THRESHOLD RESULTS FOR SEMILINEAR PARABOLIC SYSTEMS

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Abstract. In this paper, we study initial boundary value problem of semi-linear parabolic systems

$$\begin{cases} u_t - \Delta u = v^p & (x,t) \in \Omega \times (0,T), \\ v_t - \Delta v = u^q & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) \geq 0 & x \in \Omega, \\ v(x,0) = v_0(x) \geq 0 & x \in \Omega \end{cases} \tag{0.1}$$

and prove that any positive solution of its steady-state problem

$$\begin{cases} -\Delta u = v^p & x \in \Omega, \\ -\Delta v = u^q & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega \end{cases} \tag{0.2}$$

is an initial datum threshold for the existence and nonexistence of global solution to problem(0.1). For the precisely statement of this result, see Theorem 1.1 in the introduction of this paper.

Key words: Initial boundary value problem, Semi-linear parabolic systems, Threshold result, Steady-state problem.
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1. Introduction

Let Ω be a bounded domain in R^N. We consider the following initial-boundary value problem

$$\begin{cases} u_t - \Delta u = v^p & (x,t) \in \Omega \times (0,T), \\ v_t - \Delta v = u^q & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) \geq 0 & x \in \Omega, \\ v(x,0) = v_0(x) \geq 0 & x \in \Omega, \end{cases} \tag{1.1}$$

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where u_t, v_t are, respectively, the partial derivatives of $u(x,t)$ and $v(x,t)$ with respect to variable t, $\Delta = \sum\limits_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and $N \geq 2$, $p, q > 1$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}. \tag{1.2}$$

It is well known that for any $u_0(x), v_0(x) \in L^\infty(\Omega)$, problem (1.1) has an unique classical solution $(u(x,t), v(x,t))$ in a short time, which is called a local solution of problem (1.1). Let $T_{\max}$ denote the maximum existence time of $(u(x,t), v(x,t))$ as a classical solution. If $T_{\max} = +\infty$, then we say that $(u(x,t), v(x,t))$ exists globally, or problem (1.1) has global solution. If $T_{\max} < +\infty$, then we have

$$\lim_{t \to T_{\max}^-} \sup_{x \in \Omega} u(x,t) = \lim_{t \to T_{\max}^-} \sup_{x \in \Omega} v(x,t) = +\infty,$$

for which we say that $(u(x,t), v(x,t))$ blows up in a finite time (see for example [11] for more details).

It is also well known that the solution $(u(x,t), v(x,t))$ exists globally when the initial value $(u_0(x), v_0(x))$ is small enough in some sense, and blows up in a finite time when the initial value $(u_0(x), v_0(x))$ is large enough in a suitable sense (see [1][4][5][6][11] for the exact statement). However, the classification of the initial datum $(u_0(x), v_0(x))$ according to the existence or nonexistence of global solutions to problem (1.1) is still far from complete. Hence, an important task in the study of problem (1.1) is to find exact conditions on the initial datum $(u_0(x), v_0(x))$ which can ensure the existence or nonexistence of global solutions to problem (1.1). On this direction, we present here a so called threshold result for problem (1.1) by making use of its positive equilibriums. To state our result simply and precisely, we introduce some notations and definitions first. For any planar vector (a,b) and (c,d), we use $(a,b) \geq (c,d)$ to mean that $a \geq c$ and $b \geq d$, and $(a,b) = (c,d)$ to mean that $a = c$ and $b = d$. If a, b, c, d are functions of variable x, we use $(a(x), b(x)) \not\equiv (c(x), d(x))$ to mean that there exists at least one point x_0 such that $(a(x_0), b(x_0)) \neq (c(x_0), d(x_0))$. Finally, we say that $(U(x), V(x))$ is a positive equilibrium of problem (1.1) if it is a solution of the following steady-state problem related to problem (1.1).

$$\begin{cases} -\Delta U = V^p & x \in \Omega, \\ -\Delta V = U^q & x \in \Omega, \\ (U,V) > (0,0) & x \in \Omega \\ (U,V) = (0,0) & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Keeping the above notations and definitions in mind, our main result of this paper can be stated as

Theorem 1.1. *Assume that $p, q > 1$ satisfy (1.2), and that $(U(x), V(x))$ is an arbitrary smooth solution of problem (1.3). Then there holds*

(i) If $\big(0,0\big) \leq \big(u_0(x), v_0(x)\big) \leq \big(U(x), V(x)\big)$ and $\big(u_0(x), v_0(x)\big) \not\equiv \big(U(x), V(x)\big)$, then problem (1.1) has a global solution $(u(x,t), v(x,t))$. Moreover, $\lim\limits_{t\to\infty}(u(x,t), v(x,t)) = (0,0)$.

