# A BEREZIN-LI-YAU TYPE INEQUALITY FOR THE FRACTIONAL LAPLACIAN ON A BOUNDED DOMAIN

SELMA YILDIRIM YOLCU AND TÜRKAY YOLCU



Abstract. A Berezin-Li-Yau type inequality for $(-\Delta)^{\alpha/2}|_{\Omega}$, the fractional Laplacian operators restriced to a bounded domain $\Omega \subset \mathbb{R}^d$ for $\alpha \in (0,2]$, $d \geq 2$, has not been known so far. First we positively answer this question. Second, we provide an improvement to this inequality consistent with the work in [13, 14] by using a pure analytical approach.

## 1. Introduction

The purpose of this article is two-fold: First, we state a Berezin-Li-Yau type inequality for $(-\Delta)^{\alpha/2}|_{\Omega}$, the fractional Laplacian operators restricted to $\Omega$, where $\alpha \in (0,2]$ and $\Omega$ is a bounded domain in $\mathbb{R}^d$ with $d \geq 2$. The second main goal of this article, aligned with the work in [13, 14], is to provide a refinement of the Berezin-Li-Yau type inequality we obtain.

Throughout this article, unless otherwise clearly indicated, we also assume that $|\Omega|$ denotes the volume of the set $\Omega$, $t \mapsto \chi_A(t)$ denotes the characteristic function defined to be 1 when $t \in A$ and 0 when $t \in A^c$, the complement of $A$. Moreover, we assume that $B_R := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \leq R\}$ denotes the ball of radius $R$ in $\mathbb{R}^d$, and $w_d$ denotes the volume of $d$ dimensional unit ball $B_1$ in $\mathbb{R}^d$ given by

$$w_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}.$$

In this setting, the surface area of the unit ball $B_1$ in $\mathbb{R}^d$ is $dw_d$.

One defines the fractional Laplacian $(-\Delta)^{\alpha/2}$ by

$$(-\Delta)^{\alpha/2} u(\mathbf{x}) = p.v. \int_{\mathbb{R}^d} \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{d+\alpha}} \, d\mathbf{y}, \tag{1.1}$$

where $p.v.$ denotes the principal value and $u : \mathbb{R}^d \to \mathbb{R}$. With the aid of the Fourier transform, fractional Laplacians restricted to $\Omega$, denoted by $(-\Delta)^{\alpha/2}|_{\Omega}$, can be conveniently defined to be a pseudo differential operator as follows

$$(-\Delta)^{\alpha/2}|_{\Omega} u := \mathcal{F}^{-1}[|\xi|^{\alpha} \mathcal{F}[u \chi_{\Omega}]]. \tag{1.2}$$

Here, $\mathcal{F}[u]$ denotes the Fourier transform of a function $u : \mathbb{R}^d \to \mathbb{R}$ and is defined by

$$\mathcal{F}[u](\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot\xi} u(\mathbf{x}) \, d\mathbf{x}.$$

We refer the reader to the book [10] or the article [15] for the proof of the equivalence between (1.1) and (1.2). Moreover, the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ can be considered as the infinitesimal generator of the symmetric $\alpha$-stable process, which is defined as follows:

**Definition 1.1.** ([1],[2]) A symmetric $\alpha$-stable process of order $\alpha \in (0,2]$ is a stochastic process with stationary and independent increments and with the transition density (i.e., convolution kernel) $p_\alpha(t,\mathbf{x},\mathbf{y}) = p_\alpha(t,\mathbf{x}-\mathbf{y})$ given by

$$\int_{\mathbb{R}^d} e^{i\xi\cdot\mathbf{y}} p_\alpha(t,\mathbf{y})\, d\mathbf{y} = e^{-t|\xi|^\alpha},$$

with $t>0$ and when $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$.

Two important examples of symmetric $\alpha$-stable processes are Brownian motion, which is obtained by setting $\alpha = 2$, and the Cauchy process, which is obtained by setting $\alpha = 1$. Moreover, the transition density in the case of the Brownian motion is given by

$$p_2(t,\mathbf{x},\mathbf{y}) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right), \qquad t>0, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d.$$

and the transition density in the case of the Cauchy process is represented by

$$p_1(t,\mathbf{x},\mathbf{y}) = \frac{c_d t}{(t^2 + |\mathbf{x}-\mathbf{y}|^2)^{(d+1)/2}}, \qquad t>0, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d$$

where $c_d = \pi^{-\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)$ is the semiclassical constant that appears in the Weyl estimate for the eigenvalues of the Laplacian. Therefore, the infinitesimal generator of the Brownian motion for paths that are killed upon leaving the domain $\Omega$ is the Dirichlet Laplacian, and the generator of the Cauchy process with the corresponding killing condition on $\partial\Omega$ is $(-\Delta)^{1/2}|_\Omega$. For more interesting results involving stable processes and Cauchy processes, please refer to the papers [1], [2], [3] ,[5] and references therein.

