Derived equivalences for Cohen-Macaulay Auslander algebras

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Abstract

Let A and B be Gorenstein Artin algebras of Cohen-Macaulay finite type. We prove that, if A and B are derived equivalent, then their Cohen-Macaulay Auslander algebras are also derived equivalent.

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1 Introduction

Triangulated categories and derived categories were introduced by Grothendieck and Verdier [28]. Today, they have widely been used in many branches: algebraic geometry, stable homotopy theory, representation theory, etc. In the representation theory of algebras, we will restrict our attention to the equivalences of derived categories, that is, derived equivalences. Derived equivalences have been shown to preserve many invariants and provide new connection. For instance, Hochschild homology and cohomology [27], finiteness of finitistic dimension [24] have been shown to be invariant under derived equivalences. Moreover, derived equivalences are related to cluster categories and cluster tilting objects [5]. As is known, Rickard's Morita theory for derived categories leaves something to be desired, though, as for some pairs of rings, or algebras, it is currently difficult, sometimes even impossible to verify whether there exists a tilting complex. It is of interest to construct a new derived equivalence from given one by finding a suitable tilting complex. Rickard [26, 27] used tensor products and trivial extensions to get new derived equivalences. In the recent years, Hu and Xi have provided various techniques to construct new derived equivalences. In [15] they established an amazing connection between derived equivalences and Auslander-Reiten sequences via BB-tilting modules, and obtained derived equivalences from Auslander-Reiten triangles. In [17] they constructed new derived equivalences between Φ-Auslander-Yoneda algebras from a given almost ν-stable equivalence.

In [17, Corollary 3.13] Hu and Xi proved that, if two representation finite self-injective Artin algebras are derived equivalent, then their Auslander algebras are derived equivalent. In this paper, we generalize their result and prove that, if two Cohen-Macaulay finite Gorenstein Artin algebras are derived equivalent, then their Cohen-Macaulay Auslander algebras are also derived equivalent.

This paper is organized as follows. In Section 2, we review some facts on derived categories and derived equivalences. In Section 3, we state and prove our main result.

2 Preliminaries

In this section, we shall recall some definitions and notations on derived categories and derived equivalences.

Let $\mathscr{A}$ be an abelian category. For two morphisms $\alpha : X \to Y$ and $\beta : Y \to Z$, their composition is denoted by $\alpha\beta$. An object $X \in \mathscr{A}$ is called a additive generator for $\mathscr{A}$ if $\mathrm{add}\,(X) = \mathscr{A}$, where $\mathrm{add}\,(X)$ is the additive subcategory of $\mathscr{A}$ consisting of all direct summands of finite direct sums of the copies of X. A complex $X^\bullet = (X^i, d_X^i)$ over $\mathscr{A}$ is a sequence of objects X^i and morphisms d_X^i in $\mathscr{A}$ of the form: $\cdots \to X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+1} \to \cdots$, such that $d^i d^{i+1} = 0$ for all $i \in \mathbb{Z}$. If $X^\bullet = (X^i, d_X^i)$ and $Y^\bullet = (Y^i, d_Y^i)$ are two complexes, then a morphism $f^\bullet : X^\bullet \to Y^\bullet$ is a sequence of morphisms $f^i : X^i \to Y^i$ of $\mathscr{A}$ such that $d_X^i f^{i+1} = f^i d_Y^i$ for all $i \in \mathbb{Z}$. The map

$f^\bullet$ is called a chain map between $X^\bullet$ and $Y^\bullet$. The category of complexes over $\mathscr{A}$ with chain maps is denoted by $\mathrm{C}(\mathscr{A})$. The homotopy category of complexes over $\mathscr{A}$ is denoted by $\mathrm{K}(\mathscr{A})$ and the derived category of complexes is denoted by $\mathrm{D}(\mathscr{A})$.

Let R be a commutative Artin ring. And let A be an Artin R-algebra. We denote by A-mod the category of finitely generated left A-modules. The full subcategory of A-mod consisting of projective modules is denoted by ${}_A\mathcal{P}$. Recall that a homomorphism $f : X \to Y$ of A-modules is called a radical map provided that for any A-module Z and homomorphisms $g : Y \to Z$ and $h : Z \to X$, the composition hfg is not an isomorphism. A complex of A-modules is called a radical complex if its differential maps are radical maps. Let $\mathrm{K}^{\mathrm{b}}(A)$ denote the homotopy category of bounded complexes of A-modules. We denote by $\mathrm{D}^{\mathrm{b}}(A)$ by the bounded derived category of A-mod.

The fundamental theory on derived equivalences has been established. Rickard [25] gave a Morita theory for derived categories in the following theorem.

Theorem 2.1 [25, Therem 6.4] Let A and B be rings. The following conditions are equivalent.
(i) $\mathrm{D}^{\mathrm{b}}(A\text{-Mod})$ and $\mathrm{D}^{\mathrm{b}}(B\text{-Mod})$ are equivalent as triangulated categories.
(ii) $\mathrm{K}^{-}(A\text{-Proj})$ and $\mathrm{K}^{-}(B\text{-Proj})$ are equivalent as triangulated categories.
(iii) $\mathrm{K}^{\mathrm{b}}(A\text{-Proj})$ and $\mathrm{K}^{\mathrm{b}}(B\text{-Proj})$ are equivalent as triangulated categories.
(iv) $\mathrm{K}^{\mathrm{b}}({}_A\mathcal{P})$ and $\mathrm{K}^{\mathrm{b}}({}_B\mathcal{P})$ are equivalent as triangulated categories.
(v) B is isomorphic to $\mathrm{End}_{\mathrm{D}^{\mathrm{b}}(A)}(T^\bullet)$ for some complex $T^\bullet$ in $\mathrm{K}^{\mathrm{b}}({}_A\mathcal{P})$ satisfying
(1) $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(A)}(T^\bullet, T^\bullet[n]) = 0$ for all $n \neq 0$.
(2) $\mathrm{add}(T^\bullet)$, the category of direct summands of finite direct sums of copies of $T^\bullet$, generates $\mathrm{K}^{\mathrm{b}}({}_A\mathcal{P})$ as a triangulated category.

Here A-Proj is the subcategory of A-Mod consisting of all projective A-modules.

Remarks. (1) The rings A and B are said to be derived equivalent if A and B satisfy the conditions of the above theorem. The complex $T^\bullet$ in Theorem 2.1 is called a tilting complex for A.

