HYPERBOLIC SURFACE SUBGROUPS OF ONE-ENDED DOUBLES OF FREE GROUPS

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ABSTRACT. Gromov asked whether every one-ended word-hyperbolic group contains a hyperbolic surface group. We prove that every one-ended double of a free group contains a hyperbolic surface group if the free group has rank two, or every generator is used the same number of times in the list of amalgamating words. Our method is based on formulating a stronger conjecture on tilings of closed surfaces combinatorially in terms of Whitehead graphs and proving it for certain classes of graphs.

1. Introduction

A *hyperbolic surface group* is the fundamental group of a closed surface with negative Euler characteristic. Gromov [13, p. 277] raised the following remarkable question.

Question 1 (Gromov). *Does every one-ended word-hyperbolic group contain a hyperbolic surface group?*

Question 1 has been answered affirmatively for the following cases.

(1) Coxeter groups [10].
(2) Graphs of free groups with infinite cyclic edge groups with nontrivial second rational homology [5].
(3) The fundamental groups of closed hyperbolic 3-manifolds [15].

The case (2) is not resolved when the nontrivial second rational homology condition is removed. A basic, but still captivating case is when the group is given as a *double of a free group*, which is defined as follows. For a list U of nontrivial words $u_1, \ldots, u_r$ in F_n, we denote by $D_n(U)$ the fundamental group of a graph of groups, where there are

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two vertex groups isomorphic to F_n and r infinite cyclic edge groups joining the two vertex groups; here, each cyclic edge group E_i is amalgamated along the isomorphic image of $\langle u_i \rangle$ in each vertex group. We call $D_n(U)$ a *double of F_n along U*. The presentation of $D_n(U)$ is given in Section 2.1. In this paper, we discuss the groups of the form $D_n(U)$.

By specializing [5], Gordon and Wilton [11] constructed explicit families of examples of $D_n(w) = F_n *_{\langle w \rangle} F_n$ that contain hyperbolic surface groups, where w is a word in F_n. Kim and Wilton [17] formulated a different condition, called *polygonality*, for a word w in F_n. Polygonality is a combinatorial condition concerning van Kampen diagrams on closed surfaces; see Section 2.1 for a precise definition. Kim and Wilton [17] proved that polygonality of $w \in F_n$ guarantees the existence of a hyperbolic surface subgroup of $D_n(w)$, and this result generalizes to a set (or a list, allowing redundancy) of words; see [16].

Theorem 1 ([17, 16]). *If U is a polygonal list of words in F_n, then $D_n(U)$ contains a hyperbolic surface group.*

Let U be a list of words in F_n. We call U *diskbusting* if one cannot write $F_n = A * B$ in such a way that $A, B \neq \{1\}$ and each word in U is conjugate into A or B [6, 26, 25]. If U is not diskbusting, then $D_n(U)$ splits as a nontrivial free product and therefore $D_n(U)$ is not one-ended. Conversely, if $D_n(U)$ is one-ended, then U is diskbusting [28]. This observation lets us consider only the case when U is diskbusting.

The *length of U* is the sum of the lengths of the words in U. The list U is called *minimal* if the length of U is at most the length of $\phi(U)$ for every $\phi \in \mathrm{Aut}(F_n)$.

Conjecture 2 (Tiling Conjecture; see [17, 16]). *A minimal and diskbusting list of cyclically reduced words in F_n is polygonal when $n > 1$.*

If Conjecture 2 is true, an affirmative answer to Question 1 for the groups of the form $D_n(U)$ would follow from Theorem 1; see Proposition 13 for a more detailed implication of Tiling Conjecture. We note that without the hypothesis of minimality, Tiling Conjecture is no longer true [17].

We allow graphs to have parallel edges or loops. A *loop* is an edge with only one endpoint. For a graph G, the *degree* $\deg_G(v)$ of a vertex v is the number of edges incident with v, assuming that loops are counted twice. A graph is *k-regular* if every vertex has degree k. A *cycle* is a (finite) 2-regular connected graph. Let $\delta_G(v)$ be the set of non-loop edges incident with v. For a set X of vertices, we write $\delta_G(X)$ to denote the set of edges with exactly one endpoint in X. We write $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. For two vertices x and y of a graph G, the *local edge-connectivity* $\lambda_G(x, y)$ is the maximum number of pairwise edge-disjoint paths from x to y in G. We omit the subscript G in $\deg_G$, δ_G, and λ_G if the underlying graph G is clear from the context.

We say that an involution $\mu : V(G) \to V(G)$ is *fixed point free* if $\mu(x) \neq x$ for every vertex x of G. A fixed point free involution of $V(G)$ can be regarded as a partition of $V(G)$ into pairs of vertices. For a fixed point free involution μ on the vertex set $V(G)$, we will be mostly interested in the case when

$$\lambda(v, \mu(v)) = \deg(v)$$

for every vertex $v \in V(G)$. If so, then we can easily deduce that $\deg(v) = \deg(\mu(v))$ for each vertex v and G has no loops.

A graph is *non-acyclic* if the graph contains at least one cycle. Using Whitehead graphs, we will restate Conjecture 2 combinatorially as follows.

Conjecture 3. *Let $G = (V, E)$ be a non-acyclic graph with a fixed point free involution $\mu : V \to V$ and a bijection $\sigma_v : \delta(v) \to \delta(\mu(v))$ for every vertex v such that $\lambda(v, \mu(v)) = \deg(v)$ and $\sigma_{\mu(v)} = \sigma_v^{-1}$. Then there exists a nonempty list of cycles of G such that for each pair of edges e and f incident with a vertex v, the number of cycles in the list containing both e and f is equal to the number of cycles in the list containing both $\sigma_v(e)$ and $\sigma_v(f)$. Moreover, the list can be required to contain at least one cycle of length greater than two if G has a connected component which has at least four vertices.*

In Section 4, we will prove Conjecture 3 for regular graphs. This amounts to proving Conjecture 2 with the additional assumption that every generator of F_n is used the same number of times in U; more precisely speaking, this is the case when the number of occurrences of a or a^{-1} as a letter of a word in U is the same for every generator a of F_n.

Theorem 4. *Let U be a list of cyclically reduced words in F_n such that every generator of F_n is used the same number of times in U. If U is minimal and diskbusting, then U is polygonal; in particular, $D_n(U)$ contains a hyperbolic surface group.*

Even the minimality assumption can be lifted for rank-two free groups:

Corollary 5. *Let U be a list of cyclically reduced words in F_2 such that every generator of F_2 is used the same number of times in U. Then U is diskbusting if and only if U is polygonal; in this case, $D_2(U)$ contains a hyperbolic surface group.*

In Section 5, we prove Conjecture 3 for 4-vertex graphs. This will answer Question 1 for the groups of the form $D_2(U)$:

Theorem 6. *A minimal and diskbusting list of cyclically reduced words in F_2 is polygonal.*

Corollary 7. *Let U be a list of words in F_2. Then $D_2(U)$ contains a hyperbolic surface group if and only if U is diskbusting.*

Note that Theorems 4 and 6 do not depend on whether $D_n(U)$ is word-hyperbolic or not. We also note that there is a polynomial-time algorithm to decide whether a list of words in a free group is diskbusting [27, 26, 25, 22].

The above theorems will be proved as follows. In Section 2, we will summarize some of the known results relating polygonality to hyperbolic surface subgroups of doubles of free groups. In Section 3, we will prove the equivalence between Conjectures 2 and 3. In Section 4, we will prove Conjecture 3 for subdivisions of regular graphs. In Section 5, we will verify Conjecture 3 for subdivisions of 4-vertex graphs. Section 6 discusses other aspects of Tiling Conjecture and a related question.

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2. Polygonality and Doubles of Free Groups

The original proof [17] of Theorem 1 relies on the subgroup separability of free groups and the normal form theorem for graphs of groups. In this section, we give a self-contained, alternative proof for convenience. Then we describe a consequence of Tiling Conjecture.

2.1. **Basic definitions and notations.** Throughout this paper, we will let F_n be a free group of rank $n > 1$ and $\mathcal{A}_n = \{a_1, a_2, \ldots, a_n\}$ be a free basis of F_n. Each word in F_n can be written as $w = x_1 x_2 \cdots x_l$ where $x_i \in \mathcal{A}_n \cup \mathcal{A}_n^{-1}$; each x_i is called as a *letter* of w, and the subscript of x_i is taken modulo l. We say that w is *cyclically reduced* if $x_{i+1} \neq x_i^{-1}$ for each $i = 1, 2, \ldots, l$. With respect to the given basis $\mathcal{A}_n$, we denote the Cayley graph of F_n by $\mathrm{Cayley}(F_n)$. There is a natural free action of F_n on $\mathrm{Cayley}(F_n)$, so that $\mathrm{Cayley}(F_n)/F_n$ is a bouquet of circles. Let $\alpha_1, \ldots, \alpha_n$ denote the oriented circles in $\mathrm{Cayley}(F_n)/F_n$ corresponding to $a_1, \ldots, a_n$. The loop obtained by a concatenation $\alpha_{i_1}^{m_1} \alpha_{i_2}^{m_2} \cdots \alpha_{i_k}^{m_k}$ where $m_i \in \mathbb{Z}$ is said to *read* the word $a_{i_1}^{m_1} a_{i_2}^{m_2} \cdots a_{i_k}^{m_k}$.

Given a list U of nontrivial words $u_1, u_2, \ldots, u_r$ in F_n, take two copies Γ and Γ' of $\mathrm{Cayley}(F_n)/F_n$. To Γ and Γ', we glue a cylinder along the copies of the closed curve reading u_i, for each i. Let $X_n(U)$ be the resulting space and let $D_n(U) = \pi_1(X_n(U))$ be the fundamental group of $X_n(U)$. We call $D_n(U)$ a *double* [1]. If we let $\mathcal{B}_n$ and $V = \{v_1, \ldots, v_r\}$ denote the copies of $\mathcal{A}_n$ and U respectively, then a presentation of $D_n(U)$ is given as:

$$D_n(U) \cong \langle \mathcal{A}_n, \mathcal{B}_n, t_2, t_3, \ldots, t_r \mid u_1 = v_1, u_i^{t_i} = v_i \text{ for } i = 2, \ldots, r \rangle.$$

Since the isomorphism type of $D_n(U)$ does not change if some words in U are replaced by their conjugates, we may always assume that every word in U is cyclically reduced.