Let $\varrho_j$ and $u_j$ denote the $j$th eigenvalue and the corresponding normalized eigenvector of $(-\Delta)^{\alpha/2}|_\Omega$, respectively. Eigenvalues $\varrho_j$ (including multiplicities) satisfy

$$0 < \varrho_1 < \varrho_2 \le \varrho_3 \le \cdots \le \varrho_j \le \cdots \to \infty.$$

In Section 2, we prove the following Berezin-Li-Yau type inequality:

**Theorem 1.2.** *(Berezin-Li-Yau type inequality for the fractional Laplacian on $\Omega$) The eigenvalues $\varrho_j$ of $(-\Delta)^{\alpha/2}|_\Omega$ satisfy*

$$\sum_{j=1}^{k} \varrho_j \ge \frac{d}{d+\alpha}(4\pi)^{\frac{\alpha}{2}} \left(\frac{\Gamma(1+d/2)}{|\Omega|}\right)^{\frac{\alpha}{d}} k^{1+\frac{\alpha}{d}}. \tag{1.3}$$

These types of bounds have already been of more interest and investigated earlier in the context of Dirichlet Laplacian, which can be regarded as the fractional Laplacian when $\alpha = 2$, and Klein- Gordon operators $(-\Delta)^{1/2}$, $\alpha = 1$, restricted to $\Omega$. The work presented here in Section 2 is inspired by the work in [6] and [12]. Indeed, in [12] , P. Li and S.-T. Yau proved the following inequality for the eigenvalues $\lambda_j$ of the Dirichlet Laplacian on $\Omega$:

**Theorem 1.3.** *(Berezin-Li-Yau inequality) The eigenvalues $\lambda_j$ of the Dirichlet Laplacian on $\Omega$ satisfy*

$$\sum_{j=1}^{k}\lambda_j \geq \frac{d}{d+2}(4\pi)\left(\frac{\Gamma(1+\frac{d}{2})}{|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}}. \tag{1.4}$$

We prefer to call this inequality as Berezin-Li-Yau inequality instead of Li-Yau inequality because (1.4) can be obtained by a Legendre transform of an earlier result by Berezin[4] as it was first mentioned [11].

As for the case when $\alpha = 1$, E. Harrell and the first author proved an analogue of the Berezin-Li-Yau inequality for the eigenvalues of the Klein-Gordon operators $\sqrt{-\Delta}$ restricted to $\Omega$ in [6]:

**Theorem 1.4.** *(Analogue of the Berezin-Li-Yau inequality for $\alpha = 1$) Eigenvalues $\beta_k$ the Klein-Gordon operators $\sqrt{-\Delta}$ restricted to $\Omega$ satisfy*

$$\sum_{j=1}^{k}\beta_j \geq \frac{d}{d+1}(4\pi)^{\frac{1}{2}}\left(\frac{\Gamma(1+d/2)}{|\Omega|}\right)^{\frac{1}{d}} k^{1+\frac{1}{d}}. \tag{1.5}$$

Clearly, the results given in (1.4) and (1.5) can be regarded as special cases of (1.3).

Although it is not closely related to the work presented here, it is worth pointing out the result appeared in [8] in which A. Ilyin proved a type of Berezin-Li-Yau inequality for the eigenvalues of the Stokes operators. For details see [8].

In Section 3 we prove the following improvement:

**Theorem 1.5.** *(Refinement of the Berezin-Li-Yau Inequality in the Case of Fractional Laplacian $(-\Delta)^{\alpha/2}|_\Omega$ restricted to $\Omega$)*

$$\begin{aligned}\sum_{j=1}^{k}\varrho_j \;\geq\; & \frac{d(4\pi)^{\frac{\alpha}{2}}}{d+\alpha}\left(\frac{\Gamma\left(1+d/2\right)}{|\Omega|}\right)^{\frac{\alpha}{d}} k^{1+\frac{\alpha}{d}} \\ & + \frac{\ell|\Omega|^{1+\frac{2-\alpha}{d}}}{4I(\Omega)\big(4\pi\Gamma(1+d/2)\big)^{\frac{2-\alpha}{d}}} k^{1-\frac{2-\alpha}{d}}.\end{aligned} \tag{1.6}$$

*where $\ell$ is given by*

$$\ell = \min\left\{\frac{1}{12}, \frac{4\alpha d\pi^2}{(2d+2-\alpha)w_d^{\frac{4}{d}}}\right\}. \tag{1.7}$$

Improvements to the Berezin-Li-Yau inequality in (1.4) in the case of Dirichlet Laplacian have appeared recently, for example see [9],[13] and [16]. In particular, in [13], A.D. Melas proved that

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2}(4\pi)\left(\frac{\Gamma(1+d/2)}{|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}} + M_d k \frac{|\Omega|}{I(\Omega)}, \tag{1.8}$$

where the constant $M_d$ depends only on the dimension and $I(\Omega)$ denotes the second moment of inertia.

Melas's work in [13] not only motivated our work in Section 3 but also motivated the works in [7], [14]. We are particulary interested in the result appeared in [14] because our improvement is a generalization of the results in [13] and [14]. This approach uses some elementary techniques and we follow the basic strategy used there with some important differences of detail.