(ii) If $\big(u_0(x), v_0(x)\big) \geq \big(U(x), V(x)\big)$ and $\big(u_0(x), v_0(x)\big) \not\equiv \big(U(x), V(x)\big)$, then the solution $(u(x,t), v(x,t))$ of problem (1.1) blows up in a finite time.

We remark here that Theorem 1.1 is a natural generalization of results on scalar equations proved by P.L.Lions in [9] and A.A.Lacey in [10],but the method we use here is different. Roughly speaking, Theorem 1.1 says that any smooth solution of problem (1.3) is an initial datum threshold for the existence and nonexistence of global solutions to problem (1.1). It is also worth pointing out that the restriction (1.2) on the exponents p and q is optimal in the sense that problem (1.3) has no solutions for star-shaped domains Ω when (1.2) is violated (see [2]).

The plan of this paper is as follows. Section 2 devotes to prove two lemmas need in the proof of theorem 1.1. The proof of Theorem 1.1 is given in Section 3. Some further remarks are included in Section 4.

2. Preliminaries

In this section, we prove two lemmas which will be used later in the proof of our main result.

Lemma 2.1. *Let $(g(x), h(x))$ and $(U(x), V(x))$ be two distinct smooth solutions of problem (3.3). Then we have*

$$\int_\Omega g(x)U(x)(g^{q-1} - U^{q-1})\ dx = \int_\Omega h(x)V(x)(V^{p-1} - h^{p-1})\ dx.$$

Proof. This result can be found in [3]. However, for the reader's convenience, We give a proof here. Since $(g(x), h(x))$ and $(U(x), V(x))$ are solutions of problem (1.3), we have

$$\begin{cases} -\Delta g(x) = h^p & x \in \Omega, \\ -\Delta h(x) = g^q & x \in \Omega, \\ g = h = 0 & x \in \partial\Omega \end{cases} \tag{2.1}$$

and

$$\begin{cases} -\Delta U(x) = V^p & x \in \Omega, \\ -\Delta V(x) = U^q & x \in \Omega, \\ U = V = 0 & x \in \partial\Omega. \end{cases} \tag{2.2}$$

From these, we can derive

$$\int_\Omega h^p V(x)\ dx = -\int_\Omega \Delta g(x)V(x)\ dx = -\int_\Omega g(x)\Delta V\ dx = \int_\Omega g(x)U^q,$$

$$\int_\Omega hV^p\ dx = -\int_\Omega \Delta U(x)h(x)\ dx = -\int_\Omega U(x)\Delta h(x)\ dx = \int_\Omega Ug^q.$$

Consequently

$$\int_\Omega g(x)U(x)(g^{q-1} - U^{q-2})\ dx = \int_\Omega h(x)V(x)(V^{p-1} - h^{p-1})\ dx.$$

□

Lemma 2.2. *Assume that $x > 0, y > 0$, and $0 < a < 1$. Then*

$$x^a + y^a \leq 2^{1-a}(x+y)^a.$$

Proof. Let $g(t) = t^a + (1-t)^a$, $0 < t < 1$. An easy computations yields

$$g'(t) = at^{a-1} - a(1-t)^{a-1}.$$

Hence, we have

$$\begin{cases} g'(t) > 0 & 0 < t < \frac{1}{2}, \\ g'(t) = 0 & t = \frac{1}{2}, \\ g'(t) < 0 & \frac{1}{2} < t < 1 \end{cases}$$

