{3!}-\frac{1}{4!}\right)x^4+\left(\frac{1}{5!}-\frac{1}{6!}\right)x^6-\cdots \\ &= 1+\frac{x^2}{2}-\frac{3}{4!}x^4+\frac{5}{6!}x^6-\cdots+(-1)^{k-1}\frac{2k-1}{(2k)!}x^{2k}+\cdots\end{aligned} \tag{11}$$

Now, the function $\cos x + x\sin x$ can also be expanded into the following infinite product in a manner similar to what Euler [17] and Rayleigh [5] did (see also

[11, 12, 13, 19, 26]):

$$\cos x + x\sin x = \left(1+\frac{x^2}{\alpha^2}\right)\prod_{m=1}^{\infty}\left(1-\frac{x^2}{x_m^2}\right), \tag{12}$$

where $\alpha \approx 1.19967864$ satisfies $\alpha = \coth\alpha$.

In other words, $x = \pm \mathrm{i}\alpha$ are the two (and only) pure imaginary roots of $\cos x + x\sin x$. This can be verified as follows. Let us seek for the pure imaginary zeros of $\cos x + x\sin x$ by setting $x = \mathrm{i}y$, $y \in \mathbb{R}$. Then, we have

$$\begin{aligned}\cos x + x\sin x &= \cos(\mathrm{i}y) + \mathrm{i}y\sin(\mathrm{i}y)\\ &= \frac{\mathrm{e}^{\mathrm{i}(\mathrm{i}y)}+\mathrm{e}^{-\mathrm{i}(\mathrm{i}y)}}{2} + \mathrm{i}y\frac{\mathrm{e}^{\mathrm{i}(\mathrm{i}y)}-\mathrm{e}^{-\mathrm{i}(\mathrm{i}y)}}{2\mathrm{i}}\\ &= \frac{\mathrm{e}^{y}+\mathrm{e}^{-y}}{2} - y\frac{\mathrm{e}^{y}-\mathrm{e}^{-y}}{2} = 0,\end{aligned}$$

which is equivalent to $\cosh y - y\sinh y = 0$, i.e.,

$$y = \coth y. \tag{13}$$

The justification for the product formula (12) follows considerations similar to those for example in [18, Chap. 1]; see also [15]. From (12), we have

$$\begin{aligned}\cos x + x\sin x &= \prod_{m=1}^{\infty}\left(1-\frac{x^2}{x_m^2}\right) + \frac{x^2}{\alpha^2}\prod_{m=1}^{\infty}\left(1-\frac{x^2}{x_m^2}\right) \qquad (14)\\ &= 1 + \left(\frac{1}{\alpha^2} - \sum_{m=1}^{\infty}\frac{1}{x_m^2}\right)x^2 + \cdots\end{aligned}$$

Equating the corresponding coefficients of the x^2 terms of (11) and (14), we have

$$\sum_{m=1}^{\infty}\frac{1}{x_m^2} = \frac{1}{\alpha^2} - \frac{1}{2}.$$

Hence, inserting this to (9), in turn, (7) together with (8) gives us

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{\lambda_n} &= \frac{1}{\lambda_0} + \sum_{m=1}^{\infty}\frac{1}{\lambda_{2m-1}} + \sum_{m=1}^{\infty}\frac{1}{\lambda_{2m}}\\ &= \frac{1}{\lambda_0} + \frac{1}{8} + \frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{x_m^2}\\ &= \frac{1}{\lambda_0} + \frac{1}{8} + \frac{1}{4}\left(\frac{1}{\alpha^2} - \frac{1}{2}\right)\\ &= \frac{1}{\lambda_0} + \frac{1}{4\alpha^2}\\ &= 0,\end{aligned}$$

since $\lambda_0 = -4\alpha^2$, which can be verified by identifying (4) with the equation (13) via $\alpha = \sqrt{-\lambda_0}/2$. $\square$

4. Sums of Higher Powers of the Reciprocals of the Eigenvalues of Corollary 1

Furthermore, we can establish the following identities:

Theorem 3. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be the eigenvalues of the boundary value problem specified in Corollary 1. Let $K_p(x,y)$ be the pth iterated kernel of $K(x,y) = -|x-y|/2$. Then, we have*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n^p} = \int_0^1 K_p(x,x)\,\mathrm{d}x = \frac{1}{4^p}\left(S_{2p} + \frac{(-1)^p}{\alpha^{2p}}\right) + \frac{4^p - 1}{2\cdot(2p)!}|B_{2p}|, \tag{15}$$

where

$$S_{2p} := \sum_{m=1}^{\infty} \frac{1}{x_m^{2p}} = \sum_{m=1}^{\infty}\left(\frac{4}{\lambda_{2m}}\right)^p,$$

and B_{2p} is the Bernoulli number, which is defined via the generating function:

