Processes with block-associated increments

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Abstract

This paper is motivated by relations between association and independence of random variables. It is well-known that for real random variables independence implies association in the sense of Esary, Proschan and Walkup, while for random vectors this simple relationship breaks. We modify the notion of association in such a way that any vector-valued process with independent increments has also associated increments in the new sense — association between blocks.

The new notion is quite natural and admits nice characterization for some classes of processes. In particular, using the covariance interpolation formula due to Houdré, Pérez-Abreu and Surgailis, we show that within the class of multidimensional Gaussian processes block-association of increments is equivalent to supermodularity (in time) of the covariance functions.

We define also corresponding versions of weak association, positive association and negative association. It turns out that the Central Limit Theorem for weakly associated random vectors due to Burton, Dabrowski and Dehling remains valid, if the weak association is relaxed to the weak association between blocks.

1 Introduction

Random variables $X_1, X_2, \ldots, X_n$ are *associated* if

$$\mathrm{Cov}(f(X_1, X_2, \ldots, X_n), g(X_1, X_2, \ldots, X_n)) \geq 0, \tag{1}$$

for each pair of functions $f, g : \mathbb{R}^n \to \mathbb{R}^1$, which are non-decreasing in each coordinate and for which the above covariance exists. This definition, due to Esary, Proschan and Walkup [5], seems to be the most appropriate description of positive dependence phenomena encountered in various areas, e.g. reliability theory [1],[13], statistical physics [14], [15], [10], multivariate extremes [19] or random sets [9], to mention but a few. We refer to the recent monograph [2] for properties of association, an extensive list of references and more abstract formalism of associated random elements.

Our paper is motivated by relations between association and independence of random variables. It is well-known [2, Theorem 1.8], that any family of independent random variables is associated. In particular, any stochastic process $X = \{X_t\}_{t\geq 0}$ *with independent increments* has also *associated increments* in the sense of Glasserman [6]. The last statement means that for any choice of sampling points $0 < t_1 < t_2 < \ldots < t_n$ the differences

$$\Delta_1 = X_{t_1} - X_0, \quad \Delta_2 = X_{t_2} - X_{t_1}, \quad \ldots, \quad \Delta_n = X_{t_n} - X_{t_{n-1}}$$

are associated random variables.

This simple and natural relationship breaks when we pass to processes with values in $\mathbb{R}^d$, $d \geq 2$. Consider, for example, a real process $\{Z_t\}_{t\geq 0}$ with independent increments and non-degenerate marginal laws and set

$$Y_t = \begin{bmatrix} Z_t \\ -Z_t \end{bmatrix}.$$

Then $\{Y_t\}_{t\geq 0}$ retains independence of increments but cannot have associated increments: for any $t > 0$, the components of the vector

$$Y_t - Y_0 = \begin{bmatrix} Z_t - Z_0 \\ -(Z_t - Z_0) \end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

do not satisfy (1).

We aim at modifying the notion of association in such a way that

- for random variables ($d = 1$) the new notion is equivalent to association;
- any vector-valued process with independent increments has also associated increments in the new sense.

This is done in Section 2, where we introduce *association between blocks* of random variables. The idea consists in requiring association between *real* non-decreasing (in each coordinate) functions of blocks. It turns out that the modified notion of association can be easily characterized within classes of random vectors with multivariate normal or infinitely divisible distributions (like the usual association). Similarly, when applied to increments of stochastic processes, the new notion admits nice characterizations within particular classes of processes. For example, for multidimensional Gaussian processes the block-association of increments is equivalent to L-superadditivity (or supermodularity) of all covariance functions (see Theorem 3.3, Section 3). This example shows that association between blocks deals with core properties of multidimensional stochastic processes.

In a similar spirit, in Section 4 we weaken the notion of *weak association* introduced by Burton, Dabrowski and Dehling [3], *positive association* (as defined in Bulinski and Shashkin [2]) and *negative association* (due to Joag-Dev and Proschan [8]). It is interesting that obtained this way "weak association between blocks" and "positive association between blocks" coincide while their prototypes differ.

The weak association of random vectors is formally stronger than the weak association between blocks built upon coordinates of vectors. We do not know any example showing that the equality of both classes actually does not hold. On the other hand an inspection of methods based on factorization of increasing functions and used in the proof of Theorems 2.5 and 2.6 suggests that verifying whether a sequence of random vectors is "weakly associated between blocks" may be essentially easier then the corresponding procedure for "weak association". Therefore in Section 5 we restate a complete multidimensional generalization of Newman's Central Limit Theorem [14] and Newman-Wright's Invariance Principle [16] for sums of stationary associated random variables, originally proved by Burton, Dabrowski and Dehling [3] for weakly associated random vectors. The point is that this result is valid under weak association between blocks, without any change in its proof.

2 Association between blocks

In what follows when referring to vectors we mean *column* vectors.

Let us consider a family $X = \{X_i, i \in I\}$ of real-valued random variables indexed by a finite set I. Suppose that $I = \bigcup_{k=1}^n I_k$, where sets I_k are nonempty and pairwise disjoint. Equip each set I_k with some arbitrary (but fixed) linear order. Write $X(I_k)$ for vector with components $\{X_i, i \in I_k\}$. Let $|I|$ denote the cardinality of I.

We are ready to formulate our basic definition.

