

A Note on the Chevalley–Warning Theorems

D.R. Heath-Brown
Mathematical Institute, Oxford

In memory of Anatolii Karatsuba

1 Introduction

The most widely known result of Chevalley–Warning type states that if one has a polynomial over a finite field of characteristic p , and the number of variables exceeds the degree, then the number of zeros is a multiple of p . In particular, if the polynomial is homogeneous, there is at least one non-trivial zero.

There are a number of related results in the literature. Generally, let \mathbb{F}_q be a finite field, and let $\mathbf{f} = (f_1(\mathbf{x}), \dots, f_r(\mathbf{x}))$ be an r -tuple of polynomials $f_i(x_1, \dots, x_n) \in \mathbb{F}_q[x_1, \dots, x_n]$. Let d_i be the total degree of f_i , and write $d = d_1 + \dots + d_r$. For any subset $S \subseteq \mathbb{F}_q^n$ we put

$$z(\mathbf{f}; S) = z(S) := \{\mathbf{x} \in S : \mathbf{f}(\mathbf{x}) = 0\}$$

and

$$\mathcal{N}(\mathbf{f}; S) = \mathcal{N}(S) := \#z(\mathbf{f}; S).$$

Then the original 1936 result of Chevalley [2] stated that if $n > d$ and $z(\mathbb{F}_q^n)$ is non-empty, then it contains at least 2 points. Immediately afterwards it was shown by Warning [3] that if $n > d$ then

$$p \mid \mathcal{N}(\mathbb{F}_q^n), \tag{1}$$

where p is the characteristic of \mathbb{F}_q . In fact this followed from a trivial rearrangement of Chevalley’s argument. In the same paper Warning proved that if $n \geq d$, and if H_1 and H_2 are parallel affine hyperplanes in \mathbb{F}_q^n then

$$\mathcal{N}(H_1) \equiv \mathcal{N}(H_2) \pmod{p}. \tag{2}$$

As a corollary he deduced that if $z(\mathbb{F}_q^n)$ is non-empty then

$$\mathcal{N}(\mathbb{F}_q^n) \geq q^{n-d} \quad (n > d). \tag{3}$$

Later, in 1964, Ax [1] strengthened (1) by showing that indeed one has

$$q \mid \mathcal{N}(\mathbb{F}_q^n) \quad (n > d). \quad (4)$$

When n is large compared with d there are further improvements possible, as was shown by Katz [4] for example. These sharpenings of (4) lead to improvements of all the results we will establish. However we wish to focus here on the situation in which n is not much larger than d , and so will be content with using (4).

Our first result shows that there is a strengthening of (2) corresponding to Ax's improvement of (1). To stress that the affine linear spaces we encounter are not necessarily vector subspaces of \mathbb{F}_q^n we will work over $\mathbb{A}^n(\mathbb{F}_q)$, but of course the sets $z(\mathbb{A}^n(\mathbb{F}_q))$ and $z(\mathbb{F}_q^n)$ are the same.

Theorem 1 *With the notation above, we have*

$$\mathcal{N}(L_1) \equiv \mathcal{N}(L_2) \pmod{q} \quad (5)$$

for any two parallel linear spaces $L_1, L_2 \subseteq \mathbb{A}^n(\mathbb{F}_q)$ of dimension d or more.

Indeed our argument shows how one may deduce (5) from (4).

We next look at improvements to (3). If $K = \mathbb{F}_{q^k}$ has field basis $\omega_1, \dots, \omega_k$ over $F = \mathbb{F}_q$ we set

$$N_k(x_1, \dots, x_k) := N_{K/F}(x_1\omega_1 + \dots + x_k\omega_k). \quad (6)$$

This produces a form of degree k in k variables, with the property that $N_k(x_1, \dots, x_k) = 0$ with variables $x_i \in F_q$ only when the x_i are all zero. In particular, if \mathbf{f} consists of a single polynomial $f_1 = N_d$ then $z(\mathbf{f}; \mathbb{A}^n(\mathbb{F}_q))$ will be a linear space of dimension $n - d$, and we will have equality in (3). In contrast we have the following results.