(2) By [25, Corollary 8.3], two Artin R-algebras A and B are said to be derived equivalent if their derived categories $\mathrm{D}^{\mathrm{b}}(A)$ and $\mathrm{D}^{\mathrm{b}}(B)$ are equivalent as triangulated categories. By Theorem 2.1, Artin algebras A and B are derived equivalent if and only if B is isomorphic to the endomorphism algebra of a tilting complex $T^\bullet$. If $T^\bullet$ is a tilting complex for A, then there is an equivalence $F : \mathrm{D}^{\mathrm{b}}(A) \to \mathrm{D}^{\mathrm{b}}(B)$ that sends $T^\bullet$ to B. On the other hand, for each derived equivalence $F : \mathrm{D}^{\mathrm{b}}(A) \to \mathrm{D}^{\mathrm{b}}(B)$, there is an associated tilting complex $T^\bullet$ for A such that $F(T^\bullet)$ is isomorphic to B in $\mathrm{D}^{\mathrm{b}}(B)$.

3 Derived equivalences for Cohen-Macaulay Auslander Algebras

In this section, we shall prove the main result of this paper. First, let us recall the definition of Cohen-Macaulay Auslander algebras.

3.1 Cohen-Macaulay Auslander algebras

Let A be an Artin algebra. Recall that A is of finite representation type provided that there are only finitely many indecomposable finitely generated A-modules up to isomorphism. If an A-module X satisfies $\mathrm{Ext}^i_A(X, A) = 0$ for $i > 0$, then X is said to be a Cohen-Macaulay A-module. Denote by ${}_A\mathcal{X}$ the category of Cohen-Macaulay A-modules. It is easy to see that if A is a self-injective algebra, then ${}_A\mathcal{X} = A$-mod. By a $\mathrm{Hom}_A(-, X)$-exact sequence $Y^\bullet = (Y^i, d^i)$, we mean that the sequence $Y^\bullet$ itself is exact, and that $\mathrm{Hom}_A(Y^\bullet, X)$ remains to be exact. An A-module X is said to be Gorenstein projective if there is a $\mathrm{Hom}_A(-, Q)$-exact sequence

$$\cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots$$

such that $X \simeq Imd^0$, where P^i (for each i) and Q are projective A-modules. Denote by A-Gproj the subcategory of A-mod consisting of Gorenstein projective A-modules. Note that Gorenstein projective modules are Cohen-Macaulay A-modules. Following [3, Example 8.4(2)] an Artin algebra A is said to be of Cohen-Macaulay finite type provided that there are only finitely many indecomposable finitely generated Gorenstein projective A-modules up to isomorphism. It is easy to see that algebras of finite representation type are of Cohen-Macaulay finite type. Suppose that A is of Cohen-Macaulay finite type. In other words, A-Gproj has an additive generator M, that is, $\mathrm{add}\,(M) = A$-Gproj.

Definition 3.1 [6] *Suppose that an Artin algebra A is of Cohen-Macaulay finite type. Let M be an additive generator in A-Gproj. We call $\Lambda = \mathrm{End}\,(M)$ a Cohen-Macaulay Auslander algebra of A.*

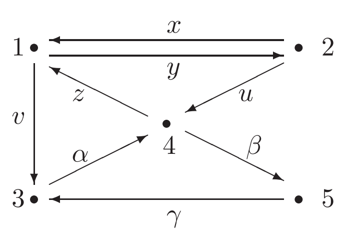
Remark. For a Cohen-Macaulay finite algebra A, its Cohen-Macaulay Auslander algebra is unique up to Morita equivalences.
Example. Let $A = k[x]/(x^2)$ and consider the Artin algebra

$$T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}.$$

Then $T_2(A)$ is a 1-Gorenstein Artin algebra of Cohen-Macaulay type [7] or [10]. $T_2(A)$ has indecomposable Gorenstein projective modules [4, p.101]:

$$M_1 = \begin{pmatrix} k \\ 0 \end{pmatrix}, M_2 = \begin{pmatrix} A \\ 0 \end{pmatrix}, M_3 = \begin{pmatrix} A \\ A \end{pmatrix}, M_4 = \begin{pmatrix} k \\ k \end{pmatrix}, M_5 = \begin{pmatrix} A \\ k \end{pmatrix}.$$

Set $M = \oplus_{1 \leq i \leq 5} M_i$. Then Cohen-Macaulay Auslander algebra $\mathrm{End}\,_{T_2(A)}(M)$ of $T_2(A)$ is given by the following quiver and relations $xy = 0 = v\alpha - yu = \alpha z = \alpha\beta\gamma$ [8].



3.2 The proof of the main result

We shall give the proof of the main result of this paper.

Suppose A and B are Artin algebras. Let $F : \mathrm{D^b}(A) \longrightarrow \mathrm{D^b}(B)$ be a derived equivalence and let $P^\bullet$ be the tilting complex associated to F. Without loss of generality, we assume that $P^\bullet$ is a radical complex of the following form

$$0 \to P^{-n} \to P^{-n+1} \to \cdots \to P^{-1} \to P^0 \to 0.$$

Then we have the following fact.

Lemma 3.2 [15, lemma 2.1] *Let $F : \mathrm{D^b}(A) \longrightarrow \mathrm{D^b}(B)$ be a derived equivalence between Artin algebras A and B. Then we have a tilting complex $\bar{P}^\bullet$ for B associated to the quasi-inverse of F of the form*

$$0 \to \bar{P}^0 \to \bar{P}^1 \to \cdots \to \bar{P}^{n-1} \to \bar{P}^n \to 0,$$

with the differential being radical maps.

Suppose that $X^\bullet$ is a complex of A-modules. We define the following truncations:
$\tau_{\geq 1}(X^\bullet) : \cdots \to 0 \to 0 \to X^1 \to X^2 \to \cdots,$
$\tau_{\leq 0}(X^\bullet) : \cdots \to X^{-1} \to X^0 \to 0 \to 0 \cdots.$
Using the properties of Cohen-Macaulay A-modules, we can prove the following lemma.

Lemma 3.3 *Let $F : \mathrm{D^b}(A) \longrightarrow \mathrm{D^b}(B)$ be a derived equivalence between Artin algebras A and B, and let G be the quasi-inverse of F. Suppose that $P^\bullet$ and $\bar{P}^\bullet$ are the tilting complexes associated to F and G, respectively. Then*

(i) For $X \in_A \mathcal{X}$, the complex $F(X)$ is isomorphic in $\mathrm{D^b}(B)$ to a radical complex $\bar{P}_X^\bullet$ of the form

$$0 \to \bar{P}_X^0 \to \bar{P}_X^1 \to \cdots \to \bar{P}_X^{n-1} \to \bar{P}_X^n \to 0$$

with $\bar{P}_X^0 \in_B \mathcal{X}$ and $\bar{P}_X^i$ projective B-modules for $1 \le i \le n$.

