Optimal stopping time problem with discontinuous reward

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Abstract

We study, for any stopping time S, the optimal stopping time problem $v(S) = \text{ess}\sup_{\theta \geq S} E[\phi(\theta)|\mathcal{F}_S]$, where the reward is given by a family $\{\phi(\theta),\ \theta \in T_0\}$ *of positive random variables* indexed by stopping times. We solve the problem under the weakest assumptions in terms of the regularity of the reward. More precisely, the reward family $\{\phi(\theta),\ \theta \in T_0\}$ is supposed to satisfy some compatibility conditions and to be upper-semicontinuous along stopping times in expectation. We give several properties of the value function. We show the existence of an optimal stopping time. Also, we obtain a characterization of the minimal and the maximal optimal stopping times.

Key word: Optimal stopping.

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Introduction

The optimal stopping time problem has been wildely studied in case of *reward* given by a right continuous left limited (RCLL) positive adapted process (ϕ_t) defined on $[0, T]$ (see for example Shiryaev (1978), El Karoui (1981), Karatzas and Shreve (1998) or Peskir and Shiryaev (2006)). If $T > 0$ is the fixed time horizon and if T_0 denotes the set of stopping times θ smaller than T, the problem consists in computing the maximal reward given by

$$v(0) = \sup\{\, E[\phi_\tau],\ \tau \in T_0 \,\}\,,$$

in finding conditions for the existence of an optimal stopping time and giving a method to compute these optimal stopping times.

Classicaly, the *value function* at time $S \in T_0$ is defined by $v(S) = \text{ess}\sup\{\, E[\phi_\tau \,|\, \mathcal{F}_S], \tau \in T_0 \text{ and } \tau \geq S \text{ a.s.} \,\}$. The value function is given by a family of random variable $\{\, v(S), S \in T_0 \,\}$. By using the right continuity of the reward (ϕ_t), it can be shown that there exists a RCLL adapted process (v_t) which *aggregates* the family of random variable $\{\, v(S), S \in T_0 \,\}$ that is such that $v_S = v(S)$ a.s. for each $S \in T_0$. This process is the *Snell envelope* of (ϕ_t), that is the smallest supermartingale process that dominates ϕ. Moreover, when the reward (ϕ_t) is continuous, the stopping time defined trajectorially by

$$\overline{\theta}(S) = \inf\{\, t \geq S,\ v_t = \phi_t \,\}$$

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is optimal.

Recall that El Karoui (1981) has introduced the more general notion of a reward given by a *family* $\{ \phi(\theta), \theta \in T_0 \}$ *of positive random variables* which satisfies some compatibility properties. In the recent paper of Kobylanski et al. (2009), this notion appears to be the appropriate one to study the d-multiple optimal stopping time problem. Moreover, in this work, Kobylanski et al. (2009) show that under quite weak assumptions (right and left continuity in expectation along stopping times of the reward), the minimal optimal stopping time for the value function at time S

$$v(S) = \text{ess sup}\{ E[\phi(\theta) \,|\, \mathcal{F}_S],\ \theta \in T_0 \,\text{and}\, \theta \geq S \,\text{a.s.} \}\,, \tag{0.1}$$

is given by

$$\theta_*(S) := \text{ess inf}\{ \theta \in T_0,\ \theta \geq S \,\text{a.s. and}\, u(\theta) = \phi(\theta)\ \text{a.s.} \}. \tag{0.2}$$

Let us emphasize that the minimal optimal stopping time $\theta_*(S)$ is no longer defined as a hitting time of processes but as an essential infimum of random variables. Also, this result allows to deal with the optimal stopping problem only in terms of admissible families of random variables. It presents the advantage that it does no longer require aggregation results. Indeed, the existence of optimal stopping times as well as the characterization of the minimal one can be done by using only the value function family and the reward family and no longer the aggregated processes. We stress on that in the multiple case, it avoids long and heavy proofs, due to some difficult aggregation problems. It allows to solve the problem under weaker assumptions than before in the unified framework of families of random variables.