Theorem 2 *Suppose that $n > d$ and that $z(\mathbb{A}^n(\mathbb{F}_q))$ is non-empty, and is not a linear subspace of $\mathbb{A}^n(\mathbb{F}_q)$. Then*

- (i) *For any q we have $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) > q^{n-d}$;*
- (ii) *If $q \geq 4$ we have $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq 2q^{n-d}$; and*
- (iii) *For any q we have $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq q^{n+1-d}/(n+2-d)$ providing that the polynomials \mathbf{f} are homogeneous.*

Our proof of this will use only combinatorial facts about $\mathbb{A}^n(\mathbb{F}_q)$, along with (5) and its consequence (4). Thus the analogues of these results would be true if one replaced $z(\mathbb{A}^n(\mathbb{F}_q))$ by any set with the property (5) for every pair of parallel d -dimensional linear spaces $\Lambda_1, \Lambda_2 \subseteq \mathbb{A}^n(\mathbb{F}_q)$. In particular we see that part (ii) is best possible in this sense, since (5) would hold if $z(\mathbb{A}^n(\mathbb{F}_q))$ were composed of two parallel linear spaces of dimension $n - d$.

It may be instructive to examine two simple examples at this point. The first shows that one cannot hope to improve part (iii) of Theorem 2 to say that $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq q^{n+1-d}$. Here we take \mathbf{f} to consist of the single polynomial $Q(x_1, \dots, x_4)N_{n-4}(x_5, \dots, x_n)$. Here N_{n-4} is a norm form of degree $n - 4$, as in (6), and

$$Q(x_1, x_2, x_3, x_4) = x_1x_2 + x_3^3 + x_3x_4 + cx_4^2$$

is a quadratic form, with the coefficient $c \in \mathbb{F}_q$ chosen so that $x_3^3 + x_3x_4 + cx_4^2$ does not factor over \mathbb{F}_q . Then $Q = 0$ has precisely $q^3 - q^2 + q$ solutions $(x_1, \dots, x_4) \in \mathbb{F}_q^4$, and since $d = n - 2$ in this case we have

$$\mathcal{N}(\mathbf{f}; \mathbb{A}^n(\mathbb{F}_q)) = q^{n+1-d}(1 - q^{-1} + q^{-2}) < q^{n+1-d}.$$

Indeed this also shows that part (ii) of the theorem cannot be extended to $q = 2$.

Our second example shows that when \mathbf{f} consists of a single polynomial f_1 , it is possible for $z(\mathbb{A}^n(\mathbb{F}_q))$ to be a linear space of dimension $n - k$ even though f_1 does not split into linear factors. For this we take $n = 4$ and choose a field basis $1, \alpha$ for \mathbb{F}_{q^2} over \mathbb{F}_q . Let N and N' denote norm forms for \mathbb{F}_{q^4} over \mathbb{F}_{q^2} and \mathbb{F}_{q^2} over \mathbb{F}_q respectively, using the basis $1, \alpha$ in the latter case. Let σ be the nontrivial automorphism of \mathbb{F}_{q^2} over \mathbb{F}_q . Then we may write $N(\mathbf{x}) = Q_1(\mathbf{x}) + \alpha Q_2(\mathbf{x})$ for certain quadratic forms Q_1, Q_2 over \mathbb{F}_q . It is clear that we may use a linear change of variables over $\overline{\mathbb{F}_q}$ to write $Q_1 + \alpha Q_2 = Y_1Y_2$ and $Q_1 + \alpha^\sigma Q_2 = Y_3Y_4$ with independent variables Y_1, \dots, Y_4 . It follows that if we take $\beta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ different from α and α^σ then $Q_1 + \beta Q_2 =$ will have rank 4. Such a β certainly exists, except when $q = 2$. We now see that the polynomial $f(\mathbf{x}) = (Q_1(\mathbf{x}) + \beta Q_2(\mathbf{x}))(Q_1(\mathbf{x}) + \beta Q_2(\mathbf{x}))^\sigma$ is defined over \mathbb{F}_q and does not split into linear factors. None the less we have $f = 0$ only for $\mathbf{x} = 0$.

Finally we will investigate the structure of $z(\mathbf{f}; \mathbb{A}^n(\mathbb{F}_q))$ geometrically.