2.2. **Whitehead graph and square complex structure on $X_n(U)$.** We briefly summarize elementary facts on CAT(0)-spaces; a standard reference for this subject is [4]. We denote by $\mathbb{E}^2$ the Euclidean plane. Let X be a geodesic metric space. For a geodesic triangle $\Delta \subseteq X$, there is a geodesic triangle $\Delta' \subseteq \mathbb{E}^2$ of the same side-lengths and a length-preserving map $f\colon \Delta \to \Delta'$. We say that X is a CAT(0)*-space* if $d_X(x, x') \leq d_{\mathbb{E}^2}(f(x), f(x'))$ for every choice of Δ, f and $x, x' \in \Delta$. A metric space X is *non-positively curved* if each point in X has a neighborhood which is a CAT(0)-space. Properties of non-positively curved spaces can be effectively used to prove the π_1-injectivity of a map:

Proposition 8 (see [4])**.** *Let X be a non-positively curved space, and $f\colon Y \to X$ be locally an isometric embedding. Then Y is non-positively curved and $f_*\colon \pi_1(Y) \to \pi_1(X)$ is injective.*

Let I denote the unit interval. A *cube complex* is a piecewise-Euclidean cell complex X inductively defined as follows: for all k, the k-skeleton $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching k-dimensional unit cubes I^k such that the restriction of each attaching map to a $(k-1)$-face of I^k is a $(k-1)$-dimensional attaching map. If $X = X^{(2)}$, we say that X is a *square* complex. A finite-dimensional cube complex is known to be a complete geodesic metric space [3]. For a cube complex X and $v \in X^{(0)}$, $\mathrm{Link}_X(x)$ is defined to be the set of unit vectors from v toward X; in particular, a link is naturally equipped with a piecewise-spherical metric. We will only consider *simple* cube complexes, in the sense that no vertex has a link containing a bigon; hence, each link will be a simplicial complex [14]. A simplicial complex L is a *flag complex* if every complete subgraph of $L^{(1)}$ is the 1-skeleton of some simplex in L. Gromov gave a combinatorial formulation of non-positive curvature for a cube complex.

Proposition 9 ([12])**.** *A finite-dimensional cube complex X is non-positively curved if and only if the link of each vertex is a flag complex.*

Recall that for a simplicial complex L and a set of vertices S in L, a *full subcomplex* L' on S is the maximal subcomplex of L whose vertex set is S. A map $f\colon Y \to X$ between cube complexes is *cubical* if f maps each cube to a cube of the same dimension. The condition for a cubical map to be locally an isometric embedding can also be combinatorially formulated in terms of the links as follows.

Proposition 10 ([7, 8])**.** *Let X and Y be cube complexes and $f\colon Y \to X$ be a cubical map. Then f is locally an isometric embedding if the following are true for each vertex $y \in Y^{(0)}$.*

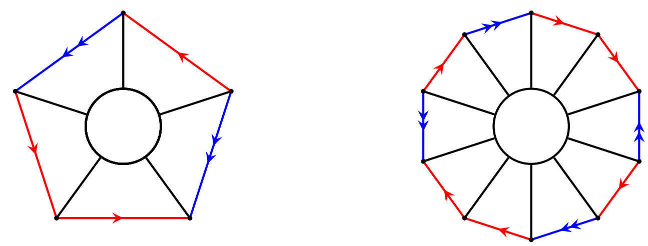
(i) *The induced map on the links* $\mathrm{Link}(f;y)\colon \mathrm{Link}_Y(y) \to \mathrm{Link}_X(f(y))$ *is injective.*
(ii) *The image of* $\mathrm{Link}(f;y)$ *is a full subcomplex of* $\mathrm{Link}_X(f(y))$.

For a word $w = x_1x_2\dots x_l \in F_n$, $x_1x_2, x_2x_3, \dots, x_{l-1}x_l, x_lx_1$ are called *length-*2 *cyclic subwords* of w. For a list U of cyclically reduced words in F_n, the *Whitehead graph $W_n(U)$ of U* is constructed as follows [27]:

(i) the vertex set of $W_n(U)$ is $\mathcal{A}_n \cup \mathcal{A}_n^{-1}$;
(ii) For each length-2 cyclic subword xy of a word in U, we add an edge joining x and y^{-1} to $W_n(U)$.

A *polygonal disk* means a topological 2-disk P equipped with a graph structure on the boundary $\partial P \approx S^1$. For a list U of cyclically reduced words $u_1, \dots, u_r$ in F_n, $Z_n(U)$ denotes the presentation 2-complex of $F_n/\langle\langle U\rangle\rangle$. We have $Z_n(U)^{(1)} = \mathrm{Cayley}(F_n)/F_n$ and for each u_i in U, a polygonal disk D_i is glued along the loop reading u_i; here, ∂D_i is a $|u_i|$-gon. Let α_j denote the oriented loop in $Z_n(U)^{(1)} = \mathrm{Cayley}(F_n)/F_n$ reading a_j. The link of the unique vertex in $Z_n(U)$ is seen to be the Whitehead graph of U, by identifying the incoming (outgoing, respectively) portion of α_j with the vertex a_j (a_j^{-1}, respectively) in $W_n(U)$.

Let us fix a point d_i in the interior of D_i and triangulate D_i so that each triangle contains d_i and one edge of ∂D_i. Remove a small open neighborhood of d_i for each i, to get a square complex Z'; see Figure 1 (a). We obtain a square complex structure on $X_n(U)$ by taking two copies of Z' and gluing the circles corresponding to the boundary of the neighborhood of each d_i. The unique vertex of $Z_n(U)$ gives two special vertices of $X_n(U)$. Note that the link of each special vertex is the barycentric subdivision $W_n(U)'$ of $W_n(U)$. Since $W_n(U)$ has no loops, $W_n(U)'$ is a bipartite graph without parallel edges. It follows from Proposition 9 that $X_n(U)$ is non-positively curved.



(a) Z' (b) S'

FIGURE 1. Square complex structures on Z' and on S'. A single and a double arrow denote the generators a and b, respectively. Figure (a) shows a punctured D_i in Z', divided into squares. Figure (b) is a punctured P_i in S', where $\partial P_i \to \mathrm{Cayley}(F)/F$ reads $(b^{-1}aba^2)^2$.

A *side-pairing* on polygonal disks $P_1, \ldots, P_m$ is an equivalence relation on the sides of $P_1, \ldots, P_m$ such that each equivalence class consists of two sides, along with a choice of a homeomorphism between the two sides of each equivalence class. For a given side-pairing $\sim$ on polygonal disks $P_1, \ldots, P_m$, one gets a closed surface $\coprod_i P_i/\sim$ by identifying the sides of P_i by $\sim$. A graph map $\phi\colon G \to \mathrm{Cayley}(F_n)/F_n$ induces an orientation and a label by $\mathcal{A}_n$ on each edge e of G, so that the oriented loop $\phi(e)$ reads the label of e. An edge labeled by a_i is called an *a_i-edge*. An *immersion* is a locally injective graph map.

Definition 11 ([17, 16])**.** Let U be a list of cyclically reduced words in F_n. Consider a side-pairing $\sim$ on some polygonal disks $P_1, P_2, \ldots, P_m$, so that we have a closed surface $S = \coprod_i P_i/\sim$ naturally equipped with a 2-dimensional CW-structure. We say U is *polygonal* if the following are true.

(i) There exists an immersion $S^{(1)} \to \mathrm{Cayley}(F_n)/F_n$ such that the composition $\partial P_i \to S^{(1)} \to \mathrm{Cayley}(F_n)/F_n$ reads a nontrivial power of a word in U for each i.
(ii) The Euler characteristic $\chi(S)$ of S is less than m.

We call S a *U-polygonal surface*.

Remark. Polygonality has been defined for a set of words [17, 16], but we generalize to a (possibly redundant) list of words. The main implication of polygonality still holds, as described in Theorem 1.

Proof of Theorem 1. As in Definition 11, let S be a closed surface obtained by a side-pairing $\sim$ on polygonal disks $P_1, P_2 \ldots, P_m$ such that $\partial P_i \to S^{(1)} \to \mathrm{Cayley}(F_n)/F_n$ is an immersion reading a nontrivial power of a word in U, and $\chi(S) < m$. Choose p_i in the interior of P_i and triangulate P_i so that p_i is the common vertex, similarly to the triangulation of D_i in $Z_n(U)$. There is a natural extension $\phi\colon S \to Z_n(U)$ of the immersion $S^{(1)} \to \mathrm{Cayley}(F)/F$. In particular, ϕ respects the triangulation and ϕ is locally injective away from $p_1, \ldots, p_m$. We obtain a square complex S' from S by taking out small open disks around $p_1, \ldots, p_m$; see Figure 1 (b). As what we have done for Z', we glue two copies of S' along the corresponding boundary components. The resulting square complex S'' is a closed surface such that $\chi(S'') = 2\chi(S') = 2(\chi(S) - m) < 0$. With the square complex structure on $X_n(U)$ described above, we have a locally injective cubical map $\phi''\colon S'' \to X_n(U)$. For a vertex $v \in S''^{(0)}$, $\mathrm{Link}(f; v)$ embeds $\mathrm{Link}_{S''}(v) \approx S^1$ onto a cycle in a link $W_n(U)'$ of $X_n(U)$. Since each cycle in $W_n(U)'$ is a full subcomplex, Propositions 8 and 10 imply that ϕ'' is locally an isometric embedding and ϕ''_* is injective. □

2.3. **Implication of Tiling Conjecture.**

Lemma 12. *The following groups contain no hyperbolic surface groups.*

(1) $G_m = \langle a, b \mid a^m = b^m \rangle$, *where* $m \in \mathbb{Z}$.
(2) $H_m = \langle a, b, t \mid (a^m)^t = b^m \rangle$, *where* $m \in \mathbb{Z}$.

Proof. (1) Our proof follows the same line as Theorem 3.5 in [10]. Suppose the fundamental group of a closed hyperbolic surface S is contained in G_m. Note that $Z = \langle a^m \rangle$ is the center of G_m. Since $\pi_1(S)$ is centerless, $\pi_1(S)$ should embed into $G_m/Z \cong \mathbb{Z}_m * \mathbb{Z}_m$; this is a contradiction.

(2) Note the following.

$$\begin{aligned} H_m &\cong \langle a, b, t, x \mid x = a^t, x^m = b^m \rangle \cong \langle a, b, t, x \mid a = x^{t^{-1}}, x^m = b^m \rangle \\ &\cong \langle b, t, x \mid x^m = b^m \rangle \cong G_m * \mathbb{Z}. \end{aligned}$$ □

If Tiling Conjecture is true, one would be able to precisely describe when doubles contain hyperbolic surface groups as follows.