In the present work, we consider the case of a one optimal stopping time problem with a discontinuous reward. More precisely, the reward is given by a family of random variables which satisfies some compatibility conditions and which is supposed to be upper-semicontinuous over stopping times in expectation. Note that these assumptions in terms of smoothness of the reward are optimal in order to ensure the existence of an optimal stopping time. Indeed, in the deterministic case, the upper-semicontinuity is the minimal assumption on a function $\phi : [0, T] \to \mathbb{R};\, t \mapsto \phi(t)$ which ensures that the supremum of ϕ is attained on any closed subset of $[0, T]$.

Under these assumptions, we show the existence of an optimal stopping time for the value function (0.1) which is given by the essential infimum $\theta_*(S)$ defined by (0.2). We stress on that the mathematical tools which are used in this proof are not sophisticated tools, as those of the general theory of processes, but just the use of well chosen supermartingale systems and an appropriate construction of penalized stopping times. We also show that $\theta_*(S)$ is the minimal optimal stopping time. Also, the stopping time given by

$$\check{\theta}(S) = \text{ess sup}\{ \theta \in T_0,\ \theta \geq S \text{ a.s. and } E[v(\theta)] = E[v(S)] \},$$

is proven to be the maximal optimal stopping time. Note that an important tool in this work is the use of the family of random variables defined by $v^+(S) = \text{ess sup}\{ E[\phi(\theta) \,|\, \mathcal{F}_S],\, \theta \in T_0$ and $\theta > S$ a.s. for each stopping time S. Some properties and links between v, v^+ and ϕ are studied in this paper.

These new results allow to solve the case of a reward process (ϕ_t) which can be much less regular than in the previous works. For instance, this allows to solve the case of a reward process given by $\phi_t = f(X_t)$, where f is upper-semicontinuous and (X_t) is a RCLL process supposed to be left continuous along stopping times. This opens a way to a large range of applications, for instance in finance.

The paper is organised as follows. In section 1, we give some first properties on v and v^+. In particular, we have $v(S) = \phi(S) \vee v^+(S)$ a.s. for each $S \in T_0$ and the family $\{ v^+(S), S \in T_0 \}$ is right continuous along stopping times. In section 2, we show the existence of an optimal stopping time under some minimal assumptions. We begin by constructing ε-optimal stopping times which are appropriate to this case. Then, these ε-optimal stopping times are shown to tend to $\theta_*(S)$ as ε tends to 0. Moreover, $\theta_*(S)$ is proven to be an optimal stopping time for $v(S)$ and even the minimal one. At last, the stopping time $\check{\theta}(S)$ is proven to be the maximal optimal stopping time. In section 3, we give some strict supermartingale conditions on v which ensure the equality between v and ϕ (locally). Secondly, we give some conditions on v and v^+ which ensure the equality between v and v^+ (for some stopping times which are specified). At last, we give a few applications of the classical Doob-Meyer decomposition in particular when the reward is right continuous in expectation, which allows to use aggregation results. In section 4, we give some examples where the reward is given by an upper semicontinuous function of a RCLL adapted process (X_t) which is left continuous in expectation. We stress on that, except in the last part, all the properties established in this work do not require any result of the general theory of processes.

Let $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions of right continuity and augmentation by the null sets $\mathcal{F} = \mathcal{F}_T$. We suppose that $\mathcal{F}_0$ contains only sets of probability 0 or 1. The time horizon is a fixed constant $T \in]0, \infty[$. We denote by T_0 the collection of stopping times of $\mathbb{F}$ with values in $[0, T]$. More generally, for any stopping times S, we denote by T_S (resp. T_{S^+}) the class of stopping times $\theta \in T_0$ with $\theta \geq S$ a.s. (resp. $\theta > S$ a.s. on $\{S < T\}$ and $\theta = T$ a.s. on $\{S = T\}$).

We also define $T_{[S,S']}$ the set of $\theta \in T_0$ with $S \leq \theta \leq S^{'}$ a.s. and $T_{]S,S']}$ the set of $\theta \in T_0$ with $S < \theta \leq S^{'}$ a.s.. Similarly, the set $T_{]S,S']}$ on A will denote the set of $\theta \in T_0$ with $S < \theta \leq S^{'}$ a.s. on A.

We use the following notation: For $t \in \mathbb{R}$ and for real valued random variables X and X_n, $n \in \mathbb{N}$, "$X_n \uparrow X$" stands for "the sequence (X_n) is nondecreasing and converges to X a.s.".

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