Theorem 3 *Given n and d there is a constant $C(n, d)$ as follows. For any set of polynomials \mathbf{f} whose degrees total d there is a corresponding set of absolutely irreducible varieties V_1, \dots, V_m defined over \mathbb{F}_q such that $z(\mathbf{f}; \mathbb{A}^n(\mathbb{F}_q))$ is precisely the set of \mathbb{F}_q -points in $V_1 \cup \dots \cup V_m$. Moreover each variety V_i has degree at most $C(n, d)$, and $m \leq C(n, d)$.*

The Lang-Weil bound [5] now gives us the following immediate corollary.

Corollary *Suppose that the largest dimension among the varieties V_i is D , and that this occurs k times. Then, enlarging $C(n, d)$ if necessary, we have*

$$|\mathcal{N}(\mathbf{f}; \mathbb{A}^n(\mathbb{F}_q)) - kq^D| \leq C(n, d)q^{D-1/2}. \quad (7)$$

Thus if $q \gg_{n,d} 1$ we have $D \geq n - d$, and indeed if $z(\mathbf{f}; \mathbb{A}^n(\mathbb{F}_q))$ is not a linear space of dimension $n - d$ then $D \geq n + 1 - d$. In the latter case we have

$$\mathcal{N}(\mathbf{f}; \mathbb{A}^n(\mathbb{F}_q)) \geq q^{n+1-d} - C(n, d)q^{n+1/2-d}$$

for $q \gg_{n,d} 1$.

One would conjecture that $D \geq n - d$ irrespective of the size of q , and it would be of interest to have a direct geometrical proof of this. Of course, the lower bounds on D follow on comparing (7) with (3) and part (iii) of Theorem 2.

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We now present our proof of Theorem 1. Given a polynomial $f(x_1, \dots, x_n)$ of total degree e there are two reasonable ways of associating a form to it. One may take $f_-(x_1, \dots, x_n)$ to be the homogeneous part of degree e , or one may define

$$f_+(x_0, \dots, x_n) = x_0^e f(x_1/x_0, \dots, x_n/x_0).$$

For a system \mathbf{f} we define \mathbf{f}_- and \mathbf{f}_+ by the above processes, using degree $e = d_i$ for each polynomial f_i . Clearly each zero of \mathbf{f} produces exactly $q - 1$ zeros of \mathbf{f}_+ with $x_0 \neq 0$; and the zeros of \mathbf{f}_+ with $x_0 = 0$ correspond precisely to the zeros of \mathbf{f}_- . Thus

$$\mathcal{N}(\mathbf{f}_+; \mathbb{F}_q^{n+1}) = (q - 1)\mathcal{N}(\mathbf{f}; \mathbb{F}_q^n) + \mathcal{N}(\mathbf{f}_-; \mathbb{F}_q^n).$$

In particular, if $n \geq d$ then (4) yields $q \mid \mathcal{N}(\mathbf{f}_+; \mathbb{F}_q^{n+1})$ and hence

$$\mathcal{N}(\mathbf{f}; \mathbb{F}_q^n) \equiv \mathcal{N}(\mathbf{f}_-; \mathbb{F}_q^n) \pmod{q}.$$

We therefore see that the value of $\mathcal{N}(\mathbf{f}; \mathbb{F}_q^n)$ modulo q only depends on the leading homogeneous parts of the polynomials f_1, \dots, f_r .

Now suppose we have parallel m -dimensional affine linear spaces L_1 and L_2 in \mathbb{F}_q^n . We may represent the restriction of the forms \mathbf{f} to L_1 by a set of polynomials $(g_1^{(1)}, \dots, g_r^{(1)})$ in m variables; and similarly for L_2 . Since L_1 and

L_2 are parallel, the leading homogeneous parts of $g_i^{(1)}$ and $g_i^{(2)}$ will be the same. We are therefore led to Theorem 1.

Our proof of Theorem 2 is based on the following two lemmas.