Before we prove our main result, we need to make some key observations. Since the set of eigenfunctions $\{u_j\}_{j=1}^{\infty}$ is an orthonormal set in $L^2(\Omega)$, the set of $\{\hat{u}_j\}_{j=1}^{\infty}$ also forms an orthonormal set in $L^2(\mathbb{R}^d)$ by using Plancherel's theorem. Set

$$U_k(\xi) := \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2 = \sum_{j=1}^{k} \left| \frac{1}{(2\pi)^{d/2}} \int_{\Omega} e^{-i\mathbf{x}\cdot\xi} u_j(\mathbf{x})\, d\mathbf{x} \right|^2. \tag{1.9}$$

Notice that the integral is taken over $\Omega$ instead of $\mathbb{R}^d$ because the support of $u$ is $\Omega$. Interchanging the sum and integral and using $\|\hat{u}_j\|_2 = 1$, we obtain

$$\int_{\mathbb{R}^d} U_k(\xi) d\xi = k. \tag{1.10}$$

Also notice that since $u_j$s form an orthonormal set in $L^2(\Omega)$, by Bessel's inequality, we get an upper bound for $U_k$:

$$U_k(\xi) \leq \frac{1}{(2\pi)^d} \int_{\Omega} \left| e^{-i\mathbf{x}\cdot\xi} \right|^2 d\mathbf{x} = \frac{|\Omega|}{(2\pi)^d}. \tag{1.11}$$

## 2. Proof of Theorem 1.2

The clincher in proving the Berezin-Li-Yau type inequality (1.3) in the case of fractional Laplacian is the following lemma, which is the adaptation of a result given in [6] ( i.e., generalization of a lemma that was attributed in [12] to Hörmander), establishing a connection between $L^\infty$ and $L^1$ norms of a function.

**Lemma 2.1.** *Let $f : \mathbb{R}^d \to [0, \infty)$ be a real valued nonnegative function in $L^\infty(\mathbb{R}^d)$. Assume that there exists a number $M > 0$ such that*

$$\int_{\mathbb{R}^d} |\xi|^\alpha f(\xi) d\xi \leq M. \tag{2.1}$$

*Then, $f \in L^1(\mathbb{R}^d)$ and*

$$\|f\|_{L^1(\mathbb{R}^d)} \leq \left(\frac{\|f\|_{L^\infty(\mathbb{R}^d)} \pi^{\frac{d}{2}}}{\Gamma(1+d/2)}\right)^{\frac{\alpha}{d+\alpha}} \left(\frac{d+\alpha}{d} M\right)^{\frac{d}{d+\alpha}}. \tag{2.2}$$

*Proof.* Let $g(\xi) = \|f\|_{L^\infty(\mathbb{R}^d)}\chi_{B_R}(\xi)$. Also, set

$$R = \left(\frac{M(d+\alpha)}{\|f\|_{L^\infty(\mathbb{R}^d)} d w_d}\right)^{\frac{1}{d+\alpha}}.$$

It is not difficult to see that $(|\xi|^\alpha - R^\alpha)(f(\xi) - g(\xi)) \geq 0$. Notice that this can be written as $R^\alpha(f(\xi) - g(\xi)) \leq |\xi|^\alpha(f(\xi) - g(\xi))$. Integrating both sides, using (2.1) and the definition of $g(\xi)$, we obtain that

$$R^\alpha \int_{\mathbb{R}^d} (f(\xi) - g(\xi))\, d\xi \leq \int_{\mathbb{R}^d} |\xi|^\alpha (f(\xi) - g(\xi))\, d\xi \leq 0.$$

Therefore, we are left with

$$\begin{aligned}
\int_{\mathbb{R}^d} f(\xi) d\xi &\leq \int_{\mathbb{R}^d} g(\xi)\, d\xi \\
&= \|f\|_{L^\infty(\mathbb{R}^d)} \int_{B_R} d\xi \\
&= \|f\|_{L^\infty(\mathbb{R}^d)} \left( d w_d \frac{R^d}{d} \right) \\
&= \|f\|_{L^\infty(\mathbb{R}^d)} w_d \left(\frac{M(d+\alpha)}{\|f\|_{L^\infty(\mathbb{R}^d)} d w_d}\right)^{\frac{d}{d+\alpha}} \\
&= \left(\frac{\|f\|_{L^\infty(\mathbb{R}^d)} \pi^{\frac{d}{2}}}{\Gamma(1+d/2)}\right)^{\frac{\alpha}{d+\alpha}} \left(\frac{d+\alpha}{d} M\right)^{\frac{d}{d+\alpha}}
\end{aligned}$$

which concludes the proof of (2.2). □

Now, we are ready to prove our first result.

***Proof of Theorem 1.2:*** Since the support of $u_i$ is $\Omega$, we observe that

$$\begin{aligned}
\varrho_j &= \langle u_j, (-\Delta)^{\alpha/2}|_\Omega u_j \rangle \\
&= \langle u_j, \mathcal{F}^{-1}[|\xi|^\alpha \mathcal{F}[u_j]] \rangle \\
&= \int_{\mathbb{R}^d} |\xi|^\alpha |\hat{u}_j(\xi)|^2\, d\xi. \qquad (2.3)
\end{aligned}$$

Now, define

$$f(\xi) := U_k(\xi) = \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2.$$

Then

$$k = \int_{\mathbb{R}^d} f(\xi)\, d\xi, \qquad \text{and} \qquad \sum_{j=1}^{k} \varrho_j = \int_{\mathbb{R}^d} |\xi|^\alpha f(\xi)\, d\xi = M.$$

Thus, (2.2) yields

$$k \leq \left( \frac{\pi^{\frac{d}{2}} \|f\|_{L^\infty(\mathbb{R}^d)}}{\Gamma(1+d/2)} \right)^{\frac{\alpha}{d+\alpha}} \left( \frac{d+\alpha}{d} \left( \sum_{j=1}^{k} \varrho_j \right) \right)^{\frac{d}{d+\alpha}}. \tag{2.4}$$