Definition 2.1. A family $X = \{X_i, i \in I\}$ is called *associated between blocks* if for all non-decreasing functions $f_k : \mathbb{R}^{|I_k|} \to \mathbb{R}$, $k = 1, 2, \ldots, n$, the random vector

$$(f_1(X(I_1)), f_2(X(I_2)), \dots, f_n(X(I_n)))$$

is associated, i.e. for all non-decreasing functions $g, h : \mathbb{R}^n \to \mathbb{R}$

$$Cov(g(f_1(X(I_1)), \dots, f_n(X(I_n))), h(f_1(X(I_1)), \dots, f_n(X(I_n)))) \geq 0 \qquad (2)$$

if the above covariance is well defined.

The very definition and basic properties of association imply the following facts.

Proposition 2.2. Let $X = \{X_i, i \in I\}$ be an associated family of random variables. Then for arbitrary partition $I = \bigcup_{k=1}^n I_k$ we have association of X between blocks based on $I_1, \dots, I_n$.

Proposition 2.3. If vectors $X(I_k)$, $k = 1, 2, \dots, n$, are independent then X is associated between blocks.

Proposition 2.4. For a fixed blocks' basis $\mathcal{J} = \{I_1, I_2, \dots, I_n\}$, the family $\mathcal{P}^+_{\mathcal{J}}$ of laws of random vectors which are associated between blocks based on $I_1, I_2, \dots, I_n$ is closed with respect to the topology of weak convergence.

Let $X = \{X_i\}_{i \in I}$ be an $|I|$-dimensional Gaussian random vector. It is well known [17] — but by no means trivial — that the non-negativity of all entries of the covariance matrix Σ of X is necessary and sufficient for association of X. We have a very similar situation for the association between blocks.

Theorem 2.5. A Gaussian random vector $X = \{X_i\}_{i \in I}$ is associated between blocks built on $I_1, I_2, \dots, I_n$ if and only if $\sigma_{kl} = Cov(X_k, X_l) \geq 0$ for all k, l which are not in the same block.

While the necessity part in the above theorem is obvious, the sufficiency does not seem to be easy unless advanced tools are used. We propose to exploit the covariance interpolation formula and the technique developed by Houdré, Pérez-Abreu and Surgailis ([7], Section 2), restated below in Proposition 2.7. Since the covariance formula is valid for general infinite dimensional distributions, we will prove Theorem 2.5 together with Theorem 2.6 given after a necessary notation is introduced.

Let $X = \{X_i\}_{i\in I}$ be an $|I|$-dimensional infinitely divisible random vector with the Lévy-Khinchin triplet (a, Σ, ν) (we write then $X \sim \mathcal{ID}(a, \Sigma, \nu)$) and the characteristic function $\varphi(t) = \varphi(t; a, \Sigma, \nu)$ given by

$$\ln \varphi(t) = i\langle t, a\rangle - \frac{1}{2}\langle \Sigma t, t\rangle + \int_{\mathbb{R}^d} \big(e^{i\langle t,u\rangle} - 1 - i\langle t, u\rangle \cdot \mathbf{1}_{\{\|u\|\le 1\}}(u)\big)\nu(du). \quad (3)$$

Recall that $a \in \mathbb{R}^{|I|}$ is a vector, $\Sigma = (\sigma_{kl})_{k,l\in I} \in \mathbb{R}^{|I|} \otimes \mathbb{R}^{|I|}$ is the covariance matrix of the Gaussian component of X and ν stands for the Lévy measure. We shall associate with ν its two-dimensional characteristics ν_{kl}. If $\pi_{kl} : \mathbb{R}^{|I|} \to \mathbb{R}^2$ are standard projections on $\mathbb{R}^2$, i.e.

$$\pi_{kl}(x_1, x_2, \ldots, x_{|I|}) = (x_k, x_l), \quad 1 \le k < l \le |I|,$$

we define ν_{kl} on $\mathbb{R}^2$ by the formula

$$\nu_{kl}(A) = (\nu \circ \pi_{kl}^{-1})\big(A \cap (\mathbb{R}^2 \setminus \{0\})\big). \quad (4)$$

Notice that ν_{kl} is a Lévy measure on $\mathbb{R}^2$, but it is not a two-dimensional projection of ν.

A combination of results by Pitt [17] and Resnick [19] states that non-negativity of all entries of Σ together with the concentration of the Lévy measure ν on $(\mathbb{R}_+)^{|I|} \cup (\mathbb{R}_-)^{|I|}$ are enough for association of X. Theorem 2.6 establishes analogous conditions for association between blocks of an infinitely divisible random vector.

Theorem 2.6. Let $X \sim \mathcal{ID}(a, \Sigma, \nu)$. If for all $k, l \in I$, which *are not in the same block*,

(i) σ_{kl} are non-negative,

(ii) the measures ν_{kl} are concentrated on $(\mathbb{R}_-)^2 \cup (\mathbb{R}_+)^2$,

then X is associated between blocks.

Let $X \sim \mathcal{ID}(a, \Sigma, \nu)$ and let φ be given by (3). Define

$$\varphi_0(r, s) = \varphi(r)\varphi(s), \quad \varphi_1(r, s) = \varphi(r + s), \quad r, s \in \mathbb{R}^{|I|}.$$

For each $\alpha \in [0, 1]$, let (Y^α, Z^α) be an infinitely divisible random vector of dimension $2|I|$ with distribution given by the characteristic function

$$\varphi_\alpha(r, s) = \varphi_0^{1-\alpha}(r, s)\varphi_1^\alpha(r, s).$$

Then for each $\alpha \in [0,1]$ we have $Y^\alpha \sim Z^\alpha \sim X$ and the vector (Y^α, Z^α) "interpolates" between independent copies Y^0, Z^0 of the vector X and the totally dependent copies $Y^1 = Z^1$ of X. We are ready to restate the covariance formula due to Houdré, Perez-Abreu and Surgailis [7].