Lemma 1 *Let $L_0 \subseteq \mathbb{A}^n(\mathbb{F}_q)$ be a linear space. Choose a linear space L of maximal dimension, k say, such that $L \supseteq L_0$ and $\mathcal{N}(L) = \mathcal{N}(L_0)$. Suppose $L' \supset L$ is a linear space of dimension $k + 1$ such that $\mathcal{N}(L')$ is minimal. Then*

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq \mathcal{N}(L) + \frac{q^{n-k} - 1}{q - 1}(\mathcal{N}(L') - \mathcal{N}(L)). \quad (8)$$

Lemma 2 *Let $S \subseteq \mathbb{A}^t(\mathbb{F}_q)$ be a set containing $t+1$ points in general position. Then*

- (i) *If $q = 2$, and if there is no 2-plane $L \subseteq \mathbb{A}^t(\mathbb{F}_q)$ meeting S in exactly 3 points, then $S = \mathbb{A}^t(\mathbb{F}_q)$.*
- (ii) *If $q \geq 3$, and if $\ell \subseteq S$ for every line ℓ meeting S in at least two points, then $S = \mathbb{A}^t(\mathbb{F}_q)$.*
- (iii) *If $q \geq 4$, and if $\#(S \cap \ell) \geq q - 1$ for every line ℓ meeting S in at least two points, then $\mathbb{A}^t(\mathbb{F}_q) \setminus S$ is contained in a hyperplane.*
- (iv) *If $m \geq 2$ is an integer, and if $\#(S \cap \ell) \geq m + 1$ for every line ℓ meeting S in at least two points, then*

$$\#S \geq \frac{m^{t+1} - 1}{m - 1}.$$

To deduce Theorem 2 we assume that $z(\mathbb{A}^n(\mathbb{F}_q))$ contains a maximal set of $t \geq n + 1 - d$ points in general position. We take $\mathbb{A}^t(\mathbb{F}_q)$ to be the space spanned by these points and apply Lemma 2 to the set S consisting of zeros of \mathbf{f} lying in $\mathbb{A}^t(\mathbb{F}_q)$. We will give the details required for the different parts of Theorem 2 in due course, but the general strategy is as follows. Lemma 2 will either provide a suitable lower bound for $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q))$, or produce a linear space L_0 to which we will apply Lemma 1. This linear space will have dimension two in the case of part (i) of Lemma 2, or dimension one otherwise. We will have arranged that $\mathcal{N}(L_0) \leq q - 1$, and since $\mathcal{N}(L) = \mathcal{N}(L_0)$ in Lemma 1 the dimension k of L can be at most d , for otherwise (5) will give us a contradiction. If $k \leq d - 2$ then (8) implies that

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq \frac{q^{n-k} - 1}{q - 1} \geq \frac{q^{n+2-d} - 1}{q - 1} \geq_{n+1-d},$$

which is a satisfactory lower bound. Thus we may assume that either L or L' has dimension d . Let $L_1 = L$ or L' as appropriate. If $\mathcal{N}(L_1)$ takes a value $v \leq q-1$ then we may deduce from (5) that $\mathcal{N}(L^*) \geq v$ for every affine d -plane L^* parallel to L_1 . Covering $\mathbb{A}^n(\mathbb{F}_q)$ with such linear spaces we deduce that $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq vq^{n-d}$. Such bounds as these will suffice in all cases for the theorem.

We begin by establishing part (i) of Theorem 2. From parts (i) and (ii) of Lemma 2 we may obtain a satisfactory bound $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq q^{n+1-d}$ unless either $q = 2$ and there is a 2-plane $L_0 \subseteq \mathbb{A}^n(\mathbb{F}_q)$ with $\mathcal{N}(L_0) = 3$, or $q \geq 3$ and there is a line $L_0 \subseteq \mathbb{A}^n(\mathbb{F}_q)$ with $2 \leq \mathcal{N}(L_0) \leq q-1$. We now apply Lemma 1. As above we must have $k \leq d$. Moreover if $k \leq d-1$ then (8) yields

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq \frac{q^{n+1-d} - 1}{q-1} > q^{n-d},$$

which is satisfactory. If $k = d$ and $q = 2$ then (8) produces

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq 3 + \frac{2^{n-d} - 1}{2-1} > 2^{n-d}.$$

Finally, if $k = d$ and $q \geq 3$ then $2 \leq \mathcal{N}(L_0) = \mathcal{N}(L) \leq q-1$, and the argument above, with $2 \leq v \leq q-1$, shows that $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq 2q^{n-d}$. Thus $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) > q^{n-d}$ in all cases.