From this, we conclude that

$$g(t) = t^a + (1-t)^a \le 2^{1-a}.$$

Substituting $t = \frac{x}{x+y}$ into the above inequality, we finally obtain that

$$x^a + y^a \le 2^{1-a}(x+y)^a.$$

□

3. Proof of Theorem 1.1

Proof of Theorem 1.1: (i) Since $(0,0) \le (u_0(x), v_0(x)) \le (U(x),\ V(x))$, and $U(x), V(x) \in L^\infty(\Omega)$, we know that problem (1.1) has a global solution $(u(x,t), v(x,t))$. Noticing that $(u_0(x), v_0(x)) \not\equiv (U(x), V(x))$, it follows from the maximum principle and the strong comparison principle that

$$(0,0) \le (u(x,t), v(x,t)) < (U(x), V(x)$$

for any $(x,t) \in \Omega \times (0, +\infty)$.

Therefore, we may assume, by replacing $(u_0(x), v_0(x))$ with $(u(x,T), v(x,T))$ for some $T > 0$ if necessary, that $(u_0(x), v_0(x)) \le (\alpha U(x), \alpha V(x))$ for some constant $0 < \alpha < 1$. Let $(g_\alpha(x), h_\alpha(x)) = (\alpha U(x), \alpha V(x))$. It is easy to verify that $(g_\alpha(x), h_\alpha(x))$ satisfies

$$\begin{cases} -\Delta g_\alpha > (h_\alpha)^p & x \in \Omega, \\ -\Delta h_\alpha > (g_\alpha)^q & x \in \Omega, \\ g_\alpha = h_\alpha = 0 & x \in \partial\Omega, \\ g_\alpha > 0, h_\alpha > 0 & x \in \Omega. \end{cases} \tag{3.1}$$

This implies that $(g_\alpha(x), h_\alpha(x))$ is a strict super-solution of the following problem

$$\begin{cases} G_t - \Delta G = H^p & (x,t) \in \Omega \times (0,T), \\ H_t - \Delta H = G^q & (x,t) \in \Omega \times (0,T), \\ G = H = 0 & (x,t) \in \partial\Omega \times [0,T], \\ G(x,0) = g_\alpha \ge 0 & x \in \Omega, \\ H(x,0) = h_\alpha \ge 0 & x \in \Omega. \end{cases} \tag{3.2}$$

Let $(G(x,t), H(x,t))$ be the solution of (3.2). By strong comparison principle we know that $(G(x,t), H(x,t))$ is strictly decreasing with respect to t, and $(0,0) \le (G(x,t), H(x,t)) \le (U(x), V(x))$. Therefore, $(G(x,t), H(x,t))$ exists globally. Moreover, there are some functions $g(x)$ and $h(x)$ such that

$$\lim_{t\to\infty} G(x,t) = g(x), \quad \lim_{t\to\infty} H(x,t) = h(x).$$

uniformly on Ω, and $(g(x), h(x))$ is a smooth solution of the following problem.

$$\begin{cases} -\Delta g = h^p & x \in \Omega, \\ -\Delta h = g^q & x \in \Omega, \\ (g,h) = (0,0) & x \in \partial\Omega. \end{cases} \tag{3.3}$$

From this, we conclude that $(g(x), h(x)) \equiv (0,0)$. Otherwise, by strong maximum principle, we have $(g(x), h(x)) > (0,0)$. On the other hand, we have $(g(x), h(x)) < (U(x), V(x))$ since $(G(x,t), H(x,t))$ is strictly decreasing with respect to t. Thus

$$\int_\Omega g(x)U(x)(g^{q-1} - U^{q-1})\, dx < 0, \quad \int_\Omega h(x)V(x)(V^{p-1} - h^{p-1})\, dx > 0.$$

