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四元数矩阵的奇异 Beta 分布和 F 分布

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摘要: 应用四元数矩阵的奇异 Wishart 分布的密度函数表达式和奇异四元数矩阵奇异值分解的工具, 求得了奇异四元数矩阵变换 $\boldsymbol{X} = \boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{\mathrm{T}}$ 的 Jacobi 行列式. 利用奇异四元数矩阵的广义逆定义了四元数矩阵的奇异 Beta 分布和 F 分布, 结合奇异四元数矩阵数乘变换的 Jacobi 行列式, 给出了四元数矩阵的奇异 Beta 分布和 F 分布的密度函数表达式. 最后, 给出了满足两种分布的奇异四元数矩阵的非零特征值的联合密度函数.

关键词: 奇异四元数矩阵; Beta 分布; 特征值联合密度函数

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Generalized Beta and F distributions of quaternion matrix argument

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Abstract: This paper computed the Jacobian of transformation $\boldsymbol{X} = \boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{\mathrm{T}}$ of singular quaternion matrices by using of the singular value decomposition of quaternion matrix and the density function of singular quaternion Wishart matrix. Then we defined the Beta and F distributions of quaternion matrix argument, and gave the density functions of the Beta and F distribution and the joint density functions of the nonzero eigenvalues of the singular quaternion matrices which satisfy the Beta or F distribution.

Key words: singular quaternion matrix; Beta distribution; joint density function of eigenvalues

0 引 言

经典的多元统计分析[1,2]是建立在非奇异的实数矩阵上的. 1994年, Uhlig[3]利用奇异值分解的工具将非奇异实数矩阵扩展到奇异的情况下, 并给出了两个猜想. 1997年, Diaz–Garcia 等人[4]给出了这两个猜想的证明, 以及奇异的 Beta 分布和 F 分布的密度函数表达

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式, 还给出了最一般的正态分布和 Wishart 分布的密度函数表达式[5]. 同时, Ratnarajah, Vaillancourt 对复数矩阵的多元统计分析也取得了很多成果[6−10].

由于四元数矩阵的不可交换性, 所以对四元数随机矩阵的研究较复杂. Anderson[11] 首次建立了四元数矩阵的多元统计模型. 滕成业等人对四元数矩阵的多元统计分析有一系列的成果, 但是他们取得四元数矩阵的带状多项式的定义是有错误的[12]. 文献[13]重新定义了四元数矩阵的带状多项式, 并由此给出了四元数矩阵的相关分布的密度函数表达式. 本文在上述基础上, 给出推广的四元数矩阵的奇异 Beta 分布和 F 分布的密度函数表达式. 即定义中涉及的两个四元数 Wishart 矩阵都可以推广到奇异的矩阵.

1 引 理

记$\mathbb{H}$为四元数体, $\mathbb{H}^{n\times m}$ 为 $\mathbb{H}$ 上 $n\times m$ 矩阵全体. 对$\boldsymbol{A}\in\mathbb{H}^{n\times m}$, 以$\boldsymbol{A}^{\mathrm{T}}$表示$\boldsymbol{A}$的共轭转置. 记${}_qV_{n,m}=\{\boldsymbol{A}\in\mathbb{H}^{n\times m}|\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}=I_m\}$. 本文中其他有关记号的意义见文献[13-14]. 下面的引理取自文献*的Theorem 4.2.

引理 1.1 设四元数矩阵$\boldsymbol{X}\sim\mathbb{H}N_{N\times m}(\mu,\Sigma,\Theta)$, $\Sigma\in\mathbb{H}^{m\times m}, r(\Sigma)=r<m, \Theta\in\mathbf{R}^{N\times N}, r(\Theta)=k<N$. 称这个分布为四元数奇异正态分布, 表示为$\boldsymbol{X}\sim\mathbb{H}N^{k,r}_{N\times m}(\mu,\Sigma,\Theta)$, 当$r=m, k=N$时, 省略上标. 设$\boldsymbol{Y}\sim\mathbb{H}N^{k,r}_{N\times m}(\mu,\Sigma,\Theta)$, 设$q=\min(r,k)$. 则称$\boldsymbol{S}=\boldsymbol{Y}^{\mathrm{T}}\Theta^-\boldsymbol{Y}$ 满足四元数矩阵奇异 Wishart 分布. 记作$\boldsymbol{S}\sim\mathbb{H}W_m(q,k,\Sigma,\Omega)$, 这里 $q=\mathrm{rank}(\boldsymbol{S}), \Omega=\Sigma^-\mu^T\Theta^-\mu$. $\boldsymbol{S}$的密度函数为

$$\frac{2^{2kr}\pi^{2kq-2kr}}{\mathbb{H}\Gamma_q(2k)}\exp(\mathrm{Retr}(-2\Sigma^-\boldsymbol{S}-2\Omega))\left(\prod_{i=1}^r\lambda_i\right)^{-2k}\left(\prod_{i=1}^q\mu_i\right)^{2k-2m+1}_0F_1(2k,4\Omega\Sigma^-\boldsymbol{S})$$

$$\boldsymbol{P}_2\boldsymbol{S}\boldsymbol{P}_2^{\mathrm{T}}=\boldsymbol{P}_2\mu^{\mathrm{T}}\Theta^-\mu\boldsymbol{P}_2^{\mathrm{T}},\quad (\mathrm{a.s})$$