(ii) For $Y \in_B \mathcal{X}$, the complex $G(Y)$ is isomorphic in $\mathrm{D^b}(A)$ to a radical complex $P_Y^\bullet$ of the form

$$0 \to P_Y^{-n} \to P_Y^{-n+1} \to \cdots \to P_Y^{-1} \to P_Y^0 \to 0$$

with $P_Y^0 \in_A \mathcal{X}$ and P_Y^i projective A-modules for $-n \le i \le -1$.

Proof. We just show the first case. The proof of (ii) is similar to that of (i).

(i) For $X \in_A \mathcal{X}$, by [16, Lemma 3.1], we see that the complex $F(X)$ is isomorphic in $\mathrm{D^b}(B)$ to a complex $\bar{P}_X^\bullet$ of the form

$$0 \to \bar{P}_X^0 \to \bar{P}_X^1 \to \cdots \to \bar{P}_X^{n-1} \to \bar{P}_X^n \to 0,$$

with $\bar{P}_X^i$ projective B-modules for $i > 0$. We only need to show that $\bar{P}_X^0$ is in ${}_B\mathcal{X}$. It suffices to prove that $\mathrm{End}\,^i_B(\bar{P}_X^0, B) = 0$ for $i \ge 1$. Indeed, there exists a distinguished triangle

$$\bar{P}_X^+ \to \bar{P}_X^\bullet \to \bar{P}_X^0 \to \bar{P}_X^+[1]$$

in $\mathrm{K^b}(B)$, where $\bar{P}_X^+$ denotes the complex $\tau_{\ge 1}(\bar{P}_X^\bullet)$. For each $i \in \mathbb{Z}$, applying the functor $\mathrm{Hom}_{\mathrm{D^b}(B)}(-, B[i])$ to the above distinguished triangle, we get an exact sequence

$$\cdots \to \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^+[1], B[i]) \to \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^0, B[i]) \to \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^\bullet, B[i]) \\ \to \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^+, B[i]) \to \cdots .$$

On the other hand, $\mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^+, B[i]) \simeq \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}_X^+, B[i]) = 0$ for $i \ge 0$. By [24, lemma 2.1] and $\mathrm{End}\,^i_A(X, A) = 0$ for $i \ge 1$, we get $\mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^\bullet, B[i]) \simeq \mathrm{Hom}_{\mathrm{D^b}(A)}(X, P^\bullet[i]) = 0$ for all $i \ge 1$. Consequently, we get $\mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^0, B[i]) = 0$ for all $i \ge 1$ by the above exact sequence. Therefore,

$$\mathrm{End}\,^i_B(\bar{P}_X^0, B) \simeq \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^0, B[i] = 0, \quad \text{for} \quad i \ge 1.$$

This implies that $\bar{P}_X^0 \in_B \mathcal{X}$. □

Now we give a lemma, which is useful in the following argument.

Lemma 3.4 *Let A be an Artin algebra and $f : X \to Y$ a homomorphism of A-modules with $X, Y \in_A \mathcal{X}$. Suppose $Q^\bullet$ is a complex in $\mathrm{K^b}({}_A\mathcal{P})$. If f factors through $Q^\bullet$ in $\mathrm{D^b}(A)$, then f factors through a projective A-module.*

Proof. There is a distinguished triangle

$$\tau_{\le 0}(Q^\bullet) \to \tau_{\ge 1}(Q^\bullet) \xrightarrow{a} Q^\bullet \xrightarrow{b} \tau_{\le 0}(Q^\bullet)[1] \quad \text{in} \quad \mathrm{D^b}(A).$$

Suppose that $f = gh$, where $g : X \to Q^\bullet$ and $h : Q^\bullet \to Y$. Since $\mathrm{Hom}_{\mathrm{D^b}(A)}(\tau_{\ge 1}(Q^\bullet), Y) \simeq \mathrm{Hom}_{\mathrm{K^b}(A)}(\tau_{\ge 1}(Q^\bullet), Y) = 0$, it follows that $ah = 0$. Then there is a map $x : \tau_{\le 0}(Q^\bullet)[1] \to Y$, such that $h = bx$. Thus, we get $f = gbx$. Now, it is sufficient to show that f factors through $\tau_{\le 0}(Q^\bullet)$. Consider the following distinguished triangle

$$Q^0 \xrightarrow{c} \tau_{\le 0}(Q^\bullet) \xrightarrow{d} \tau_{\le -1}(Q^\bullet) \to Q^0[1] \quad \text{in} \quad \mathrm{D^b}(A).$$

Note that $\mathrm{Ext}_A^i(X, A) = 0$ for $i \geq 1$. By [24, Lemma 2.1], we have $\mathrm{Hom}_{\mathrm{D^b}(A)}(X, \tau_{\leq -1}(Q^\bullet)) = 0$. Thus, we get $gbd = 0$. Then there is a morphism $u : X \to Q^0$ such that $gb = uc$. Consequently, $f = ucx$, which implies that f factors through a projective A-module Q^0. $\square$

Choose an A-module $X \in_A \mathcal{X}$, by Lemma 3.3, we know that $F(X)$ is isomorphic to a radical complex of the form

$$0 \to \bar{P}_X^0 \to \bar{P}_X^1 \to \cdots \to \bar{P}_X^{n-1} \to \bar{P}_X^n \to 0$$

such that $\bar{P}_X^0 \in_B \mathcal{X}$ and $\bar{P}_X^i$ are projective B-modules for $1 \leq i \leq n$. In the following, we try to define a functor $\underline{F} : \underline{{}_A\mathcal{X}} \to \underline{{}_B\mathcal{X}}$.

Proposition 3.5 *Let $F : \mathrm{D^b}(A) \longrightarrow \mathrm{D^b}(B)$ be a derived equivalence. Then there is an additive functor $\underline{F} : \underline{{}_A\mathcal{X}} \to \underline{{}_B\mathcal{X}}$ sending X to $\bar{P}_X^0$, such that the following diagram*

$$\begin{array}{ccc} \underline{{}_A\mathcal{X}} & \xrightarrow{\mathrm{can}} & \mathrm{D^b}(A)/\mathrm{K^b}({}_A\mathcal{P}) \\ \downarrow{\scriptstyle \underline{F}} & & \downarrow{\scriptstyle F} \\ \underline{{}_B\mathcal{X}} & \xrightarrow{\mathrm{can}} & \mathrm{D^b}(B)/\mathrm{K^b}({}_B\mathcal{P}) \end{array}$$

is commutative up to natural isomorphism.