This is a contradiction with Lemma 2.1. Therefore

$$\lim_{t\to\infty} (G(x,t), H(x,t)) = (0,0).$$

Noticing that $(0,0) \le (u_0, v_0) \le (g_\alpha(x), h_\alpha(x))$, comparison principle ensures

$$(0,0) \le (u(x,t), v(x,t)) \le (G(x,t), H(x,t)).$$

By applying squeeze principle, we obtain

$$\lim_{t\to\infty} (u(x,t), v(x,t)) = (0,0).$$

(ii) we prove the conclusion (ii) of Theorem 1.1 by contradiction. To this end, we assume that $(u_0(x), v_0(x)) \ge (U(x), V(x))$, $(u_0(x), v_0(x)) \not\equiv (U(x), V(x))$ and problem (1.1) has a global solution $(u(x,t), v(x,t))$. By strong comparison principle, we have

$$(u(x,t), v(x,t)) > (U(x), V(x)),$$

for any $(x,t) \in \bar{\Omega} \times (0, +\infty)$. Therefore, we may assume, by replacing $(u_0(x), v_0(x))$ with $(u(x,T), v(x,T; u_0, v_0))$ for some $T > 0$ if necessary, that $(u_0(x), v_0(x)) \ge (\beta U(x), \beta V(x))$ for some constant $\beta > 1$. Let $(g_\beta, h_\beta) = (\beta U(x), \beta V(x))$. It is easy to verify that (g_β, h_β) satisfies

$$\begin{cases} -\Delta g_\beta < (h_\beta)^p & x \in \Omega, \\ -\Delta h_\beta < (g_\beta)^q & x \in \Omega, \\ g_\beta = h_\beta = 0 & x \in \partial\Omega. \end{cases} \tag{3.4}$$

Hence, (g_β, h_β) is a strict sub-solution of the following problem

$$\begin{cases} G_t - \Delta G = H^p & (x,t) \in \Omega \times (0,T), \\ H_t - \Delta H = G^q & (x,t) \in \Omega \times (0,T), \\ G = H = 0 & (x,t) \in \partial\Omega \times [0,T], \\ G(x,0) = g_\beta \ge 0 & x \in \Omega, \\ H(x,0) = h_\beta \ge 0 & x \in \Omega. \end{cases} \tag{3.5}$$

Let $(G(x,t), H(x,t))$ be the solution of problem (3.5). Then it follows from the comparison principle that

$$(G(x,t), H(x,t)) \le (u(x,t), v(x,t))$$

for any (x,t) due to $(g_\beta(x), h_\beta(x)) \le \big(u_0, v_0\big)$. Consequently, $(G(x,t), H(x,t))$ exists globally and is strictly increasing with respect to t.

Let

$$\varphi(t)=\int_\Omega G(x,t)H(x,t)\ dx.$$

$$E(t)=\int_\Omega \nabla G\nabla H\ dx-\frac{1}{p+1}\int_\Omega H^{p+1}\ dx-\frac{1}{q+1}\int_\Omega G^{q+1}\ dx,$$

By making use of (3.5), we can verify that $\varphi(t)$ and $E(t)$ satisfy

$$\frac{d\varphi}{dt}=-2E(t)+\frac{p-1}{p+1}\int_\Omega H^{p+1}\ dx+\frac{q-1}{q+1}\int_\Omega G^{q+1}\ dx,$$

$$\begin{aligned}\frac{dE(t)}{dt}&=\textstyle\int_\Omega \nabla G_t\nabla H\ dx+\int_\Omega \nabla H_t\nabla G\ dx-\int_\Omega H^pH_t\ dx-\int_\Omega G^qG_t\ dx\\ &=-2\textstyle\int_\Omega G_tH_t\ dx\le 0.\end{aligned}$$

Let $\gamma=\frac{(p+1)(q+1)}{p+q+2}$. It follows from the assumption $p>1$ and $q>1$ that

$$\gamma>1,\quad \frac{q+1}{\gamma}>1,\quad \frac{p+1}{\gamma}>1,\quad \frac{\gamma}{q+1}+\frac{\gamma}{p+1}=1.$$