After rearranging the terms, we obtain

$$\sum_{j=1}^{k} \varrho_j \geq \frac{d}{d+\alpha} \left( \frac{\Gamma(1+d/2)}{\pi^{\frac{d}{2}} \|f\|_{L^\infty(\mathbb{R}^d)}} \right)^{\frac{\alpha}{d}} k^{1+\frac{\alpha}{d}}. \tag{2.5}$$

Moreover, we already have an estimate for $\|f\|_{L^\infty(\mathbb{R}^d)}$. Indeed, in view of (1.11), for any $\xi \in \mathbb{R}^d$ we have

$$f(\xi) = \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2 \leq \frac{|\Omega|}{(2\pi)^d} \tag{2.6}$$

After substituting this into (2.5) and simplifying the expressions, we obtain (1.3). □

## 3. Proof of Theorem 1.5

First, we find an estimate for $|\nabla U_k|$. Notice that

$$\sum_{j=1}^{k} |\nabla \hat{u}_j(\xi)|^2 \leq \frac{1}{(2\pi)^d} \int_\Omega \left| i\mathbf{x} e^{-i\mathbf{x}\cdot\xi} \right|^2 \, d\mathbf{x} = \frac{I(\Omega)}{(2\pi)^d}, \tag{3.1}$$

where $I(\Omega)$, the moment of inertia, is defined by

$$I(\Omega) = \min_{\mathbf{y} \in \mathbb{R}^d} \int_\Omega |\mathbf{x} - \mathbf{y}|^2 \, d\mathbf{x}.$$

After translation, we may assume that

$$I(\Omega) = \int_\Omega |\mathbf{x}|^2 \, d\mathbf{x}.$$

Furthermore, by applying Hölder's inequality, and invoking (1.11) and (3.1), we obtain that for every $\xi$,

$$\begin{aligned} |\nabla U_k(\xi)| &\leq 2 \left( \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2 \right)^{1/2} \left( \sum_{j=1}^{k} |\nabla \hat{u}_j(\xi)|^2 \right)^{1/2} \\ &\leq 2(2\pi)^{-d} \sqrt{|\Omega| I(\Omega)} \qquad (3.2) \\ &:= m. \qquad (3.3) \end{aligned}$$

Also, we may find a lower bound for $m$. Indeed, let $R$ be the number such that $|\Omega| = w_d R^d$. Then,

$$I(\Omega) \geq \int_{B_R} |\mathbf{x}|^2 \, d\mathbf{x} = \frac{d w_d}{d+2} R^{d+2} = \frac{d}{d+2} w_d^{-\frac{2}{d}} |\Omega|^{\frac{d+2}{d}}.$$

Hence, we have

$$\begin{aligned} m &= 2(2\pi)^{-d}\sqrt{|\Omega| I(\Omega)} \\ &\geq 2(2\pi)^{-d}\sqrt{\frac{d}{d+2} w_d^{-2/d} |\Omega|^{(2d+2)/d}} \\ &\geq (2\pi)^{-d} w_d^{-1/d} |\Omega|^{(d+1)/d}. \end{aligned} \tag{3.4}$$

Let $U_k^*(\xi)$ denote the decreasing radial rearrangement of $U_k(\xi)$. Therefore, by approximating $U_k$, we may assume that there exists a real valued absolutely continuous function $\zeta : [0,\infty) \to [0, |\Omega|/(2\pi)^d]$ such that $U_k^*(\xi) = \zeta(|\xi|)$. Also, we define the distribution function $\mu_k$ by

$$\mu_k(r) := |\{U_k(\xi) > r\}| = |\{U_k^*(\xi) > r\}|.$$

Then, $\mu_k(\zeta(t)) = w_d t^d$. Invoking the coarea formula in view of (1.11), we have

$$\mu_k(r) = \int_r^\infty \int_{\{U_k^{-1}(s)\}} \frac{1}{|\nabla U_k|} d\sigma\, ds = \int_r^{|\Omega|/(2\pi)^d} \int_{\{U_k = s\}} \frac{1}{|\nabla U_k|} d\sigma\, ds,$$

where $\sigma$ is the $(d-1)$ dimensional Hausdorff measure. The isoperimetric inequality,

$$\sigma(\partial S) \geq d w_d^{1/d} |\bar{S}|^{(d-1)/d}, \qquad S \subset \mathbb{R}^d,$$

gives

$$\begin{aligned} \frac{d w_d t^{d-1}}{\zeta'(t)} &= \mu'(\zeta(t)) \\ &= -\int_{\{U_k = \zeta(t)\}} \frac{1}{|\nabla U_k|} d\sigma \\ &\leq -\frac{1}{m}\sigma(\{U_k = \zeta(t)\}) \\ &\leq -\frac{1}{m} d w_d^{1/d} \mu_k(\zeta(t))^{(d-1)/d} \\ &= -\frac{1}{m} d w_d t^{d-1}. \end{aligned}$$

where $m = 2(2\pi)^{-d}\sqrt{|\Omega| I(\Omega)}$ due to (3.2). This inequality together with $\zeta' < 0$ simply yields

$$0 \leq -\zeta'(t) \leq m,$$

which is the required assumption in Lemma 3.3.