Proof. The idea of the proof is similar to that of [15, Proposition 3.4]. For convenience, we give the details here.

For each $f : X \to Y$ in ${}_A\mathcal{X}$, we denote by $\underline{f}$ the image of f in $\underline{{}_A\mathcal{X}}$. By Lemma 3.3, we have a distinguished triangle

$$\bar{P}_X^+ \overset{i_X}{\to} F(X) \overset{j_X}{\to} \bar{P}_X^0 \overset{m_X}{\to} \bar{P}_X^+[1] \quad \text{in} \quad \mathrm{D^b}(B).$$

Moreover, for each $f : X \to Y$ in ${}_A\mathcal{X}$, there is a commutative diagram

$$\begin{array}{ccccccc} \bar{P}_X^+ & \xrightarrow{i_X} & F(X) & \xrightarrow{j_X} & \bar{P}_X^0 & \xrightarrow{m_X} & \bar{P}_X^+[1] \\ \downarrow{\scriptstyle \alpha_f} & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle \beta_f} & & \downarrow{\scriptstyle \alpha_{f[1]}} \\ \bar{P}_Y^+ & \xrightarrow{i_Y} & F(Y) & \xrightarrow{j_Y} & \bar{P}_Y^0 & \xrightarrow{m_Y} & \bar{P}_Y^+[1]. \end{array}$$

Since $\mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^+, \bar{P}_Y^0) \simeq \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}_X^+, \bar{P}_Y^0) = 0$, it follows that $i_X F(f) j_Y = 0$. Then there exists a homomorphism $\alpha_f : \bar{P}_X^+ \to \bar{P}_Y^+$. Note that B-mod is fully embedding into $\mathrm{D^b}(B)$, hence β_f is a morphism of B-modules. If there is another morphism β_f' such that $j_X \beta_f' = F(f) j_Y$, then $j_X(\beta_f - \beta_f') = 0$. Thus $\beta_f - \beta_f'$ factors through $\bar{P}_X^+[1]$, which implies that $\beta_f - \beta_f'$ factors through a projective B-module by Lemma 3.4. Therefore, the morphism $\bar{\beta_f}$ is uniquely determined by f.

Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in ${}_A\mathcal{X}$. Then we have $F(fg) j_Z = j_X \beta_{fg}$ and $F(fg) j_Z = j_X \beta_f \beta_g$. By the uniqueness of $\underline{\beta_{fg}}$, we have $\underline{\beta_{fg}} = \underline{\beta_f}\ \underline{\beta_g}$. Moreover, if f factors through a projective A-module, then β_f also factors through a projective B-module.

For each $X \in_A \mathcal{X}$, we define $\underline{F}(X) := \bar{P}_X^0$. Set $\underline{F}(\underline{f}) = \underline{\beta_f}$, for each $\underline{f} \in \mathrm{Hom}_{\underline{\mathcal{X}_A}}(X, Y)$. Then F is well-defined and an additive functor.

To complete the proof of the lemma, it remains to show that $j_X : F(X) \to \underline{F}(X)$ is a natural isomorphism in $\mathrm{D^b}(B)/\mathrm{K^b}({}_B\mathcal{P})$. Since $\bar{P}_X^+$ is in $\mathrm{K^b}({}_B\mathcal{P})$, then $j_X : F(X) \overset{\simeq}{\to} \underline{F}(X)$ in $\mathrm{D^b}(B)/\mathrm{K^b}({}_B\mathcal{P})$. The following commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{j_X} & \underline{F}(X) = \bar{P}_X^0 \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle \beta_f} \\ F(Y) & \xrightarrow{j_Y} & \underline{F}(Y) = \bar{P}_Y^0 \end{array}$$

shows that $j_X : F(X) \to \underline{F}(X)$ is a natural isomorphism in $\mathrm{D^b}(B)/\mathrm{K^b}({}_B\mathcal{P})$. $\square$

The following lemma is quoted from [15] which will be used frequently.

Lemma 3.6 [15, Lemma 2.2] *Let R be an arbitrary ring, and let R-Mod be the category of all left A-modules. Suppose $X^\bullet$ is a bounded above complex and $Y^\bullet$ is a bounded below complex over R-Mod. Let m be an integer. If X^i is projective for all $i > m$ and $Y^j = 0$ for all $j < m$, then* $\mathrm{Hom}_{\mathrm{K}(R\text{-Mod})}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{R\text{-}Mod})}(X^\bullet, Y^\bullet)$.

Let A be an Artin algebra and let X be in ${}_A\mathcal{X}$ which is not a projective A-module. Set $\Lambda = \mathrm{End}_A(A \oplus X)$, $N = B \oplus \underline{F}(X)$ and $\Gamma = \mathrm{End}_B(N)$. Let $\bar{T}^\bullet$ be the complex $\bar{P}^\bullet \oplus \bar{P}^\bullet_X$. Then $\bar{T}^\bullet$ is in $\mathrm{K}^{\mathrm{b}}(\mathrm{add}\,_B N)$.

The proof of the following lemma is different from [17, Lemma 3.6], and in fact extends Hu and Xi's original methods for the self-injective case.

Lemma 3.7 *Keep the notations above. We have the following statements.*
(1) $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(\mathrm{add}\,_B N)}(\bar{T}^\bullet, \bar{T}^\bullet[i]) = 0$ *for* $i \neq 0$.
(2) $\mathrm{add}\,\bar{T}^\bullet$ *generates* $\mathrm{K}^{\mathrm{b}}(\mathrm{add}\,_B N)$ *as a triangulated category.*

Proof. (1) Decompose the complex $\bar{T}^\bullet$ as $\bar{P}^\bullet \oplus \bar{P}^\bullet_X$. Then we have the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{T}^\bullet, \bar{T}^\bullet[i]) &\simeq \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^\bullet \oplus \bar{P}^\bullet_X, (\bar{P}^\bullet \oplus \bar{P}^\bullet_X)[i]) \simeq \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^\bullet, \bar{P}^\bullet[i]) \oplus \\ &\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^\bullet, \bar{P}^\bullet_X[i]) \oplus \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet[i]) \oplus \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[i]).\end{aligned}$$

The proof falls naturally into three parts.