Proposition 2.7. For any functions $\psi_1, \psi_2 \in \mathcal{C}_b^1(\mathbb{R}^{|I|})$ (continuously differentiable with bounded derivatives)

$$Cov(\psi_1(X), \psi_2(X)) =$$
$$= \int_0^1 \mathrm{E}\left(\langle \Sigma \nabla \psi_1(Y^\alpha), \nabla \psi_2(Z^\alpha) \rangle + \int_{\mathbb{R}^{|I|}} \Delta_u \psi_1(Y^\alpha) \Delta_u \psi_2(Z^\alpha)\, \nu(du) \right) d\alpha,$$

where ∇ is the gradient operator and $\Delta_u \psi(x) = \psi(x+u) - \psi(x)$.

Now we can return to the proof of Theorem 2.6, keeping in mind that it is enough to study (1) only for functions from $C_b^1(\mathbb{R}^{|I|})$ (see [17]).

Proof. Choose non-decreasing and C_b^1 functions $f_i : \mathbb{R}^{|I_i|} \to \mathbb{R}$, $k = 1, 2, \ldots, n$ and denote by F the mapping from $\mathbb{R}^{|I|}$ into $\mathbb{R}^n$ given by

$$F(x_k, k \in I) = (f_1(x_k, k \in I_1), \ldots, f_n(x_k, k \in I_n)).$$

We will identify the functions f_i with their corresponding extensions $\widetilde{f_i}(x) = f_i(\pi_{I_i}(x))$.

Let $g, h : \mathbb{R}^n \to \mathbb{R}$ be non-decreasing and C_b^1. Our goal is to establish the sign of the covariance

$$\begin{aligned} \mathrm{Cov}(g(F(X)), h(F(X))) = & \\ = \int_0^1 & \mathrm{E}\big(\langle \Sigma \nabla (g \circ F)(Y^\alpha), \nabla (h \circ F)(Z^\alpha) \rangle + \\ & + \int_{\mathbb{R}^{|I|}} \Delta_u (g \circ F)(Y^\alpha) \Delta_u (h \circ F)(Z^\alpha)\, \nu(du) \big)\, d\alpha. \end{aligned} \tag{5}$$

Applying the chain rule we get that $\nabla(g \circ F)(y)$ is the product of the transposed matrix of partial derivatives of F and the vector $(\nabla g)(F(y))$. The first from these factors is the matrix with n columns and $|I|$ rows, with non-zero elements only for $k \in I_i$ (i is the kolumn and k is the row number). So

$$\big(\nabla(g \circ F)(y)\big)_k = \begin{cases} \dfrac{\partial f_i}{\partial x_k}(y) \dfrac{\partial g}{\partial v_i}(F(y)) & \text{if } k \in I_i,\ i = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the scalar product in the covariance formula has the following form.

$$
\begin{aligned}
&\langle \Sigma\nabla(g\circ F)(y), \nabla(h\circ F)(z)\rangle = \\
&\quad = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k\in I_i}\sum_{l\in I_j}\sigma_{kl}\frac{\partial f_i}{\partial x_k}(y)\frac{\partial g}{\partial v_i}(F(y))\frac{\partial f_j}{\partial x_l}(z)\frac{\partial h}{\partial v_j}(F(z)) = \\
&\quad = \sum_{i=1}^{n}\frac{\partial g}{\partial v_i}(F(y))\frac{\partial h}{\partial y_i}(F(z))\left(\sum_{k\in I_i}\sum_{l\in I_i}\sigma_{kl}\frac{\partial f_i}{\partial x_k}(y)\frac{\partial f_i}{\partial x_l}(z)\right) + \qquad (6)\\
&\qquad + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{k\in I_i}\sum_{l\in I_j}\sigma_{kl}\frac{\partial f_i}{\partial x_k}(y)\frac{\partial g}{\partial v_i}(F(y))\frac{\partial f_j}{\partial x_l}(z)\frac{\partial h}{\partial v_j}(F(z)). \qquad (7)
\end{aligned}
$$

The expression in line (6) is non-negative because the partial derivatives are non-negative and

$$\sum_{k\in I_i}\sum_{l\in I_i}\sigma_{kl}\frac{\partial f_i}{\partial x_k}(y)\frac{\partial f_i}{\partial x_l}(z) \geq 0$$

due to the fact that σ_{kl} for $k, l \in I_i$ are entries of the covariance matrix of the vector $X(I_i)$. The expression in line (7) is non-negative for all partial derivatives are non-negative and $\sigma_{kl} \geq 0$ if k, l are not in the same block.