By Hölder's inequality, Young's inequality, and Lemma 2.2, we have

$$\begin{aligned}\varphi(t)&\le \tfrac{q+1}{\gamma}\textstyle\int_\Omega G^{\frac{q+1}{\gamma}}\ dx+\tfrac{p+1}{\gamma}\int_\Omega H^{\frac{p+1}{\gamma}}\ dx\\ &\le \tfrac{\max\{p,q\}+1}{\gamma}|\Omega|^{1-\frac{1}{\gamma}}\big((\textstyle\int_\Omega G^{q+1}\ dx)^{\frac{1}{\gamma}}+(\int_\Omega H^{p+1}\ dx)^{\frac{1}{\gamma}}\big)\\ &\le \tfrac{\max\{p,q\}+1}{\gamma}|\Omega|^{1-\frac{1}{\gamma}}2^{1-\frac{1}{\gamma}}[\textstyle\int_\Omega G^{q+1}\ dx+\int_\Omega H^{p+1}\ dx]^{\frac{1}{\gamma}}.\end{aligned}$$

Hence, there exists a positive constant C such that

$$\frac{d\varphi}{dt}\ge -2E(t)+C\varphi^\gamma(t).$$

Since $E(t)$ is decreasing in t, we have $E(t)\le E(0)$ for any $t>0$. Consequently,

$$\frac{d\varphi}{dt}\ge -2E(0)+C\varphi^\gamma(t).$$

From this, we may conclude that

$$\sup_{t\ge 0}\int_\Omega GH\ dx<+\infty.$$

Otherwise, we have $\int_\Omega GH\ dx\to+\infty,\ as\ t\to\infty$ due to $\int_\Omega GH\ dx=\varphi(t)$ is strictly increasing in t. Hence, there exists a constant $T>0$ large enough such that

$$\frac{d}{dt}\int_\Omega GH\ dx\ge\frac{C}{2}(\int_\Omega GH\ dx)^\gamma,$$

for any $t>T$. This implies that $(G(x,t),H(x,t))$ must blow up in a finite time which contradicts the fact that $(G(x,t),H(x,t))$ is a global solution of problem (3.5).

Let

$$T(t)=\int_\Omega G^{q+1}\ dx+\int_\Omega H^{p+1}\ dx.$$

Then, $T(t)$ is strictly increasing in t because $(G(x,t),H(x,t))$ does. Thus, for any $t>0$, we have

$$C \geq \int_t^{t+1} \tfrac{d}{ds} \int_\Omega GH\ dx\ ds = -2\int_t^{t+1} E(s)ds + \tfrac{p-1}{p+1}\int_t^{t+1}\int_\Omega G^{p+1}\ dx\ ds$$

$$+\tfrac{q-1}{q+1}\int_t^{t+1}\int_\Omega H^{q+1}\ dx\ ds,$$

$$\geq -2E(0) + \min\{\tfrac{p-1}{p+1}, \tfrac{q-1}{q+1}\}T(t).$$

From this, we can easily see that

$$\sup_{t\geq 0} T(t) < +\infty.$$

Consequently, there are functions $g(x)\in L^{p+1}(\Omega)$ and $h(x)\in L^{q+1}(\Omega)$ such that

$$G(x,t)\to g(x) \quad \text{weakly in} \quad L^{p+1}(\Omega),$$

$$H(x,t)\to h(x) \quad \text{weakly in} \quad L^{q+1}(\Omega).$$

Multiplying the first and the second equation in (3.5) by φ and ψ respectively, and integrating the result equations on $[t,t+1]$, we obtain

$$\int_\Omega [G(x,t+1)-G(x,t)]\varphi\ dx\ ds + \int_t^{t+1}\int_\Omega G(-\Delta\varphi)\ dx\ ds = \int_t^{t+1}\int_\Omega H^p\varphi\ dx\ ds,$$

$$\int_\Omega [H(x,t+1)-H(x,t)]\psi\ dx\ ds + \int_t^{t+1}\int_\Omega H(-\Delta\psi)\ dx\ ds = \int_t^{t+1}\int_\Omega G^p\psi\ dx\ ds,$$

Passing to the limit as $t\to\infty$, we find that

$$\int_\Omega g(-\Delta\varphi)\ dx = \int_\Omega h^p\varphi\ dx,$$

$$\int_\Omega h(-\Delta\psi)\ dx = \int_\Omega g^p\psi\ dx.$$

This implies that $(g(x),h(x))$ is a L^1 solution of problem (1.3) (For the definition of the L^1 solution, we refer to [11]).