(a) Since $\bar{P}^\bullet$ is a tilting complex over B, we have $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^\bullet, \bar{P}^\bullet[i]) = 0$ for all $i \neq 0$. Furthermore,

$$\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^\bullet, \bar{P}^\bullet_X[i]) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^\bullet, \bar{P}^\bullet_X[i]) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(A)}(A, X[i]) = 0 \quad \text{for all} \quad i \neq 0.$$

(b) We claim that $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet[i]) = 0$ for $i \neq 0$.

Indeed, applying the functors $\mathrm{Hom}_{\mathrm{K}(B)}(-, \bar{P}^\bullet[i])$ and $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(-, \bar{P}^\bullet[i])$ to the distinguished triangle $\bar{P}^+_X \to \bar{P}^\bullet_X \to \bar{P}^0_X \to \bar{P}^+_X[1]$ in $\mathrm{K}^{\mathrm{b}}(B)$, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^+_X[1], \bar{P}^\bullet[i]) & \longrightarrow & \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^\bullet[i]) & \longrightarrow & \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(A)}(\bar{P}^\bullet_X, \bar{P}^\bullet[i]) & \longrightarrow & \cdots \\
\downarrow \simeq & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^+_X[1], \bar{P}^\bullet[i]) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(A)}(\bar{P}^0_X, \bar{P}^\bullet[i]) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(A)}(\bar{P}^\bullet_X, \bar{P}^\bullet[i]) & \longrightarrow & \cdots .
\end{array}$$

Note that

$$\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^\bullet) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^\bullet).$$

Indeed, since $P^\bullet$ is a bounded complex, it suffices to show that for the complex $\bar{P}^\bullet$ of length 2 of the form $0 \to \bar{P}^0 \to \bar{P}^1 \to 0$, we get

$$\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^\bullet) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^\bullet).$$

In this case, we have a distinguished triangle

$$\bar{P}^1 \to \bar{P}^\bullet \to \bar{P}^0 \to \bar{P}^1[1] \quad \text{in} \quad \mathrm{K}^{\mathrm{b}}(B).$$

Applying the functors $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^0_X, -)$ and $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^0_X, -)$ to the distinguished triangle $\bar{P}^1 \to \bar{P}^\bullet \to \bar{P}^0 \to \bar{P}^1[1]$, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^1) & \longrightarrow & \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^\bullet) & \longrightarrow & \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^0_X, P^0) & \longrightarrow & \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^1[1]) \\
\downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \\
\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^1) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^\bullet) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^0_X, P^0) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(B)}(\bar{P}^0_X, \bar{P}^1[1]).
\end{array}$$

Since $\mathrm{End}^i_B(\bar{P}^0_X, B) = 0$ for $i \geq 1$, it follows that $\mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}^0_X, \bar{P}^1[1]) = 0$. Moreover, $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^0_X, \bar{P}^1[1]) = 0$. Thus for $i \neq 0$, we have

$$\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet[i]) \simeq \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet[i]) \simeq \mathrm{Hom}_{\mathrm{D^b}(A)}(X, A[i]) = 0.$$

(c) We claim that $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[i]) = 0$ for $i \neq 0$.

Indeed, it follows that $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[i]) = 0$ for $i < 0$ by Lemma 3.6. It suffices to show that $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[i]) = 0$ for $i > 0$. Note that there is a distinguished triangle

$$(\star) \qquad \bar{P}^+_X \to \bar{P}^\bullet_X \to \bar{P}^0_X \to \bar{P}^+_X[1] \quad \text{in} \quad \mathrm{K^b}(B), \text{ where } \bar{P}^+_X \text{ denotes the complex } \tau_{\geq 1}(\bar{P}^\bullet_X).$$

Applying the functor $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, -)$ to $(\star)$, we get a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^+_X[i]) \to \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[i]) \to \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^0_X[i]) \to \cdots (\star\star).$$

From the distinguished triangle $\bar{P}^+_X \to \bar{P}^\bullet_X \to \bar{P}^0_X \to \bar{P}^+_X[1]$, we conclude that $H^i(G(\bar{P}^+_X)) = 0$ for $i > 1$ and $G(\bar{P}^+_X)$ is a radical complex $Q^\bullet_X$ of the form

$$\cdots \to Q^{-1}_X \to Q^0_X \to Q^1_X \to 0.$$

Applying the functors $\mathrm{Hom}_{\mathrm{K^b}(B)}(-, \bar{P}^+_X[i])$ and $\mathrm{Hom}_{\mathrm{K^b}(B)}(-, \bar{P}^+_X[i])$ to $(\star)$ again, we have the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^+_X[1], \bar{P}^+_X[i]) & \longrightarrow & \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^0_X, \bar{P}^+_X[i]) & \longrightarrow & \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^+_X[i]) & \longrightarrow & \cdots \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
\mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}^+_X[1], \bar{P}^+_X[i]) & \longrightarrow & \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}^0_X, \bar{P}^+_X[i]) & \longrightarrow & \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}^\bullet_X, \bar{P}^+_X[i]) & \longrightarrow & \cdots .
\end{array}$$

Therefore,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^+_X[i]) \simeq \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}^\bullet_X, \bar{P}^+_X[i]) &\simeq \mathrm{Hom}_{\mathrm{D^b}(A)}(G(\bar{P}^\bullet_X), G(\bar{P}^+_X[i])) \\
&\simeq \mathrm{Hom}_{\mathrm{D^b}(A)}(X, G(\bar{P}^+_X)[i]).
\end{aligned}$$

By [24, lemma 2.1], it follows that $\mathrm{Hom}_{\mathrm{D^b}(A)}(X, G(\bar{P}^+_X)[i]) = 0$ for all $i > 1$. Consequently, $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^+_X[i]) = 0$ for $i > 1$. Since $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^0_X[i]) = 0$ for $i > 0$ by shifting, it follows that $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[i]) = 0$ for $i > 1$ by the long exact sequence $(\star\star)$. It remains to prove that $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[1]) = 0$. To get $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[1]) = 0$, it suffices to show that the map

$$(\maltese) \qquad \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^0_X) \to \mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^+_X[1]) \quad \text{is} \quad \text{surjective.}$$