其中 $\mu_1,\cdots,\mu_q$ 为 $\boldsymbol{S}$ 的非零特征值; $\boldsymbol{P}_1\Sigma\boldsymbol{P}_1^{\mathrm{T}}=\mathrm{diag}(\lambda_1,\cdots,\lambda_r)$, $(\boldsymbol{P}_1|\boldsymbol{P}_2)$ 为广义酉阵. ${}_0F_1(\cdot)$是四元数矩阵的超几何函数[13]. 用$\boldsymbol{A}^-$表示矩阵$\boldsymbol{A}$的广义逆. Retr $(\cdot)$表示迹$\mathrm{tr}(\cdot)$的实部.

引理 1.2 若四元数矩阵 $\boldsymbol{X},\boldsymbol{Y}$有下列关系 $\boldsymbol{X}=\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{\mathrm{T}}$,其中$\boldsymbol{B}$是$m\times q$阶秩为$q$的四元数矩阵. $\boldsymbol{X}$是$m\times m$阶的秩为n的半正定四元数矩阵, $\boldsymbol{Y}$是$q\times q$阶的秩为n的半正定四元数矩阵. 我们有$\boldsymbol{X}=\boldsymbol{G}_1\Delta_{\boldsymbol{X}}\boldsymbol{G}_1^{\mathrm{T}}$, $\boldsymbol{Y}=\boldsymbol{H}_1\Delta_{\boldsymbol{Y}}\boldsymbol{H}_1^{\mathrm{T}}$,其中$\boldsymbol{G}_1\in{}_qV_{n,m}$, $\boldsymbol{H}_1\in{}_qV_{n,q}$. $\Delta_{\boldsymbol{X}},\Delta_{\boldsymbol{Y}}$是$n\times n$阶对角矩阵, $\Delta_{\boldsymbol{X}}=\mathrm{diag}(x_1,\cdots,x_n)$和$\Delta_{\boldsymbol{Y}}=\mathrm{diag}(y_1,\cdots,y_n)$,且$x_1>\cdots>x_n>0$, $y_1>\cdots>y_n>0$.则

$$(d\boldsymbol{X})=|\Delta_{\boldsymbol{B}}|^{4n}|\Delta_{\boldsymbol{X}}|^{2m-2n-1}|\Delta_{\boldsymbol{Y}}|^{-2q+2n+1}(d\boldsymbol{Y}).$$

证明 假定$\boldsymbol{Y}\sim\mathbb{H}W_q^n(n,\boldsymbol{I})$, 则$\boldsymbol{X}=\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{\mathrm{T}}\sim\mathbb{H}W_m^n(n,\Xi), \Xi=\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}$. 由于rank $\boldsymbol{B}=q$, 故$\boldsymbol{B}^-\boldsymbol{B}=\boldsymbol{I}_q$. 由引理1.1, 有

$$\begin{aligned}
g_{\boldsymbol{Y}}(\boldsymbol{Y}) &= \frac{2^{2qn}\pi^{2n^2-2nq}|\Delta_{\boldsymbol{Y}}|^{2n-2q+1}}{\mathbb{H}\Gamma_n(2n)}\exp(\mathrm{Retr}(-2\boldsymbol{Y})),\\
f_{\boldsymbol{X}}(\boldsymbol{X}) &= \frac{2^{2qn}\pi^{2n^2-2qn}|\Delta_{\boldsymbol{X}}|^{2n-2m+1}}{\mathbb{H}\Gamma_n(2n)|\Delta_{\boldsymbol{B}}^2|^{2n}}\exp(\mathrm{Retr}(-2(\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}})^-\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{\mathrm{T}})\\
&= \frac{2^{2qn}\pi^{2n^2-2qn}|\Delta_{\boldsymbol{X}}|^{2n-2m+1}}{\mathbb{H}\Gamma_n(2n)|\Delta_{\boldsymbol{B}}^2|^{2n}}\exp(\mathrm{Retr}(-2(\boldsymbol{B}^-)^{\mathrm{T}}\boldsymbol{B}^-\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{\mathrm{T}}))\\
&= \frac{2^{2qn}\pi^{2n^2-2qn}|\Delta_{\boldsymbol{X}}|^{2n-2m+1}}{\mathbb{H}\Gamma_n(2n)|\Delta_{\boldsymbol{B}}|^{4n}}\exp(\mathrm{Retr}(-2\boldsymbol{Y})).
\end{aligned}$$

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*LI F, XUE YF. The density functions of the singular quaternion normal matrix and the singular quaternion wishart matrix[J]. Communications in Statistics Theory and Methods. (已接受发表)

由上述等式, 可以得到结论.