The key point in proving the improvement is to use the following lemma suitably:

**Lemma 3.1.** *For $t > 0$, $s > 0$, $2 \leq d \in \mathbb{N}$, $0 < \alpha \leq 2$ we have the following inequality:*

$$t^{d+\alpha} \geq \frac{d+\alpha}{d} t^d s^\alpha - \frac{\alpha}{d} s^{d+\alpha} + \frac{\alpha}{d} s^{d+\alpha-2}(t-s)^2 \tag{3.5}$$

*Proof.* We will sketch the proof as it is not elementary. First, let's see that

$$dz^{d+\alpha} - (d+\alpha)z^d + \alpha - \alpha(z-1)^2 \geq 0. \tag{3.6}$$

Indeed, let $\alpha = \frac{m}{n}$ be a rational number with $\gcd(m,n) = 1$ and $m, n \in \mathbb{N}$. It is not difficult to see that

$$dz^{d+\frac{m}{n}} - \left(d + \frac{m}{n}\right) z^d + \frac{m}{n} - \frac{m}{n}(z-1)^2 = \frac{m}{n} z \left(z^{\frac{1}{n}} - 1\right)^2 [A + B + C],$$

where

$$A = \sum_{k=1}^{n} 2kz^{\frac{k-1}{n}} \geq 0,$$

$$B = \sum_{k=1}^{(d-2)n} (2n+k) z^{1+\frac{k-1}{n}} \geq 0,$$

$$C = \sum_{k=1}^{m-1} dn \left(1 - \frac{k}{m}\right) z^{d-1+\frac{(k-1)}{n}} \geq 0.$$

To prove (3.6) for a real number $\alpha$ define $f(\alpha) := dz^{d+\alpha} - (d+\alpha)z^d + \alpha - \alpha(z-1)^2$. Thus, we have $f(m/n) \geq 0$ for every $m, n \in \mathbb{N}$. By the density of rational numbers in real numbers, for a real number $\alpha$, we can find a sequence $\{\alpha_{m,n}\}$ of rational numbers such that $\alpha_{m,n} \to \alpha$ as $n, m \to \infty$. Since $f$ is a continuous function, we therefore have

$$f(\alpha) = f\left(\lim_{n,m\to\infty} \alpha_{n,m}\right) = \lim_{n,m\to\infty} f(\alpha_{n,m}) \geq 0.$$

This proves the inequality stated in (3.6) for any real number $\alpha > 0$, particularly, $0 < \alpha \leq 2$. Now, let $z = t/s$. Multiplying (3.6) through by $\frac{1}{d}s^{d+\alpha}$ and rearranging the terms, we deduce the inequality stated in (3.5). □

The following result is elementary but very crucial because it helps us make a connection between two integrals defined in (3.15).

**Lemma 3.2.** *Suppose that* $v : [0, \infty) \to [0, 1]$ *such that*

$$0 \leq v \leq 1 \quad \textit{and} \quad \int_0^\infty v(t)\, dt = 1. \tag{3.7}$$

*Then, there exists* $\delta \geq 0$ *so that*

$$\int_\delta^{\delta+1} t^d\, dt = \int_0^\infty t^d v(t)\, dt \tag{3.8}$$

*Moreover, we have*

$$\int_\delta^{\delta+1} t^{d+\alpha}\, dt \leq \int_0^\infty t^{d+\alpha} v(t)\, dt \tag{3.9}$$

*Proof.* Define $f : [0,\infty) \to (0,\infty)$ by

$$f(z) = \int_z^{z+1} t^d \, dt.$$

Observe that

$$\big(t^d - 1\big)\big(v(t) - \chi_{[0,1]}(t)\big) \geq 0 \quad \text{on} \quad [0,\infty). \tag{3.10}$$

Integrating (3.10) on $[0,\infty)$ gives

$$\int_0^\infty t^d v(t)\, dt \geq 1/(d+1) = f(0).$$

Since $f$ is continuous and non-decreasing and $f(z) \to \infty$ as $z \to \infty$, by the Intermediate Value Theorem, there exists $\delta \geq 0$ such that

$$f(\delta) = \int_0^\infty t^d v(t)\, dt.$$

This proves (3.8). To see the proof of (3.9) we consider the following function

$$\Lambda(t) = t^{d+\alpha} - c_1 t^d + c_2$$

where $c_1 \geq 0$ and $c_2 \geq 0$ are chosen so that $\Lambda(\delta) = 0$ and $\Lambda(\delta+1) = 0$ and $\Lambda$ remains negative on $(\delta, \delta+1)$ and positive on $[0,\infty)\backslash[\delta,\delta+1]$. Notice that

$$\Lambda(t)\,\big(\chi_{[\delta,\delta+1]}(t) - v(t)\big) \leq 0 \quad \text{on} \quad [0,\infty). \tag{3.11}$$

Integration of (3.11) on $[0,\infty)$ together with (3.7) gives

$$\int_\delta^{\delta+1} t^{d+\alpha}\, dt \leq \int_0^\infty t^{d+\alpha} v(t)\, dt - c_1 \left( \int_0^\infty t^d v(t)\, dt - \int_\delta^{\delta+1} t^d \, dt \right),$$

which combined with (3.8) yields the required inequality (3.9). This completes the proof. □

In light of Lemma 3.1 and Lemma 3.2, we obtain the following core result for obtaining Theorem 1.5.