From the above argument, we have the following commutative diagram

$$\begin{array}{ccccccc}
Q^\bullet_X & \xrightarrow{a} & X & \longrightarrow & M(a) & \longrightarrow & Q^\bullet_X[1] \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
G(\bar{P}^+_X) & \longrightarrow & G(\bar{P}^\bullet_X) & \longrightarrow & G(\bar{P}^0_X) & \longrightarrow & G(\bar{P}^+_X)[1]
\end{array}$$

in $\mathrm{D^b}(A)$, where all the vertical maps are isomorphisms, and the morphism a is chosen in $\mathrm{K^b}(A)$ such that the first square is commutative. Applying the functor $\mathrm{Hom}_{\mathrm{K^b}(A)}(X, -)$ to the first horizontal distinguished triangle, we get an exact sequence

$$\mathrm{Hom}_{\mathrm{K^b}(A)}(X, M(a)) \to \mathrm{Hom}_{\mathrm{K^b}(A)}(X, Q^\bullet_X[1]) \to 0, \quad \text{since} \quad \mathrm{Hom}_{\mathrm{K^b}(A)}(X, X[1]) = 0.$$

We have the following formulas.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^0_X) \overset{(*)}{\simeq} \mathrm{Hom}_{\mathrm{D^b}(B)}(\bar{P}^\bullet_X, \bar{P}^0_X) &\simeq \mathrm{Hom}_{\mathrm{D^b}(A)}(G(\bar{P}^\bullet_X), G(\bar{P}^0_X)) \\
&\simeq \mathrm{Hom}_{\mathrm{D^b}(A)}(X, M(a))
\end{aligned}$$

and

$$\begin{aligned}\operatorname{Hom}_{\mathrm{K^b}(B)}(\bar{P}_X^\bullet, \bar{P}_X^+[1]) \overset{(**)}{\simeq} \operatorname{Hom}_{\mathrm{D^b}(B)}(\bar{P}_X^\bullet, \bar{P}_X^+[1]) &\simeq \operatorname{Hom}_{\mathrm{D^b}(A)}(G(\bar{P}_X^\bullet), G(\bar{P}_X^+)[1]) \\ &\simeq \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^\bullet[1]).\end{aligned}$$

The isomorphisms $(*)$ and $(**)$ are deduced by Lemma 3.6. Then we have the following commutative diagram

$$\begin{array}{ccc}\operatorname{Hom}_{\mathrm{K^b}(B)}(\bar{P}_X^\bullet, \bar{P}_X^0) & \longrightarrow & \operatorname{Hom}_{\mathrm{K^b}(B)}(\bar{P}_X^\bullet, \bar{P}_X^+) \\ \downarrow \simeq & & \downarrow \simeq \\ \operatorname{Hom}_{\mathrm{D^b}(A)}(X, M(a)) & \longrightarrow & \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q^+[1]).\end{array}$$

From the above diagram, to show the map $(\maltese)$ is surjective, it is sufficient to show the map

$$\operatorname{Hom}_{\mathrm{D^b}(A)}(X, M(a)) \to \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^\bullet[1]) \quad \text{is} \quad \text{surjective.}$$

Applying the functor $\operatorname{Hom}(X,-)$ and $\operatorname{Hom}(X,-)$ to the distinguished triangle $Q^\bullet \to X \to M(a) \to Q^\bullet[1]$, we get the following commutative diagram

$$\begin{array}{ccccc}\operatorname{Hom}_{\mathrm{K^b}(A)}(X, M(a)) & \longrightarrow & \operatorname{Hom}_{\mathrm{K^b}(A)}(X, Q_X^\bullet[1]) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathrm{D^b}(A)}(X, M(a)) & \longrightarrow & \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^\bullet[1]) & \longrightarrow & \operatorname{Hom}_{\mathrm{D^b}(A)}(X, X[1]).\end{array}$$

Thus, to get the map

$$\operatorname{Hom}_{\mathrm{D^b}(A)}(X, M(a)) \to \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^\bullet[1]) \quad \text{is} \quad \text{surjective,}$$

it suffices to show the following isomorphisms

$$(i) \quad \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^\bullet[1]) \simeq \operatorname{Hom}_{\mathrm{K^b}(A)}(X, Q_X^\bullet[1])$$

and

$$(ii) \quad \operatorname{Hom}_{\mathrm{D^b}(A)}(X, M(a)) \simeq \operatorname{Hom}_{\mathrm{K^b}(A)}(X, M(a)).$$

Firstly, we show that

$$(i) \quad \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^\bullet[1]) \simeq \operatorname{Hom}_{\mathrm{K^b}(A)}(X, Q_X^\bullet[1]).$$

Indeed, it suffices to show that for the complex $Q_X^\bullet$ of the form $0 \to Q_X^{-1} \to Q_X^0 \to 0$, we get (i). There is a distinguished triangle

$$(\clubsuit) \qquad Q_X^{-1} \to Q_X^0 \to Q_X^\bullet \to Q_X^{-1}[1] \quad \text{in} \quad \mathrm{K^b}(A).$$

Applying the functors $\operatorname{Hom}_{\mathrm{K^b}(A)}(X,-)$, $\operatorname{Hom}_{\mathrm{D^b}(A)}(X,-)$ to $(\clubsuit)$, we obtain the following commutative diagram

$$\begin{array}{ccccccc}\operatorname{Hom}_{\mathrm{K^b}(A)}(X, Q_X^{-1}) & \longrightarrow & \operatorname{Hom}_{\mathrm{K^b}(A)}(X, Q_X^0) & \longrightarrow & \operatorname{Hom}_{\mathrm{K^b}(A)}(X, Q_X^\bullet) & \longrightarrow & \operatorname{Hom}_{\mathrm{K^b}(A)}(X, P^{-1}[1]) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^{-1}) & \longrightarrow & \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^0) & \longrightarrow & \operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^\bullet) & \longrightarrow & \operatorname{Hom}_{\mathrm{D^b}(A)}(X, P^{-1}[1]).\end{array}$$

Since $\operatorname{End}_A^i(X,A) = 0$ for $i \geq 1$, it follows that $\operatorname{Hom}_{\mathrm{D^b}(A)}(X, P^{-1}[1]) = 0$. Moreover, $\operatorname{Hom}_{\mathrm{K^b}(A)}(X, P^{-1}[1]) = 0$. We thus get $\operatorname{Hom}_{\mathrm{D^b}(A)}(X, Q_X^\bullet[1]) \simeq \operatorname{Hom}_{\mathrm{K^b}(A)}(X, Q_X^\bullet[1])$. Next, we prove that

$$(ii) \quad \operatorname{Hom}_{\mathrm{D^b}(A)}(X, M(a)) \simeq \operatorname{Hom}_{\mathrm{K^b}(A)}(X, M(a)).$$