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Moreover, S_{2p} satisfies the following recursion formula:

$$\sum_{\ell=1}^{n+1} \frac{(-1)^{n-\ell+1}\,(2\,(n-\ell+1)-1)}{(2\,(n-\ell+1))!}\left\{S_{2\ell} + \frac{(-1)^\ell}{\alpha^{2\ell}}\right\} = \frac{(-1)^n}{2(2n)!}. \tag{16}$$

PROOF. The first equality in (15) connecting the sum of the powers of the eigenvalues and the trace of the iterated kernel is the standard fact and its proof can be found in, e.g., [25, Sec. 15]. Now, to prove the second equality, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{\lambda_n^p} &= \frac{1}{\lambda_0^p} + \sum_{m=1}^{\infty} \frac{1}{\lambda_{2m-1}^p} + \sum_{m=1}^{\infty} \frac{1}{\lambda_{2m}^p} \\
&= \left(\frac{-1}{4\alpha^2}\right)^p + \frac{1}{\pi^{2p}} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^{2p}} + \frac{1}{4^p} \sum_{m=1}^{\infty} \frac{1}{x_m^{2p}} \\
&= \frac{(-1)^p}{4^p \alpha^{2p}} + \frac{1}{\pi^{2p}}\left(1 - \frac{1}{2^{2p}}\right) \sum_{m=1}^{\infty} \frac{1}{m^{2p}} + \frac{1}{4^p} S_{2p} \\
&= \frac{1}{4^p}\left\{S_{2p} + \frac{(-1)^p}{\alpha^{2p}} + \frac{4^p - 1}{\pi^{2p}} \sum_{m=1}^{\infty} \frac{1}{m^{2p}}\right\} \\
&= \frac{1}{4^p}\left(S_{2p} + \frac{(-1)^p}{\alpha^{2p}}\right) + \frac{4^p - 1}{2(2p)!}|B_{2p}|,
\end{aligned}$$

where we used the following well-known formula first obtained by Euler (see, e.g., [11, 12, 13] to derive the last equality:

$$\sum_{m=1}^{\infty} \frac{1}{m^{2p}} = \frac{(2\pi)^{2p}}{2(2p)!}|B_{2p}|.$$

Now, to prove the recursion formula (16), we follow Radoux [20] again. Taking the logarithm of the product formula (12) followed by differentiation with respect to x, we have

$$\frac{x\cos x}{\cos x + x\sin x} = \frac{\frac{2x}{\alpha}}{1+\frac{x^2}{\alpha^2}} + \sum_{m=1}^{\infty} \frac{\frac{-2x}{x_m^2}}{1-\frac{x^2}{x_m^2}},$$

which leads to

$$\frac{1}{2}\cos x = (\cos x + x\sin x)\cdot\left\{\frac{1}{\alpha^2}\frac{1}{1+\frac{x^2}{\alpha^2}} - \sum_{m=1}^{\infty}\frac{1}{x_m^2}\frac{1}{1-\frac{x^2}{x_m^2}}\right\}.$$

Expanding each term into the Maclaurin series or the geometric series, we have

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \left(\sum_{k=0}^{\infty}\frac{(-1)^{k-1}(2k-1)}{(2k)!}x^{2k}\right)\cdot\left(\sum_{\ell=0}^{\infty}\left(\frac{(-1)^{\ell}}{\alpha^{2\ell+2}} - S_{2\ell+2}\right)x^{2\ell}\right).$$

Hence, comparing the coefficients of the x^{2n} term, we have:

$$\begin{aligned}\frac{(-1)^n}{2(2n)!} &= \sum_{k=0}^{n}\frac{(-1)^{n-k-1}(2n-2k-1)}{(2n-2k)!}\left(\frac{(-1)^k}{\alpha^{2k+2}} - S_{2k+2}\right)\\ &= \sum_{\ell=1}^{n+1}\frac{(-1)^{n-\ell+1}(2(n-\ell+1)-1)}{(2(n-\ell+1))!}\left(S_{2\ell} + \frac{(-1)^{\ell}}{\alpha^{2\ell}}\right) \quad \text{via setting } \ell = k+1,\end{aligned}$$

which is (16). □

Let $A_p := \sum_{n=0}^{\infty}\frac{1}{\lambda_n^p}$. Here are the first few sums:

$$A_1 = 0; \quad A_2 = \frac{1}{24}; \quad A_3 = -\frac{1}{240}, \ldots$$

5. The Generating Function and Obtaining Recursive Formulas All at Once

In this section, we show how to obtain the recursion formulas for the A_ℓ's at once and without recourse to the knowledge of Bernoulli numbers. The main result is the following theorem.