We turn now to part (ii) of Theorem 2. By part (iii) of Lemma 2 we have $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq q^{n+1-d} - q^{n-d} \geq 2q^{n-d}$ unless there is a line $L_0 \subseteq \mathbb{A}^n(\mathbb{F}_q)$ with $2 \leq \mathcal{N}(L_0) \leq q-2$. As above, when we apply Lemma 1 we may assume that $k = d-1$ or d . If $k = d$ then $2 \leq \mathcal{N}(L) \leq q-2$ and we may apply (5) as before to conclude that $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq 2q^{n-d}$. Similarly if $k = d-1$ and $\mathcal{N}(L') - \mathcal{N}(L) = 1$ we have $3 \leq \mathcal{N}(L') \leq q-1$ and therefore $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq 3q^{n-d}$. Finally, if $k = d-1$ and $\mathcal{N}(L') - \mathcal{N}(L) \geq 2$ we deduce from (8) that

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq 2 \frac{q^{n+1-d} - 1}{q-1} \geq 2q^{n-d}.$$

This establishes the required bound in all cases.

To prove the third claim of Theorem 2 we apply Lemma 2 part (iv) with an integer $m \leq q-1$ to be chosen in due course. Thus

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq \frac{m^{n+2-d} - 1}{m-1} \geq m^{n+1-d}$$

unless there is a line with $2 \leq \mathcal{N}(L_0) \leq m$. In the latter case we deduce from Lemma 1 that

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq \frac{q^{n-k} - 1}{q-1} (\mathcal{N}(L') - m),$$

and by the previous argument it suffices to consider the values $k = d - 1$ and $k = d$. If $k = d - 1$ we deduce that $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq q^{n-d}(q - m)$, unless

$$m + 1 \leq \mathcal{N}(L') \leq q - 1.$$

However, in the latter case we conclude by our standard argument using (5) that $\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq (m + 1)q^{n-d}$. In the homogeneous case the value $k = d$ cannot occur. To show this we consider two cases. If $k = d$ and $\mathbf{0} \in L$ then $q - 1 \mid \mathcal{N}(L) - 1$, since if $\mathbf{x} \in z(L)$ then every scalar multiple of \mathbf{x} is also in $z(L)$. This however is impossible since $2 \leq \mathcal{N}(L_0) = \mathcal{N}(L) \leq m$. On the other hand, if $\mathbf{0} \notin L$ we consider the $(d + 1)$ -dimensional linear space $L' := \langle L, \mathbf{0} \rangle$. Here we find that $\mathcal{N}(L') = 1 + (q - 1)\mathcal{N}(L)$. According to (4) we will have $q \mid \mathcal{N}(L')$, whence $\mathcal{N}(L) \equiv 1 \pmod{q}$. This again is impossible, since $2 \leq \mathcal{N}(L) \leq m$. It follows that one of the inequalities

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq m^{n+1-d},$$

or

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq q^{n-d}(m + 1),$$

or

$$\mathcal{N}(\mathbb{A}^n(\mathbb{F}_q)) \geq (m + 1)q^{n-d}$$

must hold. The required estimate now follows on choosing

$$m = \left\lceil \frac{q}{n + 2 - d} \right\rceil,$$

and noting that

$$\left(1 - \frac{1}{v + 1}\right)^v \geq \frac{1}{v + 1}$$

for any positive integer v .

We now prove Lemmas 1 and 2. For the first of these it is enough to observe that $\mathbb{A}^n(\mathbb{F}_q)$ is the disjoint union of L together with the sets $L^* \setminus L$, with L^* running over all $(k + 1)$ -dimensional linear spaces containing L . There are $(q^{n-k} - 1)/(q - 1)$ such spaces L^* . Thus the lemma follows from the fact that $\mathcal{N}(L^* \setminus L) \geq \mathcal{N}(L') - \mathcal{N}(L)$.