Indeed, there exists a distinguished triangle

$$(\spadesuit) \qquad M(a)^0 \to M(a) \to M(a)^- \to M(a)^0[1] \quad \text{in} \quad \mathrm{K^b}(A),$$

where $M(a)^-$ denotes the truncated complex $\tau_{\leq -1}(M(a))$. Applying the homological functors $\mathrm{Hom}_{\mathrm{K^b}(A)}(X,-)$ and $\mathrm{Hom}_{\mathrm{D^b}(A)}(X,-)$ to $(\spadesuit)$, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
{}_{\mathrm{K^b}(A)}(X, M(a)^-[-1]) & \longrightarrow & {}_{\mathrm{K^b}(A)}(X, M(a)^0) & \longrightarrow & {}_{\mathrm{K^b}(A)}(X, M(a)) & \longrightarrow & {}_{\mathrm{K^b}(A)}(X, M(a)^-) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \\
{}_{\mathrm{D^b}(A)}(X, M(a)^-[-1]) & \longrightarrow & {}_{\mathrm{D^b}(A)}(X, M(a)^0) & \longrightarrow & {}_{\mathrm{D^b}(A)}(X, M(a)) & \longrightarrow & {}_{\mathrm{D^b}(A)}(X, M(a)^-).
\end{array}$$

According to (i), we have $\mathrm{Hom}_{\mathrm{K^b}(A)}(X, M(a)^-) \simeq \mathrm{Hom}_{\mathrm{D^b}(A)}(X, M(a)^-) = 0$. Therefore, $\mathrm{Hom}_{\mathrm{D^b}(A)}(X, M(a)) \simeq \mathrm{Hom}_{\mathrm{K^b}(A)}(X, M(a))$.

From the above argument, we have shown that $\mathrm{Hom}_{\mathrm{K^b}(B)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[i]) = 0$ for $i \neq 0$. Since $\mathrm{K^b}(B)$ is a full subcategory of $\mathrm{K^b}(\mathrm{add}\,_BN)$, it follows that $\mathrm{Hom}_{\mathrm{K^b}(\mathrm{add}\,_BN)}(\bar{P}^\bullet_X, \bar{P}^\bullet_X[i]) = 0$ for $i \neq 0$.

(2) Since $\bar{P}^\bullet$ is a tilting complex for B, we see that $\mathrm{add}\,\bar{P}^\bullet$ generates $\mathrm{K^b}(\mathrm{add}\,_BB)$ as triangulated category. All the terms of $\bar{P}^+_X$ are in $\mathrm{add}\,_BB$. From the distinguished triangle

$$\bar{P}^+_X \to \bar{P}^\bullet_X \to \bar{P}^0_X \to \bar{P}^+_X[1],$$

it follows that $\bar{P}^0_X$ is in the triangulated subcategory generated by $\mathrm{add}\,(\bar{P}^\bullet \oplus \bar{P}^\bullet_X)$. Therefore, $\mathrm{add}\,\bar{T}^\bullet$ generates $\mathrm{K^b}(\mathrm{add}\,_BN)$ as a triangulated category. □

Proposition 3.8 *The complex* $\mathrm{Hom}\,(N, \bar{T}^\bullet)$ *is a tiling complex over* Γ *with the endomorphism* $\mathrm{End}\,(\mathrm{Hom}\,(N, \bar{T}^\bullet)) \simeq \Lambda$. *In particular, Artin algebras* Λ *and* Γ *are derived equivalent associated with the tilting complex* $\mathrm{Hom}\,(N, \bar{T}^\bullet)$.

Proof. We have an equivalence of categories

$$\mathrm{Hom}_B(N,-) : \mathrm{add}\,_BN \xrightarrow{\simeq}{}_\Gamma \mathcal{P}.$$

We thus get an equivalence of triangulated categories induced by $\mathrm{Hom}_B(N,-)$ as follows

$$\mathrm{K^b}(\mathrm{add}\,_BN) \xrightarrow{\simeq} \mathrm{K^b}({}_\Gamma\mathcal{P}).$$

Then $\mathrm{Hom}\,(N, \bar{T}^\bullet) \in \mathrm{K^b}({}_\Gamma\mathcal{P})$. By the Lemma 3.7, we see that $\mathrm{add}\,\mathrm{Hom}\,(N, \bar{T}^\bullet)$ generates $\mathrm{K^b}({}_\Gamma\mathcal{P})$ as a triangulated category, and $\mathrm{End}\,(\mathrm{Hom}\,(N, \bar{T}^\bullet)) \simeq \mathrm{End}\,(\bar{T}^\bullet) \simeq \Lambda$. □

We have the following lemma, its proof is due to Happel [10, Lemma 4.4].

Lemma 3.9 *Suppose that* $\mathrm{inj.dim}\,_AA < \infty$. *Then the following statements are equivalent.*

(i) $\mathrm{proj.dim}\,_A(\mathrm{D}\,(\mathrm{A_A})) < \infty$.

(ii) *For* $X \in_A \mathcal{X}$, *there exists an exact sequence* $0 \to X \to P \to X' \to 0$, *with* $X' \in_A \mathcal{X}$ *and* P *a projective A-module.*

(iii) *If* $X \in_A \mathcal{X}$ *satisfies* $\mathrm{inj.dim}\,_AX < \infty$, *then* X *is a projective A-module.*

Recall that an Artin algebra A is called Gorenstein if the regular module A has finite injective dimension on both sides. If A is a Gorenstein algebra, then it follows from Lemma 7.2.8 that ${}_A\mathcal{X} = A$-Gproj, and that ${}_A\mathcal{X}$ is a Frobenius category and its category $\underline{{}_A\mathcal{X}}$ is a triangulated category.

Proposition 3.10 *Let* A *and* B *be Gorenstein Artin algebras. Suppose that* F *is a derived equivalence between* A *and* B. *Then we have the following statements.*

(1) *There is an equivalence* $\underline{F} : \underline{{}_A\mathcal{X}} \to \underline{{}_B\mathcal{X}}$.

(2) *If* A *and* B *are finite dimensional algebras over a field* k, *then there exist bimodules* ${}_AM_B$ *and* ${}_BL_A$ *such that the pair of functors*

$${}_AM_B \otimes - : A\text{-}mod \to B\text{-}mod, \; {}_BL_A \otimes - : B\text{-}mod \to A\text{-}mod$$

induces an equivalence of triangulated categories $\underline{{}_A\mathcal{X}}$ *and* $\underline{{}_B\mathcal{X}}$.

Proof. We refer to [10, Theorem 4.6] and [19, Theorem 5.4] for the proofs of (1) and (2), respectively. □

Our main result in this chapter is the following theorem.