Proposition 13. *Let* $n > 1$. *Suppose that every minimal and disk-busting list of cyclically reduced words in* F_m *is polygonal for all* $m = 2, 3, \ldots, n$. *Then for a list* U *of cyclically reduced words in* F_n, $D_n(U)$ *contains a hyperbolic surface group if and only if* F_n *cannot be written as* $F_n = G_1 * G_2 * \cdots G_n$ *in such a way that each* G_i *is infinite cyclic and each word in* U *is conjugate into one of* $G_1, \ldots, G_n$.

Proof. There exists a maximum k such that $F_n = G_1 * \cdots * G_k$ for some nontrivial groups $G_1, \ldots, G_k$ and each word in U is conjugate into one of the $G_1, \ldots, G_k$. Note that $1 \le k \le n$.

For the necessity, suppose $k < n$. Then we may assume that G_1 has rank $m > 1$. Let U_1 be the list of all the words in U conjugate into G_1. Then suitably chosen conjugates of the words in U_1 form a diskbusting list U_1' in the rank-m free group G_1. We note that $D_m(U_1') \subseteq D_n(U_1') \subseteq D_n(U_1' \cup (U \setminus U_1)) \cong D_n(U)$; here, the second inclusion can be seen by Propositions 8 and 10. From the hypothesis, a free basis $\mathcal{B}$ of G_1 can be chosen so that U_1' is polygonal as a list of words written in $\mathcal{B}$. By Theorem 1, $D_m(U_1')$ contains a hyperbolic surface group; hence, so does $D_n(U)$.

For the sufficiency, assume $k = n$ and we claim that $D_n(U)$ does not contain a hyperbolic surface group. Since we are only interested in the isomorphism type of $D_n(U)$, we may assume that each word in U is contained in one of $G_1, \ldots, G_n$, by taking conjugation if necessary. By choosing the basis $\mathcal{A}_n$ of F_n from the bases of $G_1, \ldots, G_n$, one may write $\mathcal{A}_n = \{a_1, \ldots, a_n\}$ and $U = (a_1^{m_1}, a_2^{m_2}, \ldots, a_r^{m_r})$ for some $r \leq n$ and $m_1, m_2, \ldots, m_r \neq 0$. Let $\mathcal{B}_n = \{b_1, \ldots, b_n\}$ be a copy of $\mathcal{A}_n$, so that

$$D_n(U) = \langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, t_2, \ldots, t_r \mid a_1^{m_1} = b_1^{m_1}, (a_i^{m_i})^{t_i} = b_i^{m_i} \text{ for } i = 2, 3, \ldots, r \rangle.$$

Hence, $D_n(U)$ is the free product of $\langle a_1, b_1 \mid a_1^{m_1} = b_1^{m_1} \rangle$ and the groups $\langle a_i, b_i, t_i \mid (a_i^{m_i})^{t_i} = b_i^{m_i} \rangle$ for $i = 2, \ldots, r$, as well as $\langle a_i, b_i \rangle$ for $i = r+1, \ldots, n$. The claim follows from Lemma 12. □

3. Combinatorial Formulation of Tiling Conjecture

3.1. **Graph and connecting map.** Berge [2, Section 8] gave a characterization of a minimal set of words: a set A of cyclically reduced words in F_n is not minimal if and only if for some i, there exists a set C of edges in the Whitehead graph $W_n(A)$ such that $W_n(A) \setminus C$ has no path from a_i to a_i^{-1} and $|C|$ is strictly less than $\deg_{W_n(A)}(a_i)$. By Menger's theorem [21], it follows that $A \subseteq F_n$ is minimal if and only if

$$\lambda_{W_n(A)}(a_i, a_i^{-1}) = \deg_{W_n(A)}(a_i) \text{ for each } i.$$

Also, a minimal set $A \subseteq F_n$ is diskbusting if and only if $W_n(A)$ is connected [27, 26, 25]. These results on sets of words in F_n immediately generalize to lists of words as follows.

Proposition 14 ([27, 2, 26, 25]). *A list U of cyclically reduced words in F_n is minimal and diskbusting if and only if $W_n(U)$ is connected and $\lambda_{W_n(U)}(v, v^{-1}) = \deg_{W_n(U)}(v)$ for each vertex v of $W_n(U)$.*

Let U be a list of cyclically reduced words in F_n. The Whitehead graph $W_n(U)$ is equipped with a canonical fixed point free involution μ on $\mathcal{A}_n \cup \mathcal{A}_n^{-1}$ such that $\mu(a) = a^{-1}$ for all $a \in A_n \cup \mathcal{A}_n^{-1}$. For each vertex v, the *connecting map* σ_v *associated with* $W_n(U)$ *at* v is a bijection from $\delta_{W_n(U)}(v)$ to $\delta_{W_n(U)}(\mu(v))$ [16]. For an edge e given by $x_i x_{i+1}$ in a word $w = x_1 x_2 \ldots x_l$ in U, $\sigma_{x_{i+1}^{-1}}$ maps the edge e joining x_i and x_{i+1}^{-1} to the edge f joining x_{i+1} and x_{i+2}^{-1} created by the following length-2 cyclic subword $x_{i+1}x_{i+2}$ of w. We assume that $x_{l+1} = x_1$ and $x_{l+2} = x_2$. We note that if $\sigma_{y^{-1}} \circ \sigma_{x^{-1}}(e)$ is well-defined for an edge e and vertices $x \neq y^{-1}$, then there exists a word w in U such that xy is a length-2 cyclic subword of w or w^{-1}. The proof of the following observation is now elementary.

Lemma 15. *Let U be a list of cyclically reduced words in F_n. In $W_n(U)$, consider an edge f_0 and vertices $x_1, x_2, \ldots, x_l$ where $l > 0$, such that $x_{i+1} \neq x_i^{-1}$ for $i = 1, \ldots, l$. Suppose that*

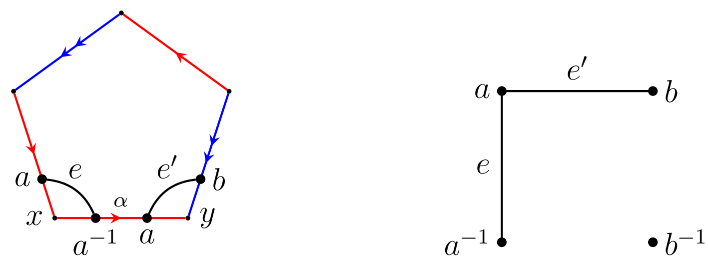
$$\sigma_{x_l^{-1}} \circ \sigma_{x_{l-1}^{-1}} \circ \cdots \circ \sigma_{x_1^{-1}}(f_0)$$

is well-defined and equal to f_0. Then $x_1 x_2 \cdots x_l$ is a nontrivial power of a cyclic conjugation of a word in U. □

Connecting maps can be described in the link of $Z_n(U)$. The presentation 2-complex $Z_n(U)$ of $F_n/\langle\langle U \rangle\rangle$ was obtained from Cayley$(F_n)/F_n$ by attaching polygonal disks D_i along the loops reading the words u_i in U. The link of a vertex p in a polygonal disk P is called the *corner* of P at p. Suppose an edge e is incident with a_i^{-1} in $W_n(U)$, where e corresponds to the corner of a vertex x in some D_j. Since we are assuming that every word in U is cyclically reduced, there exists a unique a_i-edge α outgoing from x. Choose the other endpoint y of α, and let $e' \in E(W_n(U))$ correspond to the corner of D_j at y; see Figure 2. We note that $\sigma_{a_i^{-1}}(e) = e'$ and $\sigma_{a_i}(e') = e$.

3.2. **Tiling conjecture is equivalent to Conjecture 3.** The polygonality was described in terms of Whitehead graphs [16, Propositions 17 and 21]. But this description required infinitely many graphs to be examined. In the following lemma, we obtain a simpler formulation of polygonality requiring only one finite graph to be examined.

Lemma 16. *Let U be a list of cyclically reduced words in F_n. For each vertex v of $W_n(U)$, let σ_v be the connecting map associated with $W_n(U)$ at v. Then U is polygonal if and only if $W_n(U)$ has a nonempty list of cycles such that one of the cycles has length at least three and for each pair of edges e and f incident with a vertex v, the number of cycles in*



(a) D_j (b) $W_n(U)$

FIGURE 2. Each corner of a cell D_j in $Z_n(U)$ corresponds an edge in $W_n(U)$. Here, $F_2 = \langle a, b\rangle$ and $U = \{b^{-1}aba^2\}$. In these two figures, we note that $\sigma_{a^{-1}}(e) = e'$ and $\sigma_a(e') = e$.

the list containing both e and f is equal to the number of cycles in the list containing both $\sigma_v(e)$ and $\sigma_v(f)$.

We prove the necessity part by similar arguments to [16, Propositions 17 and 21]. The sufficiency part is what we mainly need for this paper.

Proof. We denote by μ the involution on the vertices of $W_n(U)$ defined by $\mu(a_i^{\pm 1}) = a_i^{\mp 1}$.

To prove the necessity, assume U is polygonal; we can find a U-polygonal surface $S = \coprod_{1\le i\le m} P_i/\sim$ as in Definition 11. In particular, each edge in $S^{(1)}$ is oriented and labeled by $\mathcal{A}_n$. Put $S^{(0)} = \{v_1, \ldots, v_t\}$. Fix p_i in the interior of each P_i. In Section 2.2, we have seen that there exists a map $\phi\colon S \to Z_n(U)$ such that ϕ is locally injective away from $p_1, \ldots, p_m$. Since S is a closed surface and ϕ is locally injective at v_i, the image of each $\mathrm{Link}_S(v_i)$ by ϕ is a cycle, say C_i, in $W_n(U)$.

Choose a vertex $v \in W_n(U)$ and two edges e, f incident with v. Without loss of generality, we may assume that $v = a^{-1}$ for some generator $a \in \mathcal{A}_n$ and $C_1, \ldots, C_{t'}$ is the list of the cycles among $C_1, \ldots, C_t$ which contain both e and f. Then for each $i = 1, \ldots, t'$, there exists a unique a-edge e_i outgoing from v_i. Let $v_{i'}$ be the endpoint of e_i other than v_i. There exist exactly two polygonal disks Q_i and R_i sharing e_i in S, so that $\mathrm{Link}(\phi; v_i)$ sends the corner of Q_i at v_i to e, and that of R_i at v_i to f. By the definition of a connecting map, $\mathrm{Link}(\phi; v_{i'})$ maps the corners of Q_i and R_i at $v_{i'}$ to $\sigma_{a^{-1}}(e)$ and $\sigma_{a^{-1}}(f)$, respectively; see Figure 3, which is similar to [16, Figure 7]. The correspondence $e \cup f \to \sigma_{a^{-1}}(e) \cup \sigma_{a^{-1}}(f)$ defines an involution on the list of length-2 subpaths of $C_1, \ldots, C_t$. The conclusion follows.