It remains to check that the second summand in (5) is non-negative. Let us consider the following sets.

$$
\begin{aligned}
A_+ &= \{u : F(y+u) \geq F(y)\} \cap \{u : F(z+u) \geq F(z)\}\\
A_- &= \{u : F(y+u) \leq F(y)\} \cap \{u : F(z+u) \leq F(z)\}.
\end{aligned}
$$

It is easy to see that on the set $A = A_+ \cup A_-$

$$\Delta_u(g\circ F)(y)\Delta_u(h\circ F)(z) = \big(g(F(y+u)) - g(F(y))\big)\big(h(F(z+u)) - h(F(z))\big) \geq 0,$$

for both factors are at the same time either non-negative or non-positive. It follows that it is enough to prove that

$$\nu(A^c) = \nu(A_+^c \cap A_-^c) = 0, \tag{8}$$

where B^c is the complement of B. We have

$$A_+ = \bigcap_{i=1}^{n}\{u : f_i(y+u) \geq f_i(y),\ f_i(z+u) \geq f_i(z)\},$$

hence

$$A_+^c = \bigcup_{i=1}^n \{u \,:\, f_i(y+u) < f_i(y)\} \cup \{u \,:\, f_i(z+u) < f_i(z)\}$$

and similarly

$$A_-^c = \bigcup_{j=1}^n \{u \,:\, f_j(y+u) > f_j(y)\} \cup \{u \,:\, f_j(z+u) > f_j(z)\}.$$

So $A^c = \bigcup_{1\le i\ne j\le n}^n B_{ij}$, where

$$\begin{aligned} B_{ij} \;&=\; \{u \,:\, f_i(y+u) < f_i(y), f_j(y+u) > f_j(y)\} \\ &\cup\; \{u \,:\, f_i(y+u) < f_i(y), f_j(z+u) > f_j(z)\} \\ &\cup\; \{u \,:\, f_i(z+u) < f_i(z), f_j(y+u) > f_j(y)\} \\ &\cup\; \{u \,:\, f_i(z+u) < f_i(z), f_j(z+u) > f_j(z)\}. \end{aligned}$$

Since f_i's are non-decreasing, $f_i(x+u) < f_i(x)$ implies that there exists $k \in I_i$ such that $u_k < 0$. (If u were in $(\mathbb{R}_+)^{I_i}$ we would have $f_i(x+u) \ge f_i(x)$). Similarly, $f_i(x+u) > f_i(x)$ implies that there exists $l \in I_i$ such that $u_l > 0$. Thus we obtain that

$$B_{ij} \subset \bigcup_{k\in I_i} \bigcup_{l\in I_j} \{u \,:\, u_k < 0, u_l > 0\}.$$

But $i \ne j$ and so k and l in the above union of sets *are not in the same block*. It follows that

$$\nu(\{u \,:\, u_k < 0, u_l > 0\}) = \nu_{kl}\big((-\infty, 0) \times (0, +\infty)\big) = 0.$$

Hence $\nu(B_{ij}) = 0$ and $\nu(A^c) = 0$. $\square$

For future purposes we need a convenient reformulation of the condition imposed in Theorem 2.6 on the two-dimensional Lévy measures ν_{kl}.

Proposition 2.8. Let ν be a measure on $\mathbb{R}^{|I|}$ and let measures ν_{kl} on $\mathbb{R}^2$ be defined by (4). Then the following statements are equivalent:

(i) For all $k, l \in I$, which *are not in the same block*, the measures ν_{kl} are concentrated on $(\mathbb{R}_-)^2 \cup (\mathbb{R}_+)^2$,

(ii) The measure ν is concentrated on the set

$$S = (\mathbb{R}_+)^{|I|} \cup (\mathbb{R}_-)^{|I|} \cup \bigcup_{m=1}^{n} \Big(\{0\}^{\sum_{i=1}^{m-1} |I_i|} \times \mathbb{R}^{|I_m|} \times \{0\}^{\sum_{j=m+1}^{n} |I_j|} \Big). \qquad (9)$$

Proof. It is clear that if ν concentrates on S given in (9), then ν_{kl} satisfy (i). Thus we have to prove the implication (i) $\Rightarrow$ (ii) only. For notational convenience, let us write $k \sim l$ if k and l *are* in the same block and $k \not\sim l$ otherwise. Let us also denote $D_k^+ = \{x \in \mathbb{R}^{|I|} : x_k > 0\}$, $D_k^- = \{x \in \mathbb{R}^{|I|} : x_k < 0\}$ and

$$D = \bigcup_{(k,l)\,:\,k \not\sim l} D_k^+ \cap D_l^-.$$

Then (i) implies $\nu(D_k^+ \cap D_l^-) = 0$ for all pairs (k,l) such that $k \not\sim l$ and so

$$\nu(D) = 0. \qquad (10)$$

Hence it is enough to prove that $\Big((\mathbb{R}_-)^{|I|} \cup (R_+)^{|I|} \cup S\Big)^c \subset D$ or, equivalently, that

$$D^c \setminus (\mathbb{R}_-)^{|I|} = \bigcup_{k=1}^{|I|} D_k^+ \setminus D \subset (\mathbb{R}_+)^{|I|} \cup S. \qquad (11)$$

Notice that

$$\begin{aligned} D^c &= \bigcap_{\{k,l\}\,:\,k \not\sim l} \Big((D_k^+ \cap D_l^-) \cup (D_k^- \cap D_l^+) \Big)^c \\ &= \bigcap_{\{k,l\}\,:\,k \not\sim l} \Big(\{x \in \mathbb{R}^{|I|} : x_k \geq 0, x_l \geq 0\} \cup \{x \in \mathbb{R}^{|I|} : x_k \leq 0, x_l \leq 0\} \Big). \end{aligned}$$