Noticing that $p,q>1$ satisfy (1.2) and

$$\int_\Omega g^{q+1}\ dx < +\infty, \quad \int_\Omega h^{p+1}\ dx < +\infty,$$

it follows from the regularity theory (bootstrap method) of L^1 solution that $g,h\in L^\infty(\Omega)$ (see [11]). With L^∞ estimate in hand, we can establish the H_0^1 estimate of $g(x)$ and $h(x)$ by making use of the following facts

$$\begin{cases} -\Delta G \leq H^p & (x,t)\in\Omega\times(0,+\infty), \\ -\Delta H \leq G^q & (x,t)\in\Omega\times(0,+\infty), \\ G=H=0 & (x,t)\in\partial\Omega\times[0,+\infty). \end{cases} \tag{3.6}$$

Now, we can conclude that (g,h) is a classical solution of problem (1.3) by the standard regularity theory of elliptic differential equations (see [7]).

Since $(g_\beta(x),h_\beta(x))>(U(x),V(x))$, it follows from the strong comparison principle that

$$G(x,t)>U(x), \quad H(x,t)>V(x)$$

for any (x,t). Consequently

$$g(x) > U(x), \quad h(x) > V(x).$$

From this, we have

$$\int_\Omega g(x)U(x)(g^{q-1} - U^{q-1})\,dx > 0 \quad \text{and} \quad \int_\Omega h(x)V(x)(V^{p-1} - h^{p-1})\,dx < 0.$$

This is a contradiction with the conclusion of Lemma 2.1 and we complete the proof of Theorem 1.1 (ii). □

4. Further Remarks

The method used in the proof of theorem 1.1 can be applied to study the following inhomogeneous problem

$$\begin{cases} u_t - \Delta u = v^p + \lambda f(x) & (x,t) \in \Omega \times (0,T), \\ v_t - \Delta v = u^q + \lambda g(x) & (x,t) \in \Omega \times (0,T), \\ (u,v) = (0,0) & (x,t) \in \partial\Omega \times [0,T], \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \geq (0,0) & x \in \Omega, \end{cases} \tag{4.1}$$

where $p, q > 1$ satisfy (1.2), and $(0,0) \leq (f(x), g(x)) \not\equiv (0,0)$.

The main difference between problem (1.1) and (4.1) lies in the structure of their equilibrium sets. From lemma 2.1, we can easily see that any two distinct equilibriums of problem (1.1) must intersect. However, problem (4.1) has an unique minimal equilibrium for $\lambda > 0$ small enough which separates from other equilibriums. To state our results precisely, we consider the following steady-state problem of problem (4.1)

$$\begin{cases} -\Delta u = v^p + \lambda f(x) & x \in \Omega, \\ -\Delta v = u^q + \lambda g(x) & x \in \Omega, \\ (u,v) > (0,0) & x \in \Omega, \\ (u,v) = (0,0) & x \in \partial\Omega. \end{cases} \tag{4.2}$$

By sub-solution and sup-solution method, it is not difficult to prove the following

Lemma 4.1. *There exists a positive number λ^* such that the following two statements are true.*

(i) If $\lambda > \lambda^*$, then problem (4.2) has no solution.

(ii) If $0 < \lambda < \lambda^*$, then problem (4.2) has an unique minimal solution $(u_{\min}(x), v_{\min}(x))$ in the sense that $((u_{\min}(x), v_{\min}(x)) \leq (u(x), v(x))$ for any solution $(u(x), v(x))$ of problem (4.2). Moreover, if $(u(x), v(x)) \not\equiv (u_{\min}(x), v_{\min}(x))$, then $((u_{\min}(x), v_{\min}(x)) < (u(x), v(x))$.