Lemma 2 will require distinctly more work. We prove the four claims separately, using induction on t in each case. The various statements are all trivially true for $t = 0$ and $t = 1$. To handle the induction step for case (i) we may assume that S contains a linear space L_0 say, of dimension $t - 1$, together with a point P_0 say, not on L_0 . Now choose any point $P \notin L_0$ with

$P \neq P_0$, and aim to show that $P \in S$. We will then be able to conclude that $S = \mathbb{A}^t(\mathbb{F}_q)$ as required. Let P_1 be any point in L_0 and consider the 2-plane generated by P, P_0 and P_1 . Since $q = 2$, this plane consists of the three generators together with a fourth point P_2 say, which must belong to L_0 . Our construction has ensured that P, P_1 and P_2 are all in S , and so by our hypothesis we also have $P_1 \in S$, as required.

For part (ii) we use the same initial setup. We assume that S contains a linear space L_0 of dimension $t - 1$, along with a point P_0 not in L_0 , and we choose a point $P \notin L_0$ different from P_0 . Suppose firstly that the line ℓ generated by P_0 and P meets L_0 , at a point P_1 say. Then ℓ meets S in at least 2 points, namely P_0 and P_1 . Then, by our hypothesis, ℓ is contained in S , whence P in particular belongs to S . This deals with all points P except those which lie on the hyperplane, L_1 say, which is parallel to L_0 and which passes through P_0 . To handle such points P we begin by fixing any point P_1 on L_0 . We then consider the line ℓ generated by P and P_1 . Since $q \geq 3$ this line contains at least one point additional point P_2 say, which cannot lie in L_1 . Thus $P_1 \in S$ by what has already been proved. Hence ℓ meets S in at least two points, namely P_1 and P_2 , and our hypothesis implies that every point of ℓ belongs to S . In particular $P \in S$, as required.

The proof of part (iii) of Lemma 2 is the most involved. We will write S^c for the complement $\mathbb{A}^t(\mathbb{F}_q) \setminus S$ of S . Our strategy will be to show that if S^c also contains $t + 1$ points in general position, then both $\#S > \frac{1}{2}q^t$ and $\#S^c > \frac{1}{2}q^t$, which will provide a contradiction. We observe that the hypothesis of part (iii) is symmetric between S and S^c , since S meets ℓ in at least two points if and only if $\#(S^c \cap \ell) < q - 1$. Now let R be either S or S^c , and assume that R contains $t + 1$ points P_0, \dots, P_t in general position. For our inductive assumption we suppose that, for any linear space $L \subset \mathbb{A}^t(\mathbb{F}_q)$ of dimension $t - 1$, either $R \cap L$ fails to contain t points in general position, or $L \setminus R$ is contained in a proper linear subspace of L . Thus either $L \cap R$ or $L \cap R^c$ lies in a proper linear subspace of L . When $L = L_0$ is generated by P_1, \dots, P_t we must be in the second case. For every $P \in L_0$, the line ℓ generated by P and P_0 meets R in at least two points (namely P and P_0) and hence contains at least $q - 1$ points of R . For distinct choices of P the sets $\ell \setminus \{P_0\}$ are disjoint, whence

$$\#R \geq 1 + (q - 2)\#(L_0 \cap R). \quad (9)$$

Now suppose that every linear space L of dimension $t - 1$, parallel but not equal to L_0 , has the property that $L \cap R$ lies in a proper linear subspace of L . Then since $\mathbb{A}^t(\mathbb{F}_q)$ is a disjoint union of L_0 with the various spaces L , we see that

$$\#R \leq \#(L_0 \cap R) + (q - 1)q^{t-2}.$$

Comparing this with (9) yields $(q - 3)\#(L_0 \cap R) < (q - 1)q^{t-2}$, and since $\#(L_0 \cap R) \geq q^{t-1} - q^{t-2}$ we find that $q < 4$. Thus, under the assumption that $q \geq 4$, there must be at least one space L_1 , parallel but not equal to L_0 , for which it is $L_1 \cap R^c$ which is contained in a proper linear subspace of L_1 . If we pick any point $Q \in L_1$ and count points of R on lines from Q to L_0 we will obtain at least $(q - 2)\#(L_0 \cap R)$ points of R not lying on L_1 , by the argument that established (9). Allowing for points of $L_1 \cap R$ we find that

$$\#R \geq (q - 2)\#(L_0 \cap R) + \#(L_1 \cap R).$$

However we have arranged that $L_0 \cap R^c$ and $L_1 \cap R^c$ are both contained in proper linear subspaces, so that $\#(L_0 \cap R) \geq q^{t-1} - q^{t-2}$, and similarly for $\#(L_1 \cap R)$. It then follows that $\#R \geq (q - 1)^2 q^{t-2} > \frac{1}{2}q^t$, since $q \geq 4$. As explained above, this inequality leads to the claim made in part (iii) of the lemma.