We can decompose

$$\bigcup_{k=1}^{|I|} D_k^+ = \Big(\bigcup_{\{k,l\}\,:\,k \not\sim l} D_k^+ \cap D_l^+ \Big) \cup \left(\bigcup_{k=1}^{|I|} D_k^+ \setminus \Big(\bigcup_{\{k,l\}\,:\,k \not\sim l} D_k^+ \cap D_l^+ \Big) \right).$$

Choose $x \in D_{k_0}^+ \cap D_{l_0}^+ \cap D^c$. Since $x_{k_0} > 0$, we have $x_l \geq 0$ for all $l \not\sim k_0$. Similarly, since $l_0 \not\sim k_0$ and $x_{l_0} > 0$, we have also $x_k \geq 0$ for all k in the block of k_0. It follows that

$$\Big(\bigcup_{\{k,l\}\,:\,k \not\sim l} D_k^+ \cap D_l^+ \Big) \setminus D \subset (\mathbb{R}_+)^{|I|}.$$

Now suppose that $x \in \bigcup_{k=1}^{|I|} D_k^+ \setminus \left(\bigcup_{\{k,l\}\,:\,k \not\sim l} D_k^+ \cap D_l^+ \right)$. Then for some $k_0 \in I$ we have $x_{k_0} > 0$ and $x_l \le 0$ for all $l \not\sim k_0$. If, moreover, $x \notin D$, then $x_{k_0} > 0$ implies $x_l \ge 0$ for all $l \not\sim k_0$. The two last facts imply that $x_l = 0$ for all $l \not\sim k_0$, hence $x \in S$. $\square$

The example given by Samorodnitsky [20] shows that there exists an associated (so associated between blocks of the length 1, too) random vector with 2-dimensional infinitely divisible distribution and with Lévy measure assigning a positive mass out of the set $(\mathbb{R}_+)^2 \cup (\mathbb{R}_-)^2$. So in Theorem 2.6 the condition related to concentration of measures ν_{kl} is not necessary for association between blocks of the multidimensional vector with infinitely divisible distribution.

On the other hand there exists a natural framework proposed by Samorodnitsky *ibid.* in which the concentration of the Lévy measure on $(\mathbb{R}_+)^{|I|} \cup (\mathbb{R}_-)^{|I|}$ *is* necessary. The theorem below can be proved in much the same way as Theorem 3.1 *ibid.* or Proposition 3 in [7].

Theorem 2.9. Let $X \sim \mathcal{ID}(a, \Sigma, \nu)$. Let $\{X_t,\ t \ge 0\}$ be a Lévy process with $X_1 =_d X$. Then the following are equivalent.

(i) For every $t > 0$ and any choice of non-decreasing functions $f_1 : \mathbb{R}^{|I_1|} \to \mathbb{R}, \ldots, f_n : \mathbb{R}^{|I_n|} \to \mathbb{R}$, the vector

$$(f_1((X_t)_{I_1}), \ldots, f_n((X_t)_{I_n}))$$

is associated;

(ii) for all indices k, l which *are not in the same block*, the entries σ_{kl} of the matrix Σ are non-negative and the Lévy measures ν_{kl} concentrate on the set $(\mathbb{R}_+)^2 \cup (\mathbb{R}_-)^2$.

3 Block-association of increments of stochastic processes

Let $\{X_t = (X_t^1, X_t^2, \ldots, X_t^d), t \in \mathbb{R}\}$ be a d-dimensional stochastic process and let $0 < t_1 < t_2 < \ldots < t_n$. We can consider an nd-dimensional random vector formed by the increments

$$X_{t_1} - X_0, \quad X_{t_2} - X_{t_1}, \quad \ldots, \quad X_{t_n} - X_{t_{n-1}}.$$

Such vector has naturally distinguished blocks of the length d. The first is formed by the components of $X_{t_1} - X_0$, the second by the components of $X_{t_2} - X_{t_1}$ and so on. Hence, according to Definition 2.1, we have

Definition 3.1. A d-dimensional stochastic process $\{X_t, t \in \mathbb{R}\}$ *has block-associated increments* if for every $n \in \mathbb{N}$ and any choice of $0 < t_1 < t_2 < \ldots < t_n$ the increments

$$X_{t_1} - X_0, \quad X_{t_2} - X_{t_1}, \quad \ldots, \quad X_{t_n} - X_{t_{n-1}}$$

form the vector associated between blocks.

With such a definition we have the expected result.

Theorem 3.2. Every process with independent increments has block-associated increments.

Next we shall discuss Gaussian processes.

Theorem 3.3. Let $\{X_t, t \geq 0\}$ be a d-dimensional Gaussian process with the covariance functions $K^{k,l}(s,t) = Cov(X_s^k, X_t^l)$, $k, l = 1, \ldots, d$. The process $\{X_t, t \geq 0\}$ has block-associated increments if and only if its covariance functions are L-superadditive on $\{(s,t); s \leq t\}$, i.e.

$$K^{k,l}(s_1, t_1) - K^{k,l}(s_2, t_1) - K^{k,l}(s_1, t_2) + K^{k,l}(s_2, t_2) \geq 0$$

for all $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2$.