Theorem 4. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be the eigenvalues of the boundary problem in Corollary 1, and let $K_p(x,y)$ be the pth iterated kernel of $K(x,y) = -|x-y|/2$. Then,*

$$A_p = \sum_{n=0}^{\infty} \frac{1}{\lambda_n^p} = \int_0^1 K_p(x,x)\,\mathrm{d}x$$

satisfies the recursion formula:

$$4A_{n+1} + \sum_{k=1}^{n-1} (-1)^k \left(\frac{2}{(2k)!} - \frac{1}{(2k-1)!} \right) A_{n-k+1} = \frac{(-1)^{n+1} n}{(2n+1)!}, \quad n = 1, 2, \ldots,$$

with $A_1 = 0$.

Proof. From the statement of Corollary 1 and (12), it is clear that

$$(\cos x + x \sin x) \cdot \cos x = \left(1 + \frac{x^2}{\alpha^2}\right) \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{x_m^2}\right)$$

where $\lambda_0 = -4\alpha^2$ as defined above and where we set $x_k = \sqrt{\lambda_k}/2$, for $k = 1, 2, \ldots$. One can again justify this product formula as in Knopp [18, Chap. 1] or any standard Complex Analysis textbook which treats the Weierstrass Factor Theorem.

In terms of the eigenvalues one has, after some trigonometric substitutions,

$$\frac{1+\cos x}{2} + \frac{x}{4} \sin x = \left(1 - \frac{x^2}{\lambda_0}\right) \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{\lambda_m}\right). \tag{17}$$

Expanding the LHS into a Maclaurin series and equating lead to

$$1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{2(2k)!} - \frac{1}{4(2k-1)!} \right) x^{2k} = 1 - \left(\sum_{k=0}^{\infty} \frac{1}{\lambda_k} \right) x^2 + \left(\sum_{j,k=0}^{\infty} \frac{1}{\lambda_k \lambda_j} \right) x^4 - \ldots.$$

With $\alpha_k := (-1)^k \left(\frac{1}{2(2k)!} - \frac{1}{4(2k-1)!}\right)$ denoting the coefficients of the Maclaurin expansion, one can recourse to Speigel's formulas [19, 24]

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{\lambda_k} &= -\alpha_1 \\
\sum_{k=0}^{\infty} \frac{1}{\lambda_k^2} &= \alpha_1^2 - 2\alpha_2 \\
\sum_{k=0}^{\infty} \frac{1}{\lambda_k^3} &= 3\alpha_1\alpha_2 - 3\alpha_3 - \alpha_1^3 \\
\sum_{k=0}^{\infty} \frac{1}{\lambda_k^4} &= \alpha_1^4 - 4\alpha_1^2\alpha_2 + 2\alpha_2^2 + 4\alpha_1\alpha_3 - 4\alpha_4
\end{aligned}$$

to obtain, as above,

$$\begin{aligned}
A_1 = \sum_{k=0}^{\infty} \frac{1}{\lambda_k} &= 0 \\
A_2 = \sum_{k=0}^{\infty} \frac{1}{\lambda_k^2} &= \frac{1}{24} \\
A_3 = \sum_{k=0}^{\infty} \frac{1}{\lambda_k^3} &= -\frac{1}{240} \\
A_4 = \sum_{k=0}^{\infty} \frac{1}{\lambda_k^4} &= \frac{41}{40320}.
\end{aligned}$$

One can generate a recursion formula for the A_k sequence employing what Ismail and Muldoon [15] call, properly, the "Euler-Rayleigh" technique. The logarithmic derivative of the entire function $f(z) = \frac{1+\cos z}{2} + \frac{z}{4}\sin z$ appearing in (17) gives,

$$\frac{-\frac{\sin z}{4} + \frac{z\cos z}{4}}{\frac{1+\cos z}{2} + \frac{z\sin z}{4}} = -\frac{2z}{\lambda_0 - z^2} - 2\sum_{k=1}^{\infty} \frac{z}{\lambda_k - z^2}$$

Or, substituting $\lambda_0 = -4\alpha^2$ and $\lambda_k = 4x_k^2$, and after some manipulation,

$$\frac{-\frac{\sin 2z}{4} + \frac{z\cos 2z}{2}}{\frac{1+\cos 2z}{2} + \frac{z\sin 2z}{2}} = \frac{z}{\alpha^2 + z^2} - \sum_{k=1}^{\infty} \frac{z}{x_k^2 - z^2} =: -zG(z). \tag{18}$$