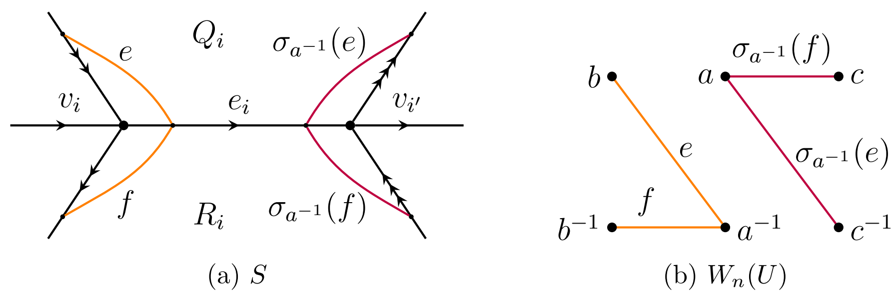


FIGURE 3. Consecutive corners in S and their images by a connecting map. $F_3 = \langle a, b, c\rangle$, and single, double and triple arrows denote the labels a, b and c, respectively.

For the sufficiency, consider a list of cycles $C_1, \ldots, C_t$ in $W_n(U)$ satisfying the given condition. For each C_i, let V_i be a polygonal disk such that ∂V_i is a cycle of the same length as C_i. We will regard ∂V_i as the dual cycle of C_i, in the sense that each edge of ∂V_i corresponds to a vertex of C_i and incident edges correspond to adjacent vertices. Choose a linear order $\prec$ on $\{(v, e) : e \in \delta(v)\}$ for each $v \in V(W_n(U))$ such that $(v, e) \prec (v, e')$ if and only if $(\mu(v), \sigma_v(e)) \prec (\mu(v), \sigma_v(e'))$. An edge g of ∂V_i will be labeled by $(a, \{e, f\})$ if the vertex v of $W_n(U)$ corresponding to g is labeled by a or a^{-1} for some $a \in \mathcal{A}_n$, and e and f are the two edges of C_i incident with v; see Figure 4 (a) and (b). Considered as a side of V_i, g will be given with a transverse orientation, which is incoming into V_i if $v \in \mathcal{A}_n$ and outgoing if $v \in \mathcal{A}_n^{-1}$. If w_e and w_f denote the vertices of g corresponding to e and f respectively, and $(v, e) \prec (v, f)$, then we shall orient g from w_f to w_e. Define a side-paring $\sim_0$ on $V_1, \ldots, V_t$ such that $\sim_0$ respects the orientations, and moreover, an incoming side labeled by $(a, \{e, f\})$ is paired with an outgoing side labeled by $(a, \{\sigma_a(e), \sigma_a(f)\})$ for each $a \in \mathcal{A}_n$ and $e, f \in \delta(a)$ where e and f are consecutive edges of some cycle C_i; the existence of such a side-pairing is guaranteed by the given condition. Consider the closed surface $S_0 = \coprod_i V_i/{\sim_0}$. Denote by η and ζ the numbers of the edges and the faces in S_0, respectively. Each edge in S_0 is shared by two faces, and each face has at least two edges; moreover, at least one face has more than two edges by the given condition. So, $2\zeta < \sum_i(\text{the number of sides in } V_i) = 2\eta$.

By the duality between C_i and V_i, each corner of V_i corresponds to an edge in C_i. Then the link of a vertex q of S_0 corresponds to the

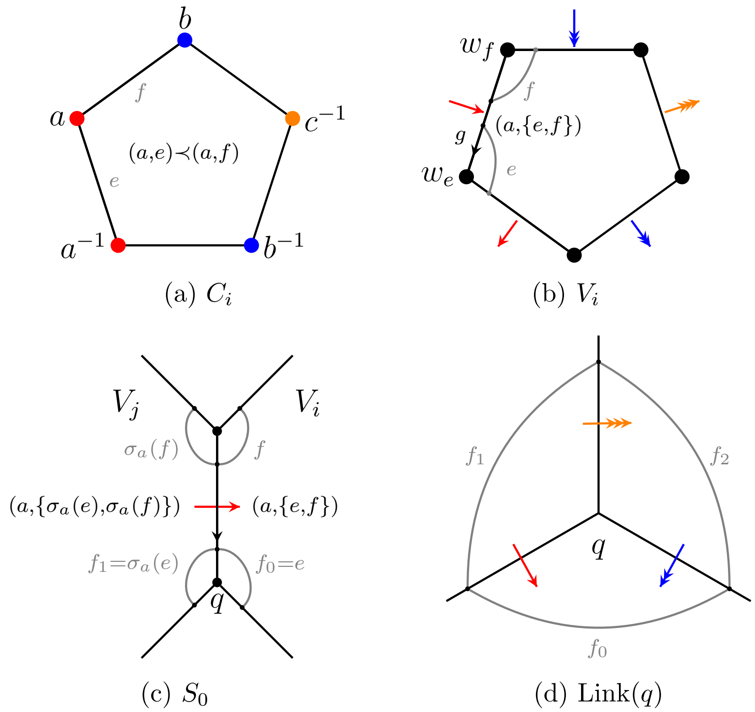


FIGURE 4. Constructing V_i and S_0 from C_i in the proof of Lemma 16. In this example, we note from (d) that $f_1 = \sigma_a(f_0)$, $f_2 = \sigma_{c^{-1}}(f_1)$ and $f_0 = \sigma_{b^{-1}}(f_2)$.

union of edges in $W_n(U)$ written as the following sequence

$$f_0, f_1 = \sigma_{x_1^{-1}}(f_0), f_2 = \sigma_{x_2^{-1}}(f_1), \ldots, f_l = \sigma_{x_l^{-1}}(f_{l-1})$$

so that $f_0 = f_l = \sigma_{x_l^{-1}} \circ \sigma_{x_{l-1}^{-1}} \circ \cdots \circ \sigma_{x_1^{-1}}(f_0)$ for some vertices $x_1, \ldots, x_l$ of $W_n(U)$; see Figure 4 (c). By Lemma 15, $x_1 \cdots x_l$ can be taken as a nontrivial power of a word in U. We will follow the boundary curve α of a small neighborhood of q with some orientation, and whenever α crosses an edge of S_0 with the first component of the label being $a \in \mathcal{A}_n$, we record a if the crossing coincides with the transverse orientation of the edge, and a^{-1} otherwise. Let $w_q \in F$ be the word obtained by this process. Then $w_q = x_1 \cdots x_l$, up to taking an inverse and cyclic conjugations.

Let S be a surface homeomorphic to S_0. We give S a 2-dimensional cell complex structure, by letting the homeomorphic image of the dual graph of $S_0^{(1)}$ to be $S^{(1)}$. In particular, the 2-cells $P_1, \ldots, P_m$ in S are the connected regions bounded by $S^{(1)}$. The transverse orientations and the first components of the labels of the sides in $V_1, \ldots, V_t$ induce orientations and labels of the sides of $P_1, \ldots, P_m$. By duality, the boundary reading of each P_i in S is of the form w_q for some vertex q of S_0; hence, ∂P_i reads a nontrivial power of a word in U. Finally, if we let ν be the number of the vertices in S_0, then

$$\chi(S) - m = \chi(S_0) - \nu = -\eta + \zeta < 0. \qquad \square$$

Proposition 14 and Lemma 16 imply the following.

Proposition 17. *Let $n' > 1$. Conjecture 2 holds for all $n = 2, \ldots, n'$ if and only if Conjecture 3 holds for graphs on $2n'$ vertices.*

Proof. (Conjecture 3 $\Rightarrow$ Conjecture 2) Let $2 \le n \le n'$ and let U be a minimal and diskbusting list of cyclically reduced words in F_n. If Conjecture 3 holds for $2n'$, then it holds for $2n$ because we can add isolated vertices. By Proposition 14, the connected graph $W_n(U)$ is equipped with the fixed point free involution $\mu(v) = v^{-1}$ on $V(W_n(U))$ and the associated connecting map σ_v at each vertex v such that $\lambda(v, \mu(v)) = \deg(v)$ and $\sigma_{\mu(v)} = \sigma_v^{-1}$. Note that $W_n(U)$ is non-acyclic; because otherwise $\deg(v) = \lambda(v, \mu(v)) \le 1$ for each vertex v and therefore $W_n(U)$ would be disconnected, as $W_n(U)$ has at least four vertices. The conclusion of Conjecture 3 along with Lemma 16 implies that U is polygonal.

(Conjecture 2 $\Rightarrow$ Conjecture 3) We let G, μ, σ_v be as in the hypothesis of Conjecture 3 such that $|V(G)| = 2n'$. Let $n = n'$. Since for each vertex v, v and $\mu(v)$ belong to the same connected component of G, we may assume that G is connected by taking a non-acyclic component of G. If $|V(G)| = 2$, then the list of all bigons is a desired collection of cycles. So we assume G is connected and $|V(G)| \ge 4$. Label the vertices of G as $a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}$ so that $a_i^{-1} = \mu(a_i)$. Then G can be regarded as the Whitehead graph of a list U of cyclically reduced words in F_n. Proposition 14 implies that U is minimal and diskbusting, as well. As we are assuming Conjecture 2 for F_n, U is polygonal. Lemma 16 completes the proof. $\square$

4. Regular Graph and Proof of Theorem 4

We will prove that Conjecture 3 holds for regular graphs. It turns out that we can prove a slightly stronger theorem.

Theorem 18. *Let $k > 1$. Let $G = (V, E)$ be a k-regular graph with a fixed point free involution $\mu : V \to V$ such that $\lambda(v, \mu(v)) = k$ for every vertex $v \in V$. Then there exists a nonempty list of cycles of G with positive integers m_1, m_2 such that every edge is in exactly m_1 cycles* *in the list and each adjacent pair of edges is contained in exactly m_2 cycles in the list.*

We obtain Theorem 4 as a corollary of Theorem 18.