Proof. Let us consider the nd-dimensional vector

$$(X_{t_1}^1 - X_0^1, \ldots, X_{t_1}^d - X_0^d, \ldots, X_{t_n}^1 - X_{t_{n-1}}^1, \ldots, X_{t_n}^d - X_{t_{n-1}}^d),$$

where $0 < t_1 < t_2 < \ldots < t_n$. As we know from Theorem 2.5, the process $\{X_t, t \geq 0\}$ has block-associated increments if and only if for all $k, l = 1, \ldots, d$ and $1 \leq i < j \leq n$, $i \neq j$ the covariances

$$\sigma_{ij}^{k,l} = \mathrm{Cov}(X_{t_i}^k - X_{t_{i-1}}^k, X_{t_j}^l - X_{t_{j-1}}^l)$$

are non-negative. But

$$\begin{aligned} 0 \leq \sigma_{ij}^{k,l} &= \mathrm{Cov}(X_{t_i}^k - X_{t_{i-1}}^k, X_{t_j}^l - X_{t_{j-1}}^l) \\ &= K^{k,l}(t_i, t_j) - K^{k,l}(t_i, t_{j-1}) - K^{k,l}(t_{i-1}, t_j) + K^{k,l}(t_{i-1}, t_{j-1}). \end{aligned}$$

$\square$

Remark 3.4. The notion of L-superadditivity is well known, see for example Marshall, Olkin [11, Ch. 6, Sect. D].

Corollary 3.5. If the covariance functions $K^{k,l}$ $(k,l=1,\ldots,d)$ of the d-dimensional Gaussian process $\{X_t, t\geq 0\}$ are continuously twice differentiable for $s\neq t$, then $\{X_t, t\geq 0\}$ has block-associated increments if and only if

$$\frac{\partial^2}{\partial s\,\partial t}K^{k,l}(s,t)\geq 0 \text{ for } s\neq t \text{ and } k,l=1,2.\ldots,d.$$

Proof. The L-superadditivity of the covariance functions is, under the corollary's assumptions, equivalent to the non-negativity of the mixed second derivatives. Indeed,

$$\begin{aligned}
&K^{k,l}(t_i,t_j)-K^{k,l}(t_i,t_{j-1})-K^{k,l}(t_{i-1},t_j)+K^{k,l}(t_{i-1},t_{j-1})=\\
&=\int_{t_{i-1}}^{t_i}\left(\frac{\partial K^{k,l}}{\partial u}(u,t_j)-\frac{\partial K^{k,l}}{\partial u}(u,t_{j-1})\right)\,du\\
&=\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\frac{\partial^2 K^{k,l}}{\partial v\,\partial u}(u,v)\,dv\,du
\end{aligned}$$

and (t_{i-1},t_i), (t_{j-1},t_j) are arbitrary disjoint intervals in $(0,+\infty)$. □

Similarly as Theorem 2.5 produced Theorem 3.3, one could also use Theorem 2.6 for writing a corresponding result for infinitely divisible processes (processes with infinitely divisible finitely dimensional distributions — see e.g. Maruyama [12] or Rajput and Rosinski [18]). We shall do that in a special case and using Proposition 2.8.

Theorem 3.6. Let $\{X_t, t\geq 0\}$ be a d-dimensional infinitely divisible stochastic process. Let us suppose that for every choice of $0=t_0<t_1<t_2<\ldots<t_n$ the distribution of $(X_0,X_{t_1},X_{t_2},\ldots,X_{t_n})$ doesn't have the Gaussian component and the support of its Lévy measure $\nu_{0,t_1,\ldots,t_n}$ is contained in the set

$$\begin{aligned}
\{(x_0,x_1,\ldots,x_n):\ & x_0\leq x_1\leq x_2\leq\ldots\leq x_n \text{ or } x_0\geq x_1\geq x_2\geq\ldots\geq x_n\\
&\text{or } x_1=x_2=\ldots=x_n \text{ or for some } m=2,3,\ldots,n\\
&x_0=x_1=\ldots=x_{m-1}, x_m=x_{m+1}=x_n\},
\end{aligned}$$

where $x_0,x_1,x_2,\ldots,x_n$ are d-dimensional vectors and $\leq$ and $\geq$ are coordinatewise inequalities.

Then $\{X_t, t\geq 0\}$ has block-associated increments.

Proof. Let $U : \mathbb{R}^{(n+1)d} \to \mathbb{R}^{nd}$ be given by the formula

$$U(x) = U(x_0, x_1, \dots, x_n) = (x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}).$$

It is well-known that if $(X_0, X_{t_1}, X_{t_2}, \dots, X_{t_n})$ has an infinitely divisible distribution with a Lévy measure $\nu_{0,t_1,\dots,t_n}$ then the vector of increments $(X_{t_1} - X_0, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ has also an infinitely divisible distribution with the Lévy measure $\nu_{0,t_1,\dots,t_n} \circ U^{-1}$ (up to an atom at 0, see e.g. Sato [21, Proposition 11.10]). For the block-association of increments it is enough that the Lévy measures $\nu_{0,t_1,\dots,t_n} \circ U^{-1}$ concentrate on $S = (\mathbb{R}_+)^{nd} \cup (\mathbb{R}_-)^{nd} \cup \bigcup_{m=1}^n (\{0\}^{(m-1)d} \times \mathbb{R}^d \times \{0\}^{(n-m)d})$ (Proposition 2.8), so for $\nu_{0,t_1,\dots,t_n}$ it is enough to concentrate on the union of sets