Finally we turn to part (iv) of Lemma 2. For the induction step we assume that we have a linear space L_0 of dimension $t - 1$, along with a point P_0 not in L_0 , such that

$$\#(S \cap L_0) \geq \frac{m^t - 1}{m - 1}$$

and $P_0 \in S$. If P is any point in $S \cap L_0$ the line ℓ generated by P and P_0 contains at least 2 points of S , and hence by our hypothesis contains at least $m + 1$ such points. For different points P the sets $\ell \setminus \{P_0\}$ are disjoint, whence

$$\#S \geq 1 + m\#(S \cap L_0) \geq 1 + m \frac{m^t - 1}{m - 1} = \frac{m^{t+1} - 1}{m - 1}.$$

This completes the induction.

We now establish Theorem 3. If any of the polynomials \mathbf{f} vanishes identically we may clearly remove it from the collection, and if any of the polynomials is a non-zero constant then $z(\mathbf{f}; \mathbb{A}^n(\mathbb{F}_q))$ is empty, so that the result is trivial. It follows that we may assume that each of the f_i has positive degree, whence r , which is the number of polynomials f_i , is at most d . Thus the zero-set of the polynomials \mathbf{f} , over $\overline{\mathbb{F}_q}$ is an algebraic variety, V say, of degree at most d^d and dimension at most n . This variety is defined over \mathbb{F}_q , and is not necessarily absolutely irreducible. Indeed V may contain irreducible components which are not defined over \mathbb{F}_q . We call such components “bad”, and our goal is to replace V by a variety V^* with no bad components, and such that $V(\mathbb{F}_q) = V^*(\mathbb{F}_q)$. We proceed to describe a general “reduction process” for our varieties. Suppose that V has at least one bad component.

Choose such a component, W say, of maximal dimension D say. Let the Galois conjugates of W be $W = W_1, \dots, W_k$. Since each of these is an irreducible component of V we see that $k \leq \deg(V)$. Any \mathbb{F}_q -point of W lies on $W' := W_1 \cap \dots \cap W_k$. Thus if we replace the components W_1, \dots, W_k of V by W' , and call the resulting variety V' , we see that $V(\mathbb{F}_q) = V'(\mathbb{F}_q)$. Moreover V' contains no bad components of dimension greater than D , and since $\dim(W') < \dim(W)$ there is one fewer bad component of dimension D . We also note that

$$\deg(W') \leq \deg(W)^k \leq \deg(V)^{\deg(V)}$$

whence

$$\deg(V') \leq \deg(V) + \deg(V)^{\deg(V)}.$$

We now repeat the reduction process, passing from V' to V'' and so on. Since the number of bad components of maximal dimension is reduced at each step, while the degree remains under control, the process will eventually terminate, and will produce a variety V^* whose degree is bounded in terms of $\deg(V)$ and n . The theorem then follows.

References

- [1] J. Ax, Zeros of polynomials over finite fields, *Amer. J. Math.*, 86 (1964), 255-261.
- [2] C. Chevalley, Démonstration d'une hypothèse de M. Artin, *Abh. Math. Sem. Univ. Hamburg*, 11 (1936), 73-75.
- [3] E. Warning, Bemerkung zur vorstehenden Arbeit von Herrn Chevalley, *Abh. Math. Sem. Univ. Hamburg*, 11 (1936), 76-83.
- [4] N.M. Katz, On a theorem of Ax, *Amer. J. Math.*, 93 (1971), 485-499.
- [5] S. Lang and A. Weil, Number of points of varieties in finite fields, *Amer. J. Math.*, 76 (1954), 819-827.

Mathematical Institute,
 24-29, St. Giles',
 Oxford
 OX1 3LB
 UK

rhb@maths.ox.ac.uk