Proof of Theorem 4. Note that $W_n(U)$ is regular. By Proposition 14, $W_n(U)$ satisfies the hypotheses of Theorem 18. Since U is minimal and diskbusting, $W_n(U)$ is connected. Since $W_n(U)$ has $2n$ vertices and $n > 1$, it has two adjacent edges e and f, not parallel to each other. By Theorem 18, there must be a cycle in the list containing both e and f and that cycle must have length at least three. Lemma 16 completes the proof. □

A graph H is a *subdivision* of G if H is obtained from G by replacing each edge by a path of length at least one. From Theorem 18, we can deduce the following.

Corollary 19. *Conjecture 3 is true for all subdivisions of k-regular graphs if $k > 1$.* □

Let us start proving Theorem 18. A graph $G = (V, E)$ is called a *k-graph* if it is k-regular and $|\delta(X)| \geq k$ for every subset X of V with $|X|$ odd. In particular if $k > 0$, then every k-graph must have an even number of vertices, because otherwise $|\delta(V(G))| \geq k$.

Why do we care about k-graphs? It turns out that every k-regular graph with the properties required by Conjecture 3 is a k-graph.

Lemma 20. *Let $G = (V, E)$ be a k-regular graph with a fixed point free involution μ such that $\lambda(v, \mu(v)) = k$ for every vertex $v \in V$. Then G is a k-graph.*

Proof. Supposes $X \subseteq V$ and $|X|$ is odd. Then there must be $x \in X$ with $\mu(x) \notin X$ because μ is an involution such that $\mu(v) \neq v$ for all $v \in V$. Then there exist k edge-disjoint paths from x to $\mu(x)$ and therefore $|\delta(X)| \geq k$. □

By the previous lemma, it is sufficient to consider k-graphs in order to prove Theorem 18. By using Edmonds' characterization of the perfect matching polytope [9], Seymour [24] showed the following theorem. This is also explained in Corollary 7.4.7 of the book by Lovász and Plummer [19]. A *matching* is a set of edges in which no two are adjacent. A *perfect matching* is a matching meeting every vertex.

Theorem 21 (Seymour [24])**.** *Every k-graph is* fractionally k-edge-colorable*. In other words, every k-graph has a nonempty list of perfect matchings M_1, M_2, ..., M_ℓ such that every edge is in exactly ℓ/k of them.*

For sets A and B, we write $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Lemma 22. *Let $k > 1$. Every k-graph has a nonempty list of cycles such that every edge appears in the same number of cycles and for each pair of adjacent edges e, f, the number of cycles in the list containing both e and f is identical.*

Proof. Let $M_1, M_2, \ldots, M_\ell$ be a nonempty list of perfect matchings of a k-graph $G = (V, E)$ such that each edge appears in ℓ/k of them. Then for distinct i, j, the set $M_i \Delta M_j$ induces a subgraph of G such that every vertex has degree 2 or 0. Thus each component of the subgraph $(V, M_i \Delta M_j)$ is a cycle. Let $C_1, C_2, \ldots, C_m$ be the list of cycles appearing as a component of the subgraph of G induced by $M_i \Delta M_j$ for each pair of distinct i and j. We allow repeated cycles. This list is nonempty because $k > 1$ and so there exist i, j such that $M_i \neq M_j$.

Since each edge is contained in exactly ℓ/k of $M_1, M_2, \ldots, M_\ell$, every edge is in exactly $\frac{\ell}{k}(\ell - \frac{\ell}{k})$ cycles in the list. For two adjacent edges e and f, since no perfect matching contains both e and f, there are $(\ell/k)^2$ cycles in $C_1, C_2, \ldots, C_m$ using both e and f. $\square$

Lemmas 20 and 22 clearly imply Theorem 18.

Proof of Corollary 5. We note that a k-regular 4-vertex graph is always a k-graph.

For the sufficiency, we recall that if U is diskbusting in F_2, then $W_2(U)$ is connected [26, 25]. Since a connected 4-vertex graph contains at least one pair of incident edges which are not parallel, Lemma 22 implies that $W_2(U)$ contains a list of cycles, not all bigons, such that each pair of incident edges appears the same number of times in the list. Lemma 16 proves the claim.

For the necessity, we note that the proof of the sufficiency part of Proposition 13 shows if U is not diskbusting in F_2, then $D_2(U)$ does not contain a hyperbolic surface group. $\square$

5. Graphs on four vertices

Let G be a graph with a fixed point free involution $\mu : V(G) \to V(G)$ and a bijection $\sigma_v : \delta(v) \to \delta(\mu(v))$ for each vertex v so that $\lambda(v, \mu(v)) = \deg(v)$ and $\sigma_{\mu(v)} = \sigma_v^{-1}$. For a vertex w of G, a permutation π on $\delta(w)$ is called *w-good* if $\{e, \sigma_w(\pi(e))\}$ is a matching of G for every edge e incident with w. Note that $\{e, f\}$ is a matching of G if and only if either $e = f$ or e, f share no vertex. In particular, if x is an edge joining w and $\mu(w)$, then $\sigma_w(\pi(x)) = x$.

A permutation π on a set X induces a permutation $\pi^{(2)}$ on 2-element subsets of X such that $\pi^{(2)}(\{x,y\}) = \{\pi(x), \pi(y)\}$ for all distinct $x, y \in X$. A w-good permutation π on $\delta(w)$ is *uniform* if $\pi^{(2)}$ has a list of orbits X_1, X_2, …, X_t satisfying the following.

(i) If $\{x,y\} \in X_i$, then x and y do not share a vertex other than w or $\mu(w)$ in G.
(ii) There is a constant $c > 0$ such that for every edge $e \in \delta(w)$,

$$|\{(X_i, F) : 1 \le i \le t,\ F \in X_i \text{ and } e \in F\}| = c.$$

The following lemma shows that in order to prove Conjecture 3 for 4-vertex graphs, it is enough to find a w-good uniform permutation on the edges incident with a vertex w of minimum degree.

Lemma 23. *Let G be a connected 4-vertex graph with a fixed point free involution $\mu : V(G) \to V(G)$ such that $\lambda(v, \mu(v)) = \deg(v)$ for each vertex v. Let w be a vertex of G with the minimum degree. Let $\sigma_w : \delta(w) \to \delta(\mu(w))$ be a bijection.*

If there is a w-good uniform permutation π on $\delta(w)$, then G admits a nonempty list of cycles satisfying the following properties.

(a) *For distinct edges $e_1, e_2 \in \delta(w)$, the number of cycles in the list containing both e_1 and e_2 is equal to the number of cycles in the list containing both $\sigma_w(e_1)$ and $\sigma_w(e_2)$.*
(b) *There is a constant $c_1 > 0$ such that each edge appears in exactly c_1 cycles in the list.*
(c) *There is a constant $c_2 > 0$ such that for a vertex $v \in V(G) \setminus \{w, \mu(w)\}$ and each pair of distinct edges $e_1, e_2 \in \delta(v)$, exactly c_2 cycles in the list contain both e_1 and e_2.*
(d) *The list contains a cycle of length at least three.*

Proof. We say that a list of cycles is *good* if it satisfies (a), (b), (c), and (d). We proceed by induction on $|E(G)|$. Let u be a vertex of G other than w and $\mu(w)$. If $\deg(u) = \deg(w)$, then the conclusion follows by Theorem 18. Therefore we may assume that $\deg(u) > \deg(w)$. There should exist an edge e joining u and $\mu(u)$. Moreover $G \setminus e$ is connected because otherwise G would not have $\deg(w)$ edge-disjoint paths from w to $\mu(w)$.

By the induction hypothesis, $G \setminus e$ has a good list of cycles C'_1, C'_2, …, C'_s. Note that we use the fact that $\deg(u) > \deg(w)$ so that $G \setminus e$ has $\deg_{G\setminus e}(v)$ edge-disjoint paths from v to $\mu(v)$ for each vertex v of $G \setminus e$. Let c'_1, c'_2 be the constants given by (b) and (c), respectively, for the list C'_1, C'_2, …, C'_s of cycles of $G \setminus e$.

Since π is w-good uniform, $\pi^{(2)}$ has a list of orbits X_1, X_2, …, X_t satisfying (i) and (ii), where each edge in $\delta(w)$ appears c times in this list.

Suppose that $\{x, y\} \in X_i$. Then $\{\pi(x), \pi(y)\} \in X_i$. If $x, y \in \delta(\mu(w))$, then we let C_{xy} be a cycle formed by two edges $x = \sigma_w(\pi(x))$ and $y = \sigma_w(\pi(y))$. If $x, y \notin \delta(\mu(w))$, let C_{xy} be a list of two cycles, one formed by three edges e, x, y, and the other formed by three edges e, $\sigma_w(\pi(x))$, $\sigma_w(\pi(y))$. If exactly one of x and y, say y, is incident with $\mu(w)$, then let C_{xy} be the cycle formed by four edges e, x, $y = \sigma_w(\pi(y))$, $\sigma_w(\pi(x))$. Since x and y never share u or $\mu(u)$ by (i), C_{xy} always consists of one or two cycles of G.

Let $C_1, C_2, \ldots, C_p$ be the list of all cycles in C_{xy} for each member $\{x, y\}$ of X_i for all $i = 1, 2, \ldots, t$. Notice that we allow repetitions of cycles.

We claim that the list $C_1, C_2, \ldots, C_p$ satisfies (a). For each occurrence of $x, y \in \delta(w)$ in a cycle in the list, there is a corresponding i such that $\{x, y\} \in X_i$. Since X_i is an orbit, there is $\{x', y'\} \in X_i$ where $\pi(x') = x$ and $\pi(y') = y$. Then the list contains cycles in $C_{x'y'}$ for X_i. This proves the claim because $\sigma_w(x) = \sigma_w(\pi(x'))$ and $\sigma_w(y) = \sigma_w(\pi(y'))$.

By (ii) of the definition of a uniform permutation, for each edge f incident with w, there are c cycles in the list $C_1, C_2, \ldots, C_p$ containing the edge f of G. Notice that whenever an edge f in C_{xy} is in $\delta(\{w, \mu(w)\})$, C_{xy} contains e and $\sigma_w(\pi(f))$ by the construction. Therefore every edge incident with w or $\mu(w)$ appears c times in the list $C_1, C_2, \ldots, C_p$.