$$\begin{aligned} &\{x : x_1 - x_0 \ge 0, x_2 - x_1 \ge 0, \dots, x_n - x_{n-1} \ge 0\} \\ &\quad \cup \ \{x : x_1 - x_0 \le 0, x_2 - x_1 \le 0, \dots, x_n - x_{n-1} \le 0\} \\ &\quad \cup \ \{x : x_2 - x_1 = 0, \dots, x_n - x_{n-1} = 0\} \\ &\quad \cup \ \bigcup_{m=2}^{n} \{x : x_1 - x_0 = 0, \dots, x_{m-1} - x_{m-2} = 0\} \\ &\qquad\qquad\qquad \cap \{x : x_{m+1} - x_m = 0, \dots, x_n - x_{n-1} = 0\} \end{aligned}$$

which equals to

$$\begin{aligned} \{x : x_0 \le x_1 \le x_2 \le \dots \le x_n \text{ or } x_0 \ge x_1 \ge x_2 \ge \dots \ge x_n \\ \text{or } x_1 = x_2 = \dots = x_n \text{ or for some } m = 2, \dots, n \\ x_0 = x_1 = \dots = x_{m-1}, x_m = x_{m+1} = \dots = x_n\}. \end{aligned}$$

□

Remark 3.7. It is clear that the finite dimensional properties of the Lévy measures $\nu_{t_0,t_1,\dots,t_n}$ can be expressed in terms of their projective limit ν (see [12]): ν must be concentrated on the union of sets consisting of non-decreasing trajectories, non-increasing trajectories and rather mysterious trajectories admitting only one jump.

4 Some other notions of relaxed association

The following notion was introduced by Burton et al. [3].

Definition 4.1. A sequence of d-dimensional random vectors $(X_1, X_2, \ldots, X_m)$ is said to be *weakly associated* if whenever π is a permutation of $\{1, 2, \ldots, m\}$, $1 \leq k < m$ and $g : \mathbb{R}^{kd} \to \mathbb{R}$, $h : \mathbb{R}^{(m-k)d} \to \mathbb{R}$ are coordinate-wise non-decreasing, then

$$\mathrm{Cov}(g(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(k)}), h(X_{\pi(k+1)}, X_{\pi(k+2)}, \ldots, X_{\pi(m)}) \geq 0,$$

if the covariance exists. A family of random vectors is weakly associated if every finite subfamily is weakly associated.

Burton et al. *ibid., Theorem 1*, provided an example of a sequence of weakly associated random variables ($d = 1$), which are not associated. Let $Y_1, Y_2, \ldots$ be such a sequence. Fix $d > 1$ and define a sequence of d-dimensional random vectors by

$$X_k = (\underbrace{Y_k, Y_k, \ldots, Y_k}_{k \text{ times}}).$$

Then it is easy to see that $X_1, X_2, \ldots$ is weakly associated but it is not associated between blocks built upon coordinates. The following definition is in the spirit of Section 2.

Definition 4.2. A family $X = \{X_i, i \in I\}$ is called *weakly associated between blocks* if for all non-decreasing functions $f_k : \mathbb{R}^{|I_k|} \to \mathbb{R}$, $k = 1, 2, \ldots, n$, the random vector

$$(f_1(X(I_1)), f_2(X(I_2)), \ldots, f_n(X(I_n)))$$

consists of weakly associated random variables.

The next definition can be found in Bulinski and Shashkin [2].

Definition 4.3. A family $\mathbf{X} = \{X_i, i \in I\}$ is called *positively associated*, if

$$Cov(g(X(A_g)), h(X(A_h))) \geq 0$$

for any disjoint sets $A_g, A_h \subseteq I$ and all non-decreasing functions $g : \mathbb{R}^{A_g} \to \mathbb{R}$, $h : \mathbb{R}^{A_h} \to \mathbb{R}$.

Clearly, for families of random variables ($d = 1$) the notions of weak association and positive association coincide. It is interesting that due to this coincidence, the notions of weak association between blocks and positive association between blocks are also the same. In fact, a definition for the latter should look as follows.

Definition 4.4. A family $\mathbf{X}$ is called *positively associated between blocks*, if for all non-decreasing functions $f_k : \mathbb{R}^{|I_k|} \to \mathbb{R}$, $k = 1, 2, \ldots, n$, the vector

$$(f_1(X(I_1)), f_2(X(I_2)), \ldots, f_n(X(I_n)))$$

is positively associated, i.e. for any disjoint finite sets $A_g, A_h \subset \{1, 2, \ldots, n\}$ and any non-decreasing functions $g : \mathbb{R}^{|A_g|} \to \mathbb{R}$, $h : \mathbb{R}^{|A_h|} \to \mathbb{R}$

$$Cov(g(f_i(X(I_i)), i \in A_g), h(f_j(X(I_j)), j \in A_h)) \geq 0, \tag{12}$$

if the covariance exists.

We see that both (12) and Definition 4.2 state that the random variables $f_1(X(I_1))$, $f_2(X(I_2)), \ldots, f_n(X(I_n))$ are weakly associated, so there is no need to define positive association between blocks.

Remark 4.5. It is easy to see that for jointly Gaussian random variables the two types of relaxed association considered in the present paper (association between blocks and weak association between blocks) coincide and are equivalent to non-negativity of covariances of random variables which *are not in the same block*.

Next we shall give a formal statement of the original form and a relaxed form of negative association due to Joag-Dev and Proschan [8].

Definition 4.6. A family $\mathbf{X} = \{X_i, i \in I\}$ is called *negatively associated* if

$$Cov(g(X(A_g)), h(X(A_h))) \leq 0$$

for any disjoint sets $A_g, A_h \subseteq I$ and all non-decreasing functions $g : \mathbb{R}^{A_g} \to \mathbb{R}$, $h : \mathbb{R}^{A_h} \to \mathbb{R}$.