Theorem 3.11 *Let A and B be Gorenstein Artin algebras of Cohen-Macaulay finite type. If A and B are derived equivalent, then the Cohen-Macaulay Auslander algebras Λ and Γ of A and B are also derived equivalent.*

Proof. In fact, if Artin algebras A and B are derived equivalent, then A is Gorenstein if and only if B is Gorenstein. By Proposition 3.10 or [3, Theorem 8.11], if Gorenstein Artin algebras A and B are derived equivalent, then A is of Cohen-Macaulay finite type if and only if B is. Let $F : \mathrm{D^b}(A) \longrightarrow \mathrm{D^b}(B)$ be a derived equivalence. Set $\Lambda = \mathrm{End}\,(A \oplus X)$ with $X = \oplus_{0\leq i\leq m}X_i$, where each X_i is indecomposable non-projective Gorenstein projective A-module. Then Λ is the Cohen-Macaulay Auslander algebra of A. By Proposition 3.10, it follows that $Y_i = \underline{F}(X_i)$ is the indecomposable non-projective Gorenstein projective B-module. Set $Y = \oplus_{0\leq i\leq m}Y_i$. Then $\Gamma = \mathrm{End}\,(B\oplus Y)$ is the Cohen-Macaulay Auslander algebra of B. Let N be the B-module $(B\oplus Y)$ and let $\bar{T}^\bullet$ be the complex $F(A \oplus X)$. Thus, we construct a tilting complex $\mathrm{Hom}\,(N, \bar{T}^\bullet)$. The result follows from Proposition 3.8. □

Remark. Let A and B be Gorenstein Artin algebras of Cohen-Macaulay finite type. According to a result of Liu and Xi [21, Theorem 1.1], we see that, if A and B are stably equivalent of Morita type, then the Cohen-Macaulay Auslander algebras of A and B are also stably equivalent of Morita type.

As a corollary of Theorem 3.11, we re-obtain the following result of Hu and Xi [17] since self-injective Artin algebras of finite representation type are Gorenstein Artin algebras of Cohen-Macaulay finite type.

Corollary 3.12 [17, Corollary 3.13] *Suppose that A and B are self-injective Artin algebras of finite representation type. If A and B are derived equivalent, then the Auslander algebras of A and B are also derived equivalent.*

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References

[1] H. Abe and M. Hoshino, Gorenstein orders associated with modules. *Comm. Algebra* 38 (2010), 165-180.

[2] M. Auslander, I. Reiten and S. O. Smalø, *Representation Thoery of Artin Algebras.* Cambridge University Press, 1995.

[3] A. Beligiannis, Cohen-Macaulay modules, (co)torsion pairs and virtually Goren- stein algebras. *J. Algebra* 288(2005), 137-211.

[4] A. Beligiannis and I. Reiten, *Homological and Homotopical Aspects of Torsion Theories.* Mem. Amer. Math. Soc., 188(2007).

[5] A. Buan, R. Marsh, M, Reineke and I. Reiten, Tilting theory and cluster combinatorics. Adv. Math. 204(2006), 572-618.

[6] X. W. Chen, Gorenstein homological algebra of Artin algebras, Postdoctoral Report, USTC, 2010.

[7] R. M. Fossum, P. A. Griffith and I. Riten, *Trivial Extensions of Abelian Categories.* Springer Lecture Notes 456, Heidelberg 1975.

[8] N. Gao and P. Zhang, Gorenstein derived categories. *J. Algebra* 323(2010), 2041-2057.

[9] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras.* Cambridge University Press, Cambridge. 1988.

[10] D. Happel, On Gorenstein algebras. *In: Representation theory of finite groups and finite-dimensional algebras* (Proc. Conf. at Bielefeld, 1991), 389-404, Progress in Math., vol. 95, Birkhäuser, Basel, 1991.

[11] M. Hoshino and Y. Kato, Tilting complexes defined by idempotents. *Comm. Algebra* 30(2002), 83-100.

[12] M. Hoshino and Y. Kato, Tilting complexes associated with a sequence of idempotents. *J. Algebra* 183(2003), 105-124.

[13] W. Hu, On iterated almost ν-stable derived equivalences. Preprint, available at : http://arxiv.org/abs/0811.0926v3, 2008.

[14] W. Hu, S. Koenig and C. C. Xi, Derived eqivalences from cohomological approximations, and mutations of perforated Yoneda algebras. In preparation.

[15] W. Hu and C. C. Xi, Almost $\mathcal{D}$-split sequences and derived equivalences. Preprint, available at : http://math.bnu.edu.cn/ ccxi/Papers/Articles/xihu-2.pdf, 2007.

[16] W. Hu and C. C. Xi, Derived equivalences and stable equivalences of Morita type, I. Preprint, available at : http://math.bnu.edu.cn/ ccxi/Papers/Articles/xihu-3.pdf, 2007, to appear in Nagoya Math. J.

[17] W. Hu and C. C. Xi, Derived equivalences for Φ-Auslander-Yoneda algebras. Preprint, available at : http://math.bnu.edu.cn/ ccxi/Papers/Articles/xihu-4.pdf, 2009.

[18] W. Hu and C. C. Xi, Derived equivalences from cohomological approximations, and mutations of perforated algebras. Preprint, 2010.

[19] Y. Kato, On derived equivalent coherent rings. *Comm. Algebra* 30(2002), 4437-4454.

[20] Y. M. Liu and C. C. Xi, Constructions of stable equivalences of Morita type for finite-dimensional algebras. II. *Math. Z.* **251**(2005), no.1, 21-39.

[21] Y. M. Liu and C. C. Xi, Constructions of stable equivalences of Morita type for finite dimensional algebras III. J. London Math. Soc. 76(2007), 567-585.

[22] A. Neeman, *Triangulated Categories.* Annals of Mathematics Studies 148, Princeton University Press, Princeton and Oxford, 2001.

[23] S. Y. Pan, Derived equivalences for Φ-Cohen Macaulay Auslander-Yoneda algebras. In preparation.

[24] S. Y. Pan and C. C. Xi, Finiteness of finitistic dimension is invariant under derived equivalences. *J. Algebra* 322(2009), 21-24.

[25] J. Rickard, Morita theory for derived categories. *J. London Math. Soc.* 39 (1989), 436-456.

[26] J. Rickard, Derived categories and stable equivalence. *J. Pure Appl. Algebra* 61 (1989), 303-317.

[27] J. Rickard, Derived equivalences as derived functors. *J. London Math. Soc.* 43 (1991), 37-48.

[28] J. Verdier, Catégories dérivées, état 0. *Lecture Notes in Math. 569*(1977), Springer, Berlin, 262-311.

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