**Lemma 3.3.** *Let $d \geq 2$ and $\zeta : [0,\infty) \to [0,\infty)$ be a decreasing, absolutely continuous function. Assume that*

$$0 \leq -\zeta'(t) \leq m, \qquad t \geq 0, \tag{3.12}$$

*where $m$ is given by (3.2). Then, for any $0 < \ell < 1/12$, we have*

$$\begin{aligned} \int_0^\infty t^{d+\alpha-1}\zeta(t)\, dt \ \ &\geq \ \frac{\zeta(0)^{-\frac{\alpha}{d}}}{d+\alpha}\left( d\int_0^\infty t^{d-1}\zeta(t)\, dt \right)^{1+\frac{\alpha}{d}} \\ &\quad + \frac{\ell\zeta(0)^{2+\frac{2-\alpha}{d}}}{m^2 d(d+\alpha)}\left( d\int_0^\infty t^{d-1}\zeta(t)\, dt \right)^{1-\frac{(2-\alpha)}{d}}. \end{aligned} \tag{3.13}$$

*Proof.* Let us first define

$$\theta(t) := \frac{1}{\zeta(0)}\zeta\left(\frac{\zeta(0)}{m}t\right). \tag{3.14}$$

Note that $\theta$ is positive, $\theta(0) = 1$ and $0 \leq -\theta'(t) \leq 1$. To ease the notation, we also set $v(t) := -\theta'(t)$ for $t \geq 0$. Hence, $0 \leq v(t) \leq 1$ for $t \geq 0$ and $\int_0^\infty v(t)\,dt = \theta(0) = 1$. Now, define

$$\kappa := \int_0^\infty t^{d-1}\theta(t)\,dt \qquad \text{and} \qquad \omega := \int_0^\infty t^{d+\alpha-1}\theta(t)\,dt. \tag{3.15}$$

If $\omega = \infty$, there is nothing to prove. Therefore, we suppose that that $\omega < +\infty$. Suppose that $x^{d+\alpha}\theta(t) \to a > 0$ as $t \to \infty$. Then for any $0 < \varepsilon \leq a$ we can find a finite $R > 0$ such that

$$0 < \frac{a-\varepsilon}{t} < t^{d+\alpha-1}\theta(t) < \frac{a+\varepsilon}{t} \tag{3.16}$$

for any $t > R$. This would yield a contradiction, because $0 \leq \omega < \infty$ and

$$\infty = \int_R^\infty t^{d+\alpha-1}\theta(t)\,dt \leq \int_0^\infty t^{d+\alpha-1}\theta(t)\,dt < \infty$$

due to (3.16). Thus, $t^{d+\alpha}\theta(t) \to 0$ as $t \to \infty$. Moreover, it is straightforward to see that $t^d\theta(t) \to 0$ as $t \to \infty$ as well. Thus, using integration by parts, we obtain

$$\int_0^\infty t^d v(x)\,dt = \kappa d, \qquad \text{and} \qquad \int_0^\infty t^{d+\alpha}v(t)\,dt = \omega(d+\alpha).$$

By Lemma 3.2 there exists $\delta \geq 0$ such that

$$\int_\delta^{\delta+1} t^d\,dt = \kappa d \tag{3.17}$$

and

$$\int_\delta^{\delta+1} t^{d+\alpha}dt \leq \int_0^\infty t^{d+\alpha}v(t)\,dt = \omega(d+\alpha). \tag{3.18}$$

Notice that (3.5) gives the key inequality in the proof of this lemma. Indeed, integrating (3.5) in $t$ from $\delta$ to $\delta+1$ we obtain

$$\int_\delta^{\delta+1} t^{d+\alpha}\,dt \geq \frac{d+\alpha}{d}s^\alpha \int_\delta^{\delta+1} t^d\,dt - \frac{\alpha}{d}s^{d+\alpha} + \frac{\alpha}{d}t^{d+\alpha-2}\int_\delta^{\delta+1}(t-s)^2\,dt \tag{3.19}$$

Notice that setting $s = (\kappa d)^{\frac{1}{d}}$ and using (3.17) and (3.18), we obtain that (3.19) yields

$$\omega(d+\alpha) \geq (\kappa d)^{1+\frac{\alpha}{d}} + \frac{\alpha}{d}(\kappa d)^{1+\frac{(\alpha-2)}{d}}\int_\delta^{\delta+1}(t-s)^2\,dt. \tag{3.20}$$

Observe that

$$\int_\delta^{\delta+1}(t-s)^2\,dt \geq \min_{\delta\in\mathbb{R}}\int_\delta^{\delta+1}(t-s)^2\,dt = \int_{s-\frac{1}{2}}^{s+\frac{1}{2}}(t-s)^2\,dt = \frac{1}{12}. \tag{3.21}$$

Therefore, combining (3.20) with (3.21) and then simplifying the terms yields

$$\omega \geq \frac{1}{d+\alpha}(\kappa d)^{1+\frac{\alpha}{d}} + \frac{\alpha}{12d(d+\alpha)}(\kappa d)^{1-\frac{(2-\alpha)}{d}}. \tag{3.22}$$