We now construct a good list of cycles for G as follows: We take c'_2 copies of $C_1, C_2, \ldots, C_p$, c copies of $C'_1, C'_2, \ldots, C'_s$, and cc'_2 copies of cycles formed by e and another edge $f \neq e$ joining u and $\mu(u)$. We claim that this is a good list of cycles of G. It is trivial to check (a). For distinct edges e_1, e_2 incident with u, the list contains cc'_2 cycles containing both of them, verifying (c). Let a be the number of edges in $\delta(u)$ incident with w or $\mu(w)$ and let b be the number of edges joining u and $\mu(u)$. By (c) on $G \setminus e$, we have $c'_1 = c'_2(a + b - 2)$. Finally to prove (b), every edge incident with w or $\mu(w)$ appears $cc'_2 + cc'_1 = cc'_2(a+b-1)$ times in the list and the edge e appears $acc'_2 + (b-1)cc'_2 = cc'_2(a+b-1)$ times in the list. An edge $f \neq e$ joining u and $\mu(u)$ appears $cc'_1 + cc'_2 = cc'_2(a + b - 1)$ times. □

5.1. **Lemma on Odd Paths and Even Cycles.** To find a w-good uniform permutation of $\delta(w)$, we need a combinatorial lemma on a set of disjoint odd paths and even cycles. The *length* of a path or a cycle is the number of its edges.

Lemma 24. *Let D be a directed graph with at least four vertices such that each component is a directed path of odd length or a directed cycle of even length. Suppose that every vertex of in-degree* 0 *or out-degree* 0 *in D is colored with red or blue, while the number of red vertices of in-degree* 0 *is equal to the number of red vertices of out-degree* 0*. We say that a graph is* good *if at most half of all the vertices are blue and at most half of all the vertices are red. We say that a directed path or a cycle is* long *if its length is at least three. A directed path or a cycle is said to be* short *if it is not long. A* R-R *path denotes a directed path starting with a red vertex and ending with a red vertex. Similarly we say* R-B *paths,* B-B *paths,* B-R *paths. A set of paths is called* monochromatic *if it has no blue vertex or no red vertex.*

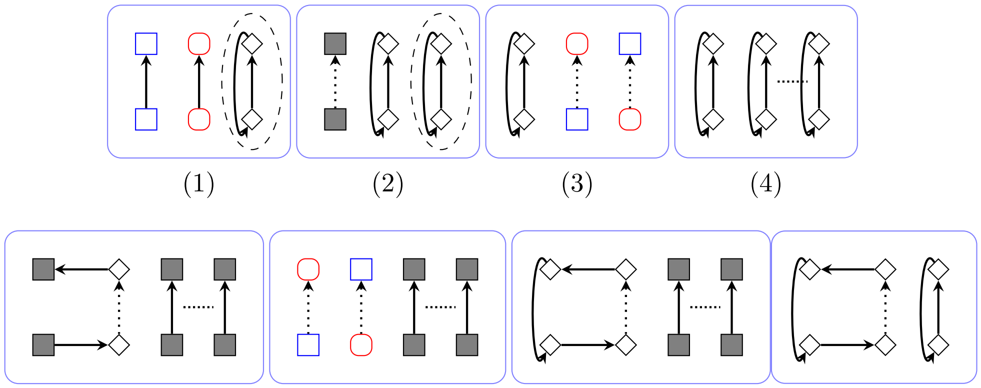
If D is good, then D can be partitioned into good subgraphs, each of which is one of eight types listed below. (See Figure 5.)

1. *A short R-R path, a short B-B path, and possibly a short cycle.*
2. *A monochromatic path and one or two short cycles.*
3. *A short cycle, a B-R path, and an R-B path.*
4. *At least two short cycles.*
5. *A long monochromatic path and monochromatic short paths, possibly none.*
6. *A B-R path, a R-B path, and monochromatic short paths, possibly none.*
7. *A long cycle and monochromatic short paths, possibly none.*
8. *A long cycle and a short cycle.*

We remark that in a subgraph of type (5), we require that the long path is monochromatic and the set of short paths monochromatic, but we allow the long path to have a color unused in short paths.

Proof. We proceed by induction on $|V(D)|$. If D has a subgraph H that is a disjoint union of a short R-R path and a short B-B path, then $D \setminus V(H)$, the subgraph obtained by removing vertices of H from D, is still good. If $D = H$, then we have nothing to prove. If $|V(D) \setminus V(H)| = 2$, then D is the disjoint union of a short R-R path, a short B-B path, and a short cycle, and therefore D is a directed graph of type (1). If $|V(D) \setminus V(H)| \geq 4$, then H is a good subgraph of type (1). Then we apply the induction hypothesis to get a partition for $D \setminus V(H)$.

Therefore we may assume that D has no pair of a short B-B path and a short R-R path. By symmetry, we may assume that D has no short R-R path. Then in each component, the number of red vertices is at most half of the number of vertices. Thus, in order to check whether some disjoint union of components is good, it is enough to count blue vertices.



(1) (2) (3) (4) (5) (6) (7) (8)

FIGURE 5. Description of eight types of good subgraphs

Suppose that D has a short cycle and a short B-B path. We are done if D is a graph of type (2). Thus we may assume that D has at least eight vertices. Let X be the set of vertices in the pair of a short cycle and a short B-B path. Then the subgraph of D induced on X is a subgraph of type (2). Because X has two blue vertices and two uncolored vertices, $D \setminus X$ is good and has at least four vertices. By the induction hypothesis, we obtain a good partition of $D \setminus X$. This together with the subgraph induced by X is a good partition of D.

We may now assume that either D has no short cycles, or D has no short B-B path.

(Case 1) Suppose that D has no short cycles. The subgraph of D consisting of all components other than short B-B paths can be partitioned into good subgraphs P_1, P_2, $\ldots$, P_k of type (5), (6), or (7), because the number of R-B paths is equal to the number of B-R paths. We claim that short B-B paths can be assigned to those subgraphs while maintaining each P_i to be good. Suppose that P_i has $2b_i$ blue vertices and $2n_i = |V(P_i)|$. Notice that b_i and n_i are integers. Let x be the number of short B-B paths in D. Since D is good, $2(2x + \sum_{i=1}^{k} 2b_i) \leq \sum_{i=1}^{k} 2n_i + 2x$ and therefore $x \leq \sum_{i=1}^{k}(n_i - 2b_i)$. Each P_i can afford to have $n_i - 2b_i$ short B-B paths to be good. Overall all $P_1, \ldots, P_k$ can afford $\sum_{i=1}^{k}(n_i - 2b_i)$ short B-B paths; thus consuming all short B-B paths. This proves the claim.

(Case 2) Suppose D has short cycles but has no short B-B paths. If D has at least two short cycles, then we can take all short cycles as a subgraph of type (4) and the subgraph of D consisting of all

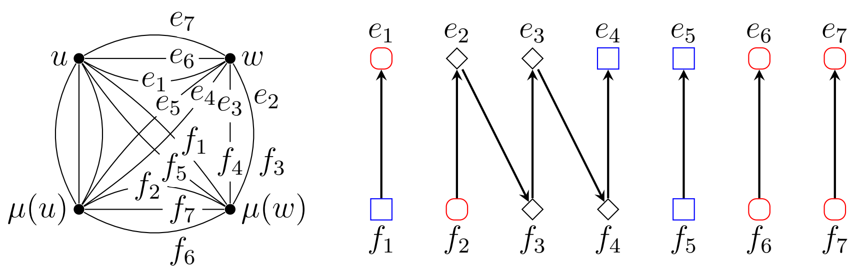


FIGURE 6. A graph and its auxiliary directed graph at w

components other than short cycles can be decomposed into subgraphs, each of which is type (5), (6), or (7).

Thus we may assume D has exactly one short cycle. Since D has at least four vertices, D must have a subgraph P consisting of components of D that is one of the following type: a monochromatic path, a long cycle, or a pair of a B-R path and an R-B path. Then P with the short cycle forms a good subgraph of type (2), (8), or (3), respectively. The subgraph of D induced by all the remaining components can be decomposed into subgraphs of type (5), (6), and (7). $\square$

5.2. **Finding a Good Uniform Permutation.** Let G be a connected 4-vertex graph with a fixed point free involution $\mu : V(G) \to V(G)$ such that $\lambda(v, \mu(v)) = \deg(v)$ for each vertex v. Let w be a vertex of G with the minimum degree and let u be a vertex of G other than w and $\mu(w)$. Let $\sigma_w : \delta(w) \to \delta(\mu(w))$ be a bijection.

Let $e_1, e_2, \ldots, e_m$ be the edges incident with w and let $f_1, f_2, \ldots, f_m$ be the edges incident with $\mu(w)$ so that $f_i = \sigma_w(e_i)$. We construct an auxiliary directed graph D on the disjoint union of $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_m\}$ as follows:

(i) For all $i \in \{1, 2, \ldots, m\}$, D has an edge from f_i to e_i.
(ii) If e_i and f_j denote the same edge in G, then D has an edge from e_i to f_j.

We have an example in Figure 6. It is easy to observe the following.

- Every vertex in $\{e_1, e_2, \ldots, e_m\}$ of D has in-degree 1.
- Every vertex in $\{f_1, f_2, \ldots, f_m\}$ of D has out-degree 1.
- A vertex e_i of D has out-degree 1 if the edge e_i of G is incident with $\mu(u)$, and out-degree 0 if otherwise.
- A vertex f_i of D has in-degree 1 if the edge f_i of G is incident with u, and in-degree 0 if otherwise.

By the degree condition, D is the disjoint union of odd directed paths and even directed cycles.

Let r be the number of edges of G joining u and w and let b be the number of edges of G joining $\mu(u)$ and w. For each i, we color e_i red if it is incident with u and blue if it is incident with $\mu(u)$. Similarly for each i, we color f_i blue if it is incident with u and red if it is incident with $\mu(u)$. Clearly there are r red vertices and b blue vertices in $\{e_1, e_2, \ldots, e_m\}$.

Let r' be the number of edges of G joining $\mu(u)$ and $\mu(w)$ and let b' be the number of edges of G joining u and $\mu(w)$. We claim that $r' = r$ and $b' = b$. Of course, there are r red vertices and b blue vertices in $\{f_1, f_2, \ldots, f_m\}$. Since $\deg w = \deg \mu(w)$ and $\deg u = \deg \mu(u)$, we have $r + b' = b + r'$ and $r + b = r' + b'$. We deduce that $r = r'$ and $b = b'$.

We also assume that G has $\deg(u)$ edge-disjoint paths from u to $\mu(u)$. Therefore $|\delta(\{u, w\})| \geq |\delta(u)|$ and $|\delta(\{u, \mu(w)\})| \geq |\delta(u)|$. This implies that $b+b+(m-r-b) \geq b+r$ and $r+r+(m-r-b) \geq b+r$. Thus

$$2r \leq m \text{ and } 2b \leq m.$$

From now on, our goal is to describe a w-good permutation π on $\delta(w)$ from a directed graph D with a few extra edges.