引理 1.3　若$\boldsymbol{X}\sim\mathbb{H}W_m^{q_1}(k_1,\Sigma,\Omega_1)$, $\boldsymbol{Y}\sim\mathbb{H}W_m^{q_2}(k_2,\Sigma,\Omega_2)$, 且$\boldsymbol{X}$与$\boldsymbol{Y}$独立. 则

$$\boldsymbol{X}+\boldsymbol{Y}\sim\mathbb{H}W_m^{q_3}(k_1+k_2,\Sigma,\Omega_3),$$

其中$\Omega_3=\Omega_1+\Omega_2$.

证明　由$\boldsymbol{X}\sim\mathbb{H}W_m^{q_1}(k_1,\Sigma,\Omega_1)$, $\boldsymbol{Y}\sim\mathbb{H}W_m^{q_2}(k_2,\Sigma,\Omega_2)$, 其中$\Sigma=\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$, $\Theta_1=\boldsymbol{B}_1\boldsymbol{B}_1^{\mathrm{T}}$, $\Theta_2=\boldsymbol{B}_2\boldsymbol{B}_2^{\mathrm{T}}$, $\mathrm{rank}(\Sigma)=r,\mathrm{rank}(\Theta_1)=k_1,\mathrm{rank}(\Theta_2)=k_2$, $\mathrm{rank}(\boldsymbol{B}_1)_{n\times k_1}=k_1,\mathrm{rank}(\boldsymbol{B}_2)_{n\times k_2}=k_2$. 则可以将$\boldsymbol{X}$, $\boldsymbol{Y}$写作

$$\begin{aligned}\boldsymbol{X}&=\boldsymbol{Z}_1^{\mathrm{T}}\Theta_1^{-}\boldsymbol{Z}_1,\quad \boldsymbol{Z}_1\sim\mathbb{H}N_{n\times m}^{k_1,r}(\mu_1,\Sigma,\Theta_1);\\ \boldsymbol{Y}&=\boldsymbol{Z}_2^{\mathrm{T}}\Theta_2^{-}\boldsymbol{Z}_2,\quad \boldsymbol{Z}_2\sim\mathbb{H}N_{n\times m}^{k_2,r}(\mu_2,\Sigma,\Theta_2).\end{aligned}$$

则

$$\begin{aligned}\boldsymbol{X}=&\quad\boldsymbol{Z}_1^{\mathrm{T}}\Theta_1^{-}\boldsymbol{Z}_1=\boldsymbol{Z}_1^{\mathrm{T}}\boldsymbol{B}_1^{-\boldsymbol{H}}\boldsymbol{B}_1^{-}\boldsymbol{Z}_1=\boldsymbol{W}_1^{\mathrm{T}}\boldsymbol{W}_1,\quad \boldsymbol{W}_1\sim N_{k_1\times m}^{k_1,r}(\boldsymbol{B}_1^{-}\mu_1,\Sigma,\boldsymbol{I}_{k_1});\\ \boldsymbol{Y}=&\quad\boldsymbol{Z}_2^{\mathrm{T}}\Theta_2^{-}\boldsymbol{Z}_2=\boldsymbol{Z}_2^{\mathrm{T}}\boldsymbol{B}_2^{-\boldsymbol{H}}\boldsymbol{B}_2^{-}\boldsymbol{Z}_2=\boldsymbol{W}_2^{\mathrm{T}}\boldsymbol{W}_2,\quad \boldsymbol{W}_2\sim N_{k_2\times m}^{k_2,r}(\boldsymbol{B}_2^{-}\mu_2,\Sigma,\boldsymbol{I}_{k_2}).\end{aligned}$$

由

$$\begin{pmatrix}\boldsymbol{W}_1\\ \boldsymbol{W}_2\end{pmatrix}\sim\mathbb{H}N_{(k_1+k_2)\times m}^{k_1+k_2,r}(\mu_3,\Sigma,\boldsymbol{I}_{k_1+k_2}),$$

$$\mu_3=\begin{pmatrix}\boldsymbol{B}_1^{-}\mu_1\\ \boldsymbol{B}_2^{-}\mu_2\end{pmatrix},\Omega_3=\Sigma^{-1}\begin{pmatrix}\boldsymbol{B}_1^{-}\mu_1\\ \boldsymbol{B}_2^{-}\mu_2\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{B}_1^{-}\mu_1\\ \boldsymbol{B}_2^{-}\mu_2\end{pmatrix}=\Omega_1+\Omega_2.$$