The function

$$G(t) = -\frac{1}{\alpha^2 + t^2} + \sum_{k=1}^{\infty} \frac{1}{x_k^2 - t^2}$$

is known as the generating function of A_ℓ. That is, one can obtain the needed recursion formula for this sequence from consideration of this function. To simplify notation, we let $M_\ell := 4^{\ell+1} A_{\ell+1}$. It is then clear that

$$M_{\ell-1} = \frac{(-1)^\ell}{\alpha^{2\ell}} + \sum_{m=1}^{\infty} \frac{1}{x_m^{2\ell}}.$$

Moreover, a straightforward calculation leads to

$$\sum_{\ell=0}^{\infty} M_\ell t^{2\ell} = G(t).$$

By (18), one then obtains

$$\frac{\sin 2t}{4t} - \frac{\cos 2t}{2} = \left(\frac{1+\cos 2t}{2} + \frac{t}{2}\sin 2t\right)\left(\sum_{\ell=0}^{\infty} M_\ell t^{2\ell}\right).$$

Expanding into power series leads to

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n n}{(2n+1)!} t^{2n} = \left(1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{2^{2k-1}}{(2k)!} - \frac{2^{2k-2}}{(2k-1)!}\right) t^{2k}\right)\left(\sum_{\ell=0}^{\infty} M_\ell t^{2\ell}\right).$$

From which one obtains $M_0 = 0$, and

$$\sum_{k+\ell=n} (-1)^k \left(\frac{2^{2k-1}}{(2k)!} - \frac{2^{2k-2}}{(2k-1)!}\right) M_\ell = \frac{(-1)^{n+1} 4^n n}{(2n+1)!}.$$

In terms of the A_ℓ's one has, $A_1 = 0$, as before,

$$4A_{n+1} + \sum_{k=1}^{n-1} (-1)^k 4^{-k+1} \left(\frac{2^{2k-1}}{(2k)!} - \frac{2^{2k-2}}{(2k-1)!}\right) A_{n-k+1} = \frac{(-1)^{n+1} n}{(2n+1)!},$$

which is the same as the desired statement of the theorem. □

Remark 5. We note that the recursion generates the following values $A_2 = 1/24$, $A_3 = -1/240$, $A_4 = 41/40320$, $A_5 = -107/725760$, etc., corresponding to what we obtained differently in Section 4.

Remark 6. As in [15], one can exploit the formulas generated for the A_ℓ's to obtain

$$-|A_{2m-1}|^{-1/(2m-1)} < \lambda_0 < -A_{2m}^{-1/(2m)} \tag{19}$$

and

$$A_{2m}/A_{2m+1} < \lambda_0 < A_{2m-1}/A_{2m}. \tag{20}$$

for $m = 1,2,3,\ldots$. These inequalities provide strict improvable bounds for the unique negative root of the transcendental equation (4) and another way of obtaining it.

6. Higher Dimensional Considerations

One of the motivations that led to the non-local BVP considered in [1] is that one is able to read the spectral data (eigenvalues, eigenfunctions) by discretizing then computing integrals involving the kernel $K(\mathbf{x},\mathbf{y})$ over a domain $\Omega \subset \mathbb{R}^d$ without imposing conditions on $\partial\Omega$. For the two dimensional case, $K(\mathbf{x},\mathbf{y})$ takes the form of a logarithmic kernel

$$K(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{y}|. \tag{21}$$

Troutman [28] gave an analytical proof for the existence of at most one negative eigenvalue and gave an upper bound estimate for it in terms of the area and transfinite diameter of Ω. (The transfinite diameter is a measure of the compactness of a domain; see [28] for the definition.) In [29], Kac offers a probabilistic proof of this fact (see also [30] and the generalization in [31]). With A_p denoting the power sum in (6) and the iterated integrals computed numerically, (19) and (20) provide a practical and improvable means of computing this negative eigenvalue for a specific domain. When the transfinite diameter of Ω is less than or equal to one, this negative eigenvalue disappears. This is the case of the unit disk. In [1], it was found that the eigenvalues of the nonlocal BVP associated with the kernel (21) are of two types, $j_{0,n}^2$, with multiplicity 3, and $j_{m-1,n}^2$ with multiplicity 2, for $m = 2,3,\ldots$, and $n = 1,2,\ldots$. Based on the values of $\sigma_{2\ell}(\nu)$, one can generate for the first few power sums. While $\sum_{k=1}^{\infty} 1/\lambda_k$ is easily seen to diverge,

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2} = 3\sigma_4(0) + 2\sum_{\nu=1}^{\infty}\sigma_4(\nu) = \frac{3}{32} + \frac{1}{8}\left(\frac{\pi^2}{6} - \frac{3}{2}\right).$$

Similarly

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^\ell} = 3\sigma_{2\ell}(0) + 2\sum_{\nu=1}^{\infty} \sigma_{2\ell}(\nu)$$

can be carried out explicitly for $\ell = 3, 4, \ldots$, but there may not be an obvious recursion scheme.

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