Note that back substitution of (3.14) gives

$$\int_0^\infty t^p \theta(t)\,dt = \frac{m^{p+1}}{\zeta(0)^{p+2}} \int_0^\infty t^p \zeta(t)\,dt. \tag{3.23}$$

which together with $p = d-1$ and $p = d+\alpha-1$ in (3.15) turns (3.22) into

$$\begin{aligned}\int_0^\infty t^{d+\alpha-1}\zeta(t)\,dt \ \geq\ & \frac{\zeta(0)^{-\frac{\alpha}{d}}}{d+\alpha}\left(d\int_0^\infty t^{d-1}\zeta(t)\,dt\right)^{1+\frac{\alpha}{d}} \\ & + \frac{\zeta(0)^{2+\frac{2-\alpha}{d}}}{12m^2 d(d+\alpha)}\left(d\int_0^\infty t^{d-1}\zeta(t)\,dt\right)^{1-\frac{(2-\alpha)}{d}},\end{aligned}$$

concluding the proof of the lemma. □

As a result of Lemma 3.3 we get the following upshot.

**Lemma 3.4.** *For any $\ell \in (0, 1/12)$, We have the following inequality*

$$\sum_{j=1}^k \varrho_j \geq \frac{d}{d+\alpha} w_d^{-\frac{\alpha}{d}} \zeta(0)^{-\frac{\alpha}{d}} k^{1+\frac{\alpha}{d}} + \frac{\ell}{m^2(d+\alpha)} w_d^{\frac{2-\alpha}{d}} \zeta(0)^{2+\frac{2-\alpha}{d}} k^{1-\frac{2-\alpha}{d}} \tag{3.24}$$

*Proof.* Due to (2.3) we have

$$\varrho_j = \int_{\mathbb{R}^d} |\xi|^\alpha |\hat{u}_j(\xi)|^2\,d\xi.$$

Using this together with the definition of $U_k$ in (1.9), we obtain

$$\sum_{j=1}^k \varrho_j = \int_{\mathbb{R}^d} |\xi|^\alpha U_k(\xi)\,d\xi. \tag{3.25}$$

On the other hand, we observe that

$$k = \int_{\mathbb{R}^d} U_k(\xi) d\xi = \int_{\mathbb{R}^d} U_k^*(\xi) d\xi = d w_d \int_0^\infty t^{d-1}\zeta(t) dt, \tag{3.26}$$

and since the map $\xi \mapsto |\xi|^\alpha$ is radial and increasing, in view of (3.25) we therefore have

$$\sum_{j=1}^k \varrho_j = \int_{\mathbb{R}^d} |\xi|^\alpha U_k(\xi) d\xi \geq \int_{\mathbb{R}^d} |\xi|^\alpha U_k^*(\xi) d\xi = d w_d \int_0^\infty t^{d+\alpha-1}\zeta(t) dt. \tag{3.27}$$

Equations (3.26), (3.27), when combined with Lemma 3.3 provide us with required the inequality in (3.24). □

Now, we see that Theorem 1.5 falls out as a by-product of Lemmas 3.1, 3.2, 3.3 and 3.4.

***Proof of Theorem 1.5:*** Set

$$C_1 := \frac{d}{d+\alpha} w_d^{-\frac{\alpha}{d}} k^{1+\frac{\alpha}{d}}, \qquad C_2 := \frac{1}{m^2(d+\alpha)} w_d^{\frac{2-\alpha}{d}} k^{1-\frac{2-\alpha}{d}}. \tag{3.28}$$

and consider $G : (0,\infty) \to \mathbb{R}$ defined by

$$G(z) = C_1 z^{-\frac{\alpha}{d}} + \ell C_2 z^{2+\frac{(2-\alpha)}{d}}.$$

It is elementary to see that $G$ is decreasing when

$$0 < z \le \left(\frac{\alpha C_1}{(2d+2-\alpha)C_2 \ell}\right)^{\frac{d}{2d+2}}.$$

In other words, $G$ is decreasing in the interval $(0,\tau]$ if

$$\ell \le \frac{(2d+2-\alpha)C_2}{\alpha C_1} \tau^{\frac{2d+2}{d}}.$$

It therefore turns out that $G$ is decreasing on $(0, |\Omega|/(2\pi)^d]$ when

$$\frac{|\Omega|}{(2\pi)^d} \le \left(\frac{\alpha d m^2 k^{\frac{2}{d}}}{\ell(2d+2-\alpha) w_d^{\frac{2}{d}}}\right)^{\frac{d}{2d+2}}.$$

In other words, in view of the lower bound for $m$ given in (3.4), we may choose $\zeta(0) \le |\Omega|/(2\pi)^d$ when

$$\ell \le \frac{\alpha d (2\pi)^2 w_d^{-\frac{4}{d}}}{2d+2-\alpha} \tag{3.29}$$

Therefore, we can replace $\zeta(0)$ with $(2\pi)^{-d}|\Omega|$ in (3.24) when we set

$$\ell = \min\left\{\frac{1}{12}, \frac{4\alpha d \pi^2}{(2d+2-\alpha) w_d^{\frac{4}{d}}}\right\} \tag{3.30}$$

Thus, substitution of $m$ given in (3.2) together with $\zeta(0) = (2\pi)^{-d}|\Omega|$ and $w_d = \pi^{\frac{d}{2}}/\Gamma\left(1+d/2\right)$ turns (3.24) into the following inequality:

$$\sum_{j=1}^{k} \varrho_j \ \ge\ \frac{d(4\pi)^{\frac{\alpha}{2}}}{d+\alpha}\left(\frac{\Gamma\left(1+d/2\right)}{|\Omega|}\right)^{\frac{\alpha}{d}} k^{1+\frac{\alpha}{d}} + \frac{\ell |\Omega|^{1+\frac{2-\alpha}{d}}}{4 I(\Omega)\big(4\pi\Gamma(1+d/2)\big)^{\frac{2-\alpha}{d}}} k^{1-\frac{2-\alpha}{d}}. \tag{3.31}$$

where $\ell$ is given by (3.30). Note that the first term on the right of (3.31) is same bound as in (1.3) and it is straightforward to check that if $\alpha = 1$ and $\alpha = 2$, we immediately obtain the asymptotic results in [14] and [13] respectively.