Let $u(x) = U(x) + u_{\min}(x)$ and $v(x) = V(x) + v_{\min}(x)$. Then, it is easy to see that (U,V) satisfies

$$\begin{cases} -\Delta U = (V + v_{\min})^p - v_{\min}^p & x \in \Omega, \\ -\Delta V = (U + u_{\min})^q - u_{\min}^q & x \in \Omega, \\ (U,V) = (0,0) & x \in \partial\Omega. \end{cases} \tag{4.3}$$

By variational method, we can prove that problem (4.3) has at least one positive solution provided that (1.2) holds (see [8]). Hence, we have

Theorem 4.1. *Assume that $p, q > 1$ satisfy (1.2). Let λ^* be the number obtained in lemma 4.1. Then, for any $\lambda \in (0, \lambda^*)$, problem (4.2) has at least two solutions, and among them there exists a minimal one.*

By the same method as that used in the proof of lemma 2.1, we can prove the following

Lemma 4.2. *Let (U_1, V_1) and (U_2, V_2) be any two smooth solutions of problem (4.3), $G(u) = \frac{(u+u_{\min})^q - u_{\min}^q}{u}$ and $H(v) = \frac{(v+v_{\min})^p - v_{\min}^p}{v}$. Then we have*

$$\int_\Omega U_1 U_2 (G(U_2) - G(U_1))\, dx = \int_\Omega V_1 V_2 (H(V_1) - H(V_2))\, dx.$$

Noting that $G(u)$ and $H(v)$ are strictly increasing in u and v respectively due to $p, q > 1$, we infer from lemma 4.2 that the following result on the structure of solution set of problem (4.2) holds

Theorem 4.2. *With the same assumption as that of theorem 4.1, problem (4.2) has at least two solutions, and among them there exists a minimal one. Moreover, any two distinct solutions of problem (4.2) which are also different from the minimal one must intersect somewhere.*

With theorem 4.2 established, by a similar argument to that used in the proof of theorem 1.1, we can reach the following

Theorem 4.3. *Assume that $p, q > 1$ satisfy (1.2). Let λ^* be the number obtained in lemma 4.1. Then, we have*

(i) If $\lambda > \lambda^*$, then, for any initial value $(u_0(x), v_0(x)) \geq (0, 0)$, the solution $(u(x,t), v(x,t))$ of problem (4.1) must blow up in a finite time.

(ii) If $0 < \lambda < \lambda^*$, and $(U(x), V(x))$ is an arbitrary smooth solution of problem(4.2) which is different from the minimal one, then problem (4.1) has a global solution $(u(x,t), v(x,t))$ with $\lim_{t\to\infty}(u(x,t), v(x,t)) = (u_{\min}(x), v_{\min}(x))$ provided that $(0,0) \leq (u_0(x), v_0(x)) \leq (U(x), V(x))$ and $(u_0(x), v_0(x)) \not\equiv (U(x), V(x))$; whereas, the solution $(u(x,t), v(x,t))$ of problem (4.1) must blow up in a finite time if $(u_0(x), v_0(x)) \geq (U(x), V(x))$ and $(u_0(x), v_0(x)) \not\equiv (U(x), V(x))$.

Finally, we point out that the method of this paper can also be applied to study the following initial-boundary value problem with Robin boundary conditions.

$$\begin{cases} u_t - \Delta u = v^p & (x,t) \in \Omega \times (0,T), \\ v_t - \Delta v = u^q & (x,t) \in \Omega \times (0,T), \\ \frac{\partial}{\partial n}(u,v) + \beta(u,v) = (0,0) & (x,t) \in \partial\Omega \times [0,T], \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \geq (0,0) & x \in \Omega, \end{cases} \tag{4.4}$$

where n is the outer unit vector normal to the boundary $\partial\Omega$ of Ω, and β is a positive constant.

By similar arguments to that used in the proof of theorem 1.1, we can also prove the following result.

Theorem 4.4. *Assume that $p, q > 1$ satisfy (1.2), and that $(U(x), V(x))$ is an arbitrary smooth positive equilibrium of problem (4.4). Then there holds*

(i) If $(0,0) \leq (u_0(x), v_0(x)) \leq (U(x), V(x))$ and $(u_0(x), v_0(x)) \not\equiv (U(x), V(x))$, then problem (4.4) has a global solution $(u(x,t), v(x,t))$. Moreover, $\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0)$.

(ii) If $(u_0(x), v_0(x)) \geq (U(x), V(x))$ *and* $(u_0(x), v_0(x)) \not\equiv (U(x), V(x))$*, then the solution* $(u(x,t), v(x,t))$ *of problem (4.4) must blows up in a finite time.*

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