Lemma 25. *Let D' be a directed graph obtained by adding one edge from each vertex of out-degree 0 to a vertex of in-degree 0 with the same color so that every vertex has in-degree 1 and out-degree 1 in D'. Let π be a permutation on $\delta(w) = \{e_1, e_2, \ldots, e_m\}$ so that $\pi(e_i) = e_j$ if and only if D' has a directed walk from e_i to e_j of length two. Then π is w-good.*

Let us call such a directed graph D' a *completion* of D. A completion of D' always exists, because the number of red vertices of in-degree 0 is equal to the number of red vertices of out-degree 0. Clearly there are $r!\,b!$ completions of D.

Proof. It is enough to show that if D' has an edge e from e_i to f_j, then $\{e_i, f_j\}$ is a matching of G. If $e \in E(D)$, then $e_i = f_j$ and therefore $\{e_i, f_j\} = \{e_i\}$ is a matching of G. If $e \notin E(D)$, then e_i and f_j should have the same color and therefore e_i and f_j do not share any vertex. □

Out of $r!\,b!$ completions of D', we wish to find a completion D' of D so that the w-good permutation induced by D' is uniform.

Lemma 26. *If D is a directed graph of type (1), (2), …, (8) described in Lemma 24, then D has a completion D' so that the induced w-good permutation is uniform.*

Proof. We claim that for each type of a directed graph D, there is a completion D' of D such that its induced w-good permutation π on $\delta(w)$ is uniform. Recall that a w-good permutation π is uniform if $\pi^{(2)}$ has a list of orbits X_1, X_2, …, X_t satisfying the following conditions:

(i) If $\{x, y\} \in X_i$, then x and y do not share a vertex other than w or $\mu(w)$ in G.
(ii) There is a constant $c > 0$ such that for every edge $e \in \delta(w)$,

$$|\{(X_i, F) : 1 \leq i \leq t,\ F \in X_i \text{ and } e \in F\}| = c.$$

Case 1: Suppose that D is of type (1) or (4) with k components. Then There is a unique completion D' of D. It is easy to verify that the list of all orbits of $\pi^{(2)}$ satisfies the conditions (i) and (ii) where $c = k - 1$.

Case 2: Suppose that D is of type (2). Then D consists of a monochromatic path P and one or two short cycles. A completion D' of D is unique, as it is obtained by adding an edge from the terminal vertex of P to the initial vertex of P. Let π be the permutation of $\delta(w)$ induced by D'. Let $x_1, x_2, \ldots, x_m$ be the edges in $\delta(w)$ that are in P such that $\pi(x_i) = x_{i+1}$ for all $i = 1, 2, \ldots, m$ where $x_{m+1} = x_1$. Let $y_1 \in \delta(w)$ be the vertex in the first short cycle such that $\pi(y_1) = y_1$. If D has two cycles, then let $y_2 \in \delta(w)$ be the vertex in the second short cycle such that $\pi(y_2) = y_2$.

Then $O_j = \{\{x_i, y_j\} : 1 \leq i \leq m\}$ is an orbit of $\pi^{(2)}$ satisfying (i). If $m > 1$, then $O_P = \{\{x_i, x_{i+1}\} : 1 \leq i \leq m\}$ is an orbit of $\pi^{(2)}$ satisfying (i) in which each x_i appears twice if $m > 2$ and each x_i appears once if $m = 2$.

If D has only one cycle, then each x_i appears once and y_1 appears m times in O_1. So if $m = 1$, then O_1 satisfies (i) and (ii). If $m = 2$, then O_1 and O_P form a list of orbits of $\pi^{(2)}$ satisfying (i) and (ii). If $m > 2$, then a list of two copies of O_1 and $(m-1)$ copies of O_P satisfies (i) and (ii).

If D has two short cycles, then in O_1 and O_2, each x_i appears twice and each y_j appears m times. Notice that $\{\{y_1, y_2\}\}$ is an orbit of $\pi^{(2)}$. If $m = 1$, then a list of O_1, O_2, and $\{\{y_1, y_2\}\}$ satisfies (i) and (ii). If $m = 2$, then a list of O_1 and O_2 satisfies (i) and (ii). If $m > 3$, then a list of two copies of O_1, two copies of O_2, and $(m-2)$ copies of O_P satisfies (i) and (ii).

Case 3: If D is of type (3), then D has a unique completion D'. Let π be the permutation of $\delta(w)$ induced by D'. Let $y \in \delta(w)$ be a vertex of D in the short cycle such that $\pi(y) = y$. Let $x_1, x_2, \ldots, x_m \in \delta(w)$ be the vertices on the long cycle in D' such that $\pi(x_i) = x_{i+1}$ for all

$i = 1, 2, \ldots, m$ where $x_{m+1} = x_1$. Since D has two paths, $m > 1$. Then $O_P = \{\{x_i, x_{i+1}\} : i = 1, 2, \ldots, m\}$ and $O_C = \{\{y, x_i\} : i = 1, 2, \ldots, m\}$ are orbits of $\pi^{(2)}$. In O_P, each x_i appears twice if $m > 2$ and once if $m = 2$. In O_C each x_i appears once and y appears m times. Now it is routine to create a list of orbits satisfying (i) and (ii) by taking copies of O_C and copies of O_P.

Case 4: Suppose that D is of type (5) having both red and blue vertices or D is of type (7) or (8). Let D' be a completion of D obtained by making each path of D to be a cycle of D'. Let $x_1, x_2, \ldots, x_m \in \delta(w)$ be vertices in the long cycle of D' so that $\pi(x_i) = x_{i+1}$ for all $i = 1, 2, \ldots, m$ where $x_{m+1} = x_1$. Let $y_1, y_2, \ldots, y_k \in \delta(w)$ be vertices in short cycles of D' such that $\pi(y_i) = y_i$. Since D is good, $k \leq m$. Let $O_j = \{\{x_i, y_j\} : i = 1, 2, \ldots, m\}$ for $j = 1, 2, \ldots, k$ and $O_P = \{\{x_i, x_{i+1}\} : i = 1, 2, \ldots, m\}$ where $x_{m+1} = x_1$. In the list of O_1, O_2, …, O_k, each x_i appears k times and each y_j appears m times. In O_P, each x_i appears twice if $m > 2$ and once if $m = 2$. To satisfy (i) and (ii), we can take a list of two copies of each O_j for $j = 1, 2, \ldots, k$ and copies of O_P.

Case 5: Suppose that D is a directed graph of type (5) not having both red and blue, or D is a directed graph of type (6). Then D has a completion D' consisting of a single cycle. Let π be the permutation of $\delta(w)$ induced by D'. Let $x_1, x_2, \ldots, x_m \in \delta(w)$ be vertices in D such that $\pi(x_i) = x_{i+1}$ for all $i = 1, 2, \ldots, m$. We $O_P = \{\{x_i, x_{i+\lfloor m/2 \rfloor}\} : i = 1, 2, \ldots, m\}$ where $x_{j+m} = x_j$ for all $j = 1, \cdots, \lfloor m/2 \rfloor$. Then in O_P, each x_i appears twice if m is odd and once if m is even. Moreover, since all the vertices of the same color appear consecutively in D' and the number of vertices of the same color is at most half of m, O_P never contains a pair $\{x_i, x_j\}$ of vertices of the same color, red or blue. Therefore O_P satisfies (i) and (ii). This completes the proof. □

Lemma 27. *There exists a completion D' of D so that the w-good permutation induced by D' is uniform.*

Proof. By Lemma 24, D can be partitioned into good subgraphs D_1, D_2, …, D_t of type (1), (2), …, (8). Lemma 26 shows that each D_i admits a completion that induces a w-good uniform permutation π_i with a list L_i of orbits of $\pi_i^{(2)}$ satisfying (i) and (ii). Let us assume that each vertex of D_i appears $c_i > 0$ times in L_i. Let $c = \mathrm{lcm}(c_1, c_2, \ldots, c_t)$. Then let L be the list of orbits obtained by taking c/c_i copies of L_i for each $i = 1, 2, \ldots, t$. Then L satisfies (i) and (ii). This proves the lemma. □

Now we are ready to prove Conjecture 3 for 4-vertex graphs. Let us state the theorem.

Theorem 28. *Let G be a connected 4-vertex graph with a fixed point free involution $\mu : V(G) \to V(G)$ and a bijection $\sigma_v : \delta(v) \to \delta(\mu(v))$ for each vertex v such that $\lambda(v, \mu(v)) = \deg(v)$ and $\sigma_{\mu(v)} = \sigma_v^{-1}$. Then G has a nonempty list of cycles satisfying the following.*

(a) *For each pair of edges e and f incident with a vertex v, the number of cycles in the list containing both e and f is equal to the number of cycles in the list containing both $\sigma_v(e)$ and $\sigma_v(f)$.*
(b) *Each edge of G appears in the same number of cycles in the list.*
(c) *The list contains a cycle of length at least three.*

Proof. Let w be a vertex of minimum degree. By Lemma 27, G has a w-good uniform permutation π on $\delta(w)$. By Lemma 23, G has a nonempty list of cycles satisfying (a), (b), and (c). □

Because of (b), we can obtain the following corollary.

Corollary 29. *Conjecture 3 is true for subdivisions of connected 4-vertex graphs.* □

Theorem 6 is an immediate consequence of Theorem 28. Corollary 7 follows from Proposition 13, since we have shown that Tiling Conjecture is true for $n = 2$.

6. Final Remarks

Minimality assumption in Tiling Conjecture. A graph G is 2-*connected* if $|V(G)| > 2$, G is connected, and $G \setminus x$ is connected for every vertex x. It is well-known that a list U of cyclically reduced words in F_n is diskbusting if and only if $W_n(\phi(U))$ is 2-connected for some $\phi \in \mathrm{Aut}(F_n)$ [26, 25]. However, the minimality assumption in Conjecture 2 cannot be weakened to the 2-connectedness of the Whitehead graph; this is equivalent to saying that $\lambda(v, \mu(v)) = \deg(v)$ in Conjecture 3 cannot be relaxed to 2-connectedness. Daniel Král' [18] kindly provided us Example 30 showing why this relaxation is not possible.

Example 30. Let G be a 4-vertex graph shown in Figure 7. For a vertex v and edges $e \in \delta(v)$ and $f \in \delta(\mu(v))$, we let $\sigma_v(e) = f$ if and only if the number written on e near v coincides with the number written on f near $\mu(v)$. Actually, G is the Whitehead graph of $a(ab^{-1})^3b^{-2}$ with the associated connecting maps σ_v. While G is 2-connected, one can verify that G does not have a list of cycles satisfying the conclusion of Conjecture 3. Note that $\lambda(a, \mu(a)) = 3 < 4 = \deg(a)$.