Definition 4.7. A family $\mathbf{X}$ is called *negatively associated between blocks* if for all non-decreasing functions $f_k : \mathbb{R}^{|I_k|} \to \mathbb{R}$, $k = 1, 2, \ldots, n$, the vector

$$(f_1(X_{I_1}), f_2(X_{I_2}), \ldots, f_n(X_{I_n}))$$

is negatively associated, i.e. for any disjoint finite sets $A_g, A_h \subset \{1, 2, \ldots, n\}$ and any non-decreasing functions $g : \mathbb{R}^{|A_g|} \to \mathbb{R}$, $h : \mathbb{R}^{|A_h|} \to \mathbb{R}$

$$Cov(g(f_i(X_{I_i}), i \in A_g), h(f_j(X_{I_j}), j \in A_h)) \leq 0),$$

if the covariance exists.

We conclude this section with definition of the corresponding notions for increments of processes.

Definition 4.8. A d-dimensional stochastic process $\{X_t, t \geq 0\}$ has *block-weakly-associated* (resp. *block-negatively-associated*) *increments* if for every $n \in \mathbb{N}$ and any choice of $0 < t_1 < t_2 < \ldots < t_n$ the increments

$$X_{t_1} - X_0, \quad X_{t_2} - X_{t_1}, \quad \ldots, \quad X_{t_n} - X_{t_{n-1}}$$

form the sequence of vectors which are weakly (resp. negatively) associated between blocks formed by the d components of each increment $X_{t_i} - X_{t_{i-1}}$.

5 Limit theorems under weak association between blocks

Let $X_1, X_2, \ldots$ be a sequence of d-dimensional random vectors. After building blocks upon the coordinates of consecutive vectors we may compare the notions of weak association of random vectors $\{X_k\}$ (Definition 4.1) and weak association between blocks (Definition 4.2). Formally the latter is weaker: in place of non-decreasing functions g and h "directly" acting on vectors:

$$g(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(k)}),\ h(X_{\pi(k+1)}, X_{\pi(k+2)}, \ldots, X_{\pi(m)}),$$

the latter definition operates with factorizations

$$g\big(f_{\pi(1)}(X_{\pi(1)}), \ldots, f_{\pi(k)}(X_{\pi(k)})\big),\ h\big(f_{\pi(k+1)}(X_{\pi(k+1)}), \ldots, f_{\pi(m)}(X_{\pi(m)})\big).$$

As already mentioned in Introduction, we are not able to exhibit any example of a sequence $\{X_k\}$, which is weakly associated between blocks, but not weakly associated. On the other hand, the computations performed in Section 2 and based on the covariance interpolation formula suggest that it might be a serious advantage to deal with factorized functions while checking whether the sequence is weakly associated between blocks. This is one reason for including the present section into the paper.

The other reason is that the complete generalization of Newman's Central Limit Theorem [14] and Newman-Wright's Invariance Principle [16] for sums of stationary associated random variables, originally proved by Burton, Dabrowski and Dehling [3] for weakly associated random vectors, remains valid under weak association between blocks, without any change in its proof. Here "complete generalization" means including as a particular case the Central Limit Theorem for i.i.d. random vectors, with covariance matrices possibly containing negative entries.

Theorem 5.1. Let $X_1, X_2, \ldots$ be a strictly stationary sequence of d-dimensional random vectors, which are *weakly associated between blocks* and let $S_n = X_1 + X_2 + \ldots + X_n$.

If $\mathrm{E}X_1 = 0$, $\mathrm{E}\|X_1\|^2 < +\infty$ and $\sum_{j=2}^{\infty} \mathrm{E}X_1^k X_j^l < +\infty$ for all $k, l = 1, \ldots, d$ (where X_j^k is the k-th component of the vector X_j), then

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathscr{D}} \mathcal{N}(0, \Sigma)$$

where $\Sigma = (\sigma_{kl})_{k,l=1\ldots,d}$ and $\sigma_{kl} = \mathrm{E}X_1^k X_1^l + 2\sum_{j=2}^{\infty} \mathrm{E}X_1^k X_j^l$.

Moreover, if

$$Y_n(t) = \frac{1}{\sqrt{n}} S_{[nt]},\ t \in \mathbb{R}^+,$$

(or $Y_n(t)$ is a polygonal interpolation between points $(k/n, S_k/\sqrt{n})$), then

$$Y_n \xrightarrow[n\to\infty]{\mathscr{D}} W_\Sigma,$$

on the function space $C(\mathbb{R}^+ : \mathbb{R}^d)$, where W_Σ is a Wiener process with covariance matrix Σ.

Proof. In their proof, Burton, Dabrowski and Dehling [3] use the weak association of the following random variables:

$$f_j(X_j) = \langle a_j, X_j \rangle = \sum_{k=1}^{d} a_j^k X_j^k,$$

where $a_j^1, a_j 2, \ldots, a_j^d \geq 0$ are suitably chosen (for tightness purposes, convergence of finite dimensional distributions etc.). Our assumption on weak association between blocks provides exactly the same information. $\square$

Remark 5.2. It is likely that also other existing limit theorems for associated random variables (see e.g. [2, Chapter 3]) can be proved under relaxed assumptions like weak association between blocks and in a similar way as Theorem 5.1. In particular, there is a work in progress towards results on convergence to stable laws with infinite variance, paralleling [4].

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