Note that we could have used the following lemma instead of Lemma 3.1

**Lemma 3.5.** *If $d \geq 1$, $\alpha > 0$, $t > 0$ and $s > 0$, then we have*

$$t^{d+\alpha} > t^d y^\alpha + \frac{\alpha}{d} t^d s^\alpha \ln((t/s)^d). \tag{3.32}$$

*Proof.* Let $f(r) = z^{r+\alpha}$. Note that $f'(r) = z^{r+\alpha}\ln(z)$ and $f''(r) = z^{r+\alpha}(\ln(z))^2 + z^{r+\alpha-1} > 0$ when $1 \neq z > 0$ Therefore, $f$ is strictly convex. Thus,

$$f(d) > f(d-\alpha) + \alpha f'(d-\alpha)$$

gives

$$z^{d+\alpha} > z^d + \frac{\alpha}{d} z^d \ln(z^d).$$

Setting $z = t/s$ and multiplying both sides by $s^{d+\alpha}$, we obtain (3.32). □

To conclude, let us also remark that one can basically employ Lemma 3.5 and adapt the strategy in Lemma 3.3 and apply Jensen's inequality to get another improvement of the Brezin-Li-Yau type inequality in Theorem 1.2. The advantage of (3.5) is to have a simple lower bound as shown in (3.21). However, both Lemmas yield the same expression on the right hand side of (1.3) at once. This work is left to the reader.

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## References

1. R. Bañuelos and T. Kulczycki, *The Cauchy process and the Steklov problem*, J. Funct. Analysis **211(2)** (2004) 355-423. 1.1, 1
2. R. Bañuelos and T. Kulczycki, *Eigenvalue gaps for the Cauchy process and a Poincaré inequality*, J. Funct. Analysis **234** (2006) 199-225. 1.1, 1
3. R. Bañuelos, T. Kulczycki and Bartłomiej Siudeja, *On the trace of symmetric stable processes on Lipschitz domains*, J. Funct. Analysis **257(10)** (2009) 3329–3352. 1
4. F. A. Berezin, *Covariant and contravariant symbols of operators, Izv. Akad. Nauk SSSR Ser. Mat.* **36**, (1972) 1134–1167. 1
5. R. Blumenthal and R.Getoor, *The asymptotic distribution of the eigenvalues for a class of Markov operators*, Pacific J. Math, **9** (1959) 399-408. 1
6. E. M. Harrell II and S. Yildirim Yolcu, *Eigenvalue inequalities for Klein-Gordon Operators*, J. Funct. Analysis, **256(12)** (2009) 3977-3995. 1, 1, 2
7. Alexei A. Ilyin, *Lower bounds for the spectrum of the Laplace and Stokes operators*, Discrete and Continuous Dynamical Systems, **28(1)**, (2010) 131-146. 1
8. Alexei A. Ilyin, *On the Spectrum of the Stokes Operator*, Funktsonal. Anal. i Prilozhen., **43(4)** (2009) 1425; English transl. in Funct. Anal. Appl., **43(4)** (2009) 254263. 1
9. H. Kovařík, S. Vugalter and T. Weidl, *Two dimensional Berezin-Li-Yau inequalities with a correction term*, Comm. Math. Phys., **287(3)** (2009), 959–981. 1
10. N. S. Landkof,*Foundations of modern potential theory*, New York: Springer-Verlag (1972). 1
11. A. Laptev and T. Weidl, *Recent results on Lieb-Thirring inequalities*, Journées "Équations aux Dérivées Partielles"(La Chapelle sur Erdre, 2000), Exp. No. XX, 14 pp., Univ. Nantes, Nantes, 2000. 1

12. P. Li and S.-T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983) 309–318. 1, 2
13. A. D. Melas, *A lower bound for sums of eigenvalues of the Laplacian*, Proceedings of the American Mathematical Society, **131(2)** (2002) 631–636 (document), 1, 1, 1, 3
14. S. Yildirim Yolcu, *An improvement to a Brezin-Li-Yau type inequality*, Proceedings of the American Mathematical Society, **138(11)** (2010) 4059-4066 (document), 1, 1, 3
15. E. Valdinoci, *From the long jump random walk to the fractional Laplacian*, ArXiv:0901.3261 1
16. T. Weidl, *Improved Berezin-Li-Yau inequalities with a remainder term, Spectral Theory of Differential Operators*, Amer. Math. Soc. Transl., **225(2)** (2008), 253–263. 1

Purdue University
*E-mail address*: syildir@math.purdue.edu
*E-mail address*: tyolcu@math.purdue.edu