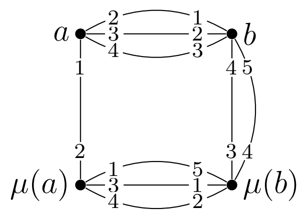


FIGURE 7. Example 30.

Control over positive degrees. The following lemma states that Conjecture 3 can be strengthened to require each edge to appear the same number of times.

Lemma 31. *Suppose Conjecture 3 is true. If G is connected and has at least four vertices, then the list of cycles in the conclusion of Conjecture 3 can be chosen so that each edge appears the same number of times.*

Proof. Let G be a given graph. We claim that G is 2-connected. Suppose not and let x be a vertex such that $G \setminus x$ is disconnected. Let C be a component of $G \setminus x$ containing $\mu(x)$ and D be a component of $G \setminus x$ other than C. Since G is connected, x has an edge incident with a vertex in D and therefore G can not have $\deg(x)$ edge-disjoint paths from x to $\mu(x)$, a contradiction. This proves the claim.

Let $e_1, e_2, \ldots, e_m$ be the list of edges of G. Let G' be a graph obtained from G by replacing each edge with a path of length m. Let v_{ij} be the j-th internal vertex of the path of G' representing e_i where $j = 1, 2, \ldots, m-1$. We extend μ of G to obtain μ' of G' so that $\mu'(v_{i,j}) = v_{j,i-1}$ and $\mu'(v_{j,i-1}) = v_{i,j}$ for all $1 \leq j < i \leq m$.

Since G is 2-connected, for each pair of edges e and f of G, there is at least one cycle containing both e and f. Thus in G', there are two edge-disjoint paths from $v_{i,j}$ to $v_{j,i-1}$ for all $1 \leq j < i \leq m$. So we can apply Conjecture 3 to G' and deduce that each edge of G is used the same number of times because the number of cycles passing $v_{i,j}$ is equal to the number of cycles passing $v_{j,i-1}$ for all $1 \leq j < i \leq m$. □

Suppose U is a polygonal list of cyclically reduced words $u_1, \ldots, u_r$ in F_n. There exists a closed U-polygonal surface S obtained by a side-pairing on polygonal disks $P_1, \ldots, P_m$ equipped with an immersion $S^{(1)} \to \text{Cayley}(F_n)/F_n$ as in Definition 11. We shall orient each ∂P_i so that each $\partial P_i \to S^{(1)} \to \text{Cayley}(F_n)/F_n$ reads a positive power of a word in U. For each u_j in U, if $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ is the list of polygonal disks whose boundaries read powers, say $u_j^{c_1}, u_j^{c_2}, \ldots, u_j^{c_k}$, of u_j, then we say that $c_1 + c_2 + \cdots + c_k$ is the *positive degree of* u_j in S.

Proposition 32. *Let U be a minimal and diskbusting list of cyclically reduced words $u_1, \ldots, u_r$ in F_n for some $n > 1$. We assume that either Conjecture 2 is true, or $n = 2$.*

1. *There exists a U-polygonal surface S such that the positive degree of every word in U is the same.*
2. *For every list of positive integers $\alpha_1, \ldots, \alpha_r$, there exists $\phi \in \mathrm{Aut}(F_n)$, a $\phi(U)$-polygonal surface S, and a constant K such that the positive degree of $\phi(u_i)$ is $K\alpha_i$ for each $i = 1, \ldots, r$.*

Proof. (1) Suppose that $W_n(U)$ has a list of cycles satisfying the conclusion of Conjecture 3 and each edge appears the same number of times, say s, in the list. So, every word in U has the positive degree s in the closed surface S which is constructed in the proof of Lemma 16. Hence, the proof follows from Part (b) of Theorem 28 and Lemma 31.

(2) We make a new list U' by duplicating each u_i for α_i times. U' might not be minimal, but one can find $\phi \in \mathrm{Aut}(F_n)$ such that $\phi(U')$ is minimal. By applying Part (1) to $\phi(U')$, there exists a $\phi(U')$-polygonal surface S where the positive degrees are the same, say K. By regarding S as a $\phi(U)$-polygonal surface, one obtains the desired result. □

Non-virtually geometric words. Let H_n denote a 3-dimensional handlebody of genus n. A word w in F_n can be realized as an embedded curve $\gamma \subseteq H_n$. A word w is said to be *virtually geometric* if there exists a finite cover $p\colon H' \to H_n$ such that $p^{-1}(\gamma)$ is homotopic to a 1-submanifold on the boundary of H' [11]. Using Dehn's lemma, Gordon and Wilton [11] proved that if $w \in F_n$ is diskbusting and virtually geometric, then $D_n(\{w\})$ contains a surface group; this also follows from the fact that a minimal diskbusting geometric word is polygonal [16]. On the other hand, Manning [20] showed that for $n > 1$, there exist minimal diskbusting words which are not virtually geometric. More precisely, he proved that if the Whitehead graph of $w \in F_n$ is non-planar, k-regular and k-edge-connected for some $k \geq 3$, then w is not virtually geometric. Note that a graph G is *k-edge-connected* if $|\delta(X)| \geq k$ for all $\emptyset \neq X \subsetneq V(G)$. Even for such w, Theorem 18 implies that $D_n(\{w\})$ contains a hyperbolic surface group.

Existence of separable surface subgroups. A subgroup H of a group G is said to be *separable* if H coincides with the intersection of all the finite-index subgroups of G containing H. If every finitely generated subgroup of G is separable, we say G is *subgroup separable.* The *Virtual Haken Conjecture* for a closed hyperbolic 3-manifold M asserts that there exists a π_1-injective, homeomorphically embedded, closed hyperbolic surface in some finite cover of M [23]; this is a main motivation for Question 1. If $\pi_1(M)$ contains a *separable* hyperbolic surface subgroup, then it is known that a closed hyperbolic surface π_1-injectively embeds into a finite cover of M [23]. So, it is natural to augment Question 1 as follows.

Question 2. *Does every one-ended word-hyperbolic group contain a separable hyperbolic surface group?*

Since $X_n(U)$ has a non-positively curved square complex structure (Section 2.2), and also decomposes a graph of free groups with cyclic edge groups, $D_n(U)$ is subgroup separable by [29].

References

[1] Gilbert Baumslag, Benjamin Fine, Charles F. Miller, III, and Douglas Troeger, *Virtual properties of cyclically pinched one-relator groups*, Internat. J. Algebra Comput. **19** (2009), no. 2, 213–227. MR 2512551 (2010c:20034)

[2] John Berge, *Heegaard documentation. Preprint (c.1990), http://www.math.uic.edu/~t3m.*

[3] Martin R. Bridson, *Geodesics and curvature in metric simplicial complexes*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 373–463. MR 1170372 (94c:57040)

[4] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038)

[5] Danny Calegari, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009. MR 2527432

[6] Richard D. Canary, *Ends of hyperbolic* 3*-manifolds*, J. Amer. Math. Soc. **6** (1993), no. 1, 1–35. MR 1166330 (93e:57019)

[7] Ruth Charney, *The Tits conjecture for locally reducible Artin groups*, Internat. J. Algebra Comput. **10** (2000), no. 6, 783–797.

[8] John Crisp and Bert Wiest, *Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups*, Algebr. Geom. Topol. **4** (2004), 439–472.

[9] Jack Edmonds, *Maximum matching and a polyhedron with* $0, 1$*-vertices*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 125–130. MR 0183532 (32 #1012)

[10] Cameron McA. Gordon, Darren D. Long, and Alan W. Reid, *Surface subgroups of Coxeter and Artin groups*, J. Pure Appl. Algebra **189** (2004), no. 1–3, 135–148. MR 2038569 (2004k:20077)

[11] Cameron McA. Gordon and Henry Wilton, *On surface subgroups of doubles of free groups*, J. London Math. Soc. (2) **82** (2010), no. 1, 17–31.

[12] Mikhail Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR 919829 (89e:20070)

[13] ———, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR 1253544 (95m:20041)

[14] Frédéric Haglund and Daniel T. Wise, *Special cube complexes*, Geom. Funct. Anal. **17** (2008), no. 5, 1551–1620. MR 2377497 (2009a:20061)
[15] Jeremy Kahn and Vladimir Markovic, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, Preprint, 10 2009.
[16] Sang-hyun Kim, *Geometricity and polygonality in free groups*, to appear at Internat. J. Algebra Comput., 2009.
[17] Sang-hyun Kim and Henry Wilton, *Polygonal words in free groups*, to appear at Q. J. of Math., 2010.
[18] Daniel Král', *Personal communication*, 2010.
[19] László Lovász and Michael D. Plummer, *Matching theory*, North-Holland Mathematics Studies, vol. 121, North-Holland Publishing Co., Amsterdam, 1986, Annals of Discrete Mathematics, 29. MR 859549 (88b:90087)
[20] Jason F. Manning, *Virtually geometric words and Whitehead's algorithm*, Math. Res. Lett. **17** (2010), no. 5, 917–925.
[21] Karl Menger, *Zur allgemeinen kurventheorie*, Fund. Math. **10** (1927), 96–115.
[22] Abdó Roig, Enric Ventura, and Pascal Weil, *On the complexity of the Whitehead minimization problem*, Internat. J. Algebra Comput. **17** (2007), no. 8, 1611–1634. MR 2378055 (2008m:20044)
[23] Peter Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) **17** (1978), no. 3, 555–565. MR 0494062 (58 #12996)
[24] Paul Seymour, *On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte*, Proc. London Math. Soc. (3) **38** (1979), no. 3, 423–460. MR 532981 (81j:05061)
[25] John R. Stallings, *Whitehead graphs on handlebodies*, Geometric group theory down under (Canberra, 1996), de Gruyter, Berlin, 1999, pp. 317–330. MR 1714852 (2001i:57028)
[26] Richard Stong, *Diskbusting elements of the free group*, Math. Res. Lett. **4** (1997), no. 2-3, 201–210. MR 1453054 (98h:20049)
[27] John H. C. Whitehead, *On certain sets of elements in a free group*, Proc. London Math. Soc. **s2–41** (1936), no. 1, 48–56.
[28] Henry Wilton, *Personal communication*, 2010.
[29] Daniel T. Wise, *Subgroup separability of graphs of free groups with cyclic edge groups*, Q. J. Math. **51** (2000), no. 1, 107–129. MR 1760573 (2001e:20026)

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