我们有

$$\boldsymbol{X}+\boldsymbol{Y}=(\boldsymbol{W}_1^{\mathrm{T}}|\,\boldsymbol{W}_2^{\mathrm{T}})\begin{pmatrix}\boldsymbol{W}_1\\ \boldsymbol{W}_2\end{pmatrix}\sim\mathbb{H}W_m^{q_3}(k_1+k_2,\Sigma,\Omega_3).$$

2　定理及推论

定理 2.1　令$\boldsymbol{A}\sim\mathbb{H}W_m^{q_1}(k_1,\Sigma)$, $\boldsymbol{B}\sim\mathbb{H}W_m^{q_2}(k_2,\Sigma)$, $\boldsymbol{A}$ 和$\boldsymbol{B}$是相互独立的. 设$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{T}$, 其中$\boldsymbol{T}$是$q_3\times m$阶秩为$q_3$的四元数矩阵. 令$\boldsymbol{U}$是自共轭的四元数矩阵, 定义为

$$\boldsymbol{A}=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{T}.$$

则$\boldsymbol{U}$和$\boldsymbol{A}+\boldsymbol{B}$相互独立, $\boldsymbol{A}+\boldsymbol{B}\sim\mathbb{H}W_m^{q_3}(k_1+k_2,\Sigma)$, $q_1,q_2\leqslant q_3$. $\boldsymbol{U}$的密度函数表达式如下,

$$\frac{\pi^{2k_1q_1+2k_2q_2-2k_1q_3-2k_2q_3}\mathbb{H}\Gamma_{q_3}(2k_1+2k_2)}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)}|\Delta_{\boldsymbol{U}}|^{2k_1-2q_3+1}|\boldsymbol{I}-\boldsymbol{U}|^{2k_2-2m+1},$$

其中$\boldsymbol{U}=\boldsymbol{H}_1\Delta_{\boldsymbol{U}}\boldsymbol{H}_1^{\mathrm{T}}$, $\boldsymbol{H}_1\in{}_qV_{q_1,q_3}$, $\Delta_{\boldsymbol{U}}=\mathrm{diag}(\lambda_1,\cdots,\lambda_{q_1})$, $1>\lambda_1>\cdots>\lambda_{q_1}>0$,

$$(\mathrm{d}\boldsymbol{U})=(2\pi^2)^{-q_1}\prod_{i=1}^{q_1}\lambda_i^{4q_3-4q_1}\prod_{j<i}^{q_1}(\lambda_j-\lambda_i)^4\bigwedge_{i=1}^{q_1}d\lambda_i\wedge(\boldsymbol{H}_1^{\mathrm{T}}d\boldsymbol{H}_1),$$

记作$\boldsymbol{U}\sim\mathbb{H}B_{q_3}(2k_1,2k_2)$.

证明 由于$\boldsymbol{A}\sim \mathbb{H}W_m^{q_1}(k_1,\Sigma)$, $\boldsymbol{B}\sim \boldsymbol{H}W_m^{q_2}(k_2,\Sigma)$, 则有

$$f_{\boldsymbol{A}}(\boldsymbol{A})=\frac{2^{2k_1r}\pi^{2k_1q_1-2k_1r}}{\mathbb{H}\Gamma_{q_1}(2k_1)|\Delta_\Sigma|^{2k_1}}\exp(\mathrm{Retr}(-2\Sigma^-\boldsymbol{A}))|\Delta_{\boldsymbol{A}}|^{2k_1-2m+1},$$
$$f_{\boldsymbol{B}}(\boldsymbol{B})=\frac{2^{2k_2r}\pi^{2k_2q_2-2k_2r}}{\mathbb{H}\Gamma_{q_2}(2k_2)|\Delta_\Sigma|^{2k_2}}\exp(\mathrm{Retr}(-2\Sigma^-\boldsymbol{B}))|\Delta_{\boldsymbol{B}}|^{2k_2-2m+1}.$$

则它们的联合密度函数为

$$\begin{aligned}&\frac{2^{2r(k_1+k_2)}\pi^{2k_1q_1+2k_2q_2-2k_1r-2k_2r}}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)|\Delta_\Sigma|^{2k_1+2k_2}}\exp(\mathrm{Retr}(-2\Sigma^-(\boldsymbol{A}+\boldsymbol{B})))|\Delta_{\boldsymbol{A}}|^{2k_1-2m+1}\\
&\quad\times|\Delta_{\boldsymbol{B}}|^{2k_2-2m+1}(\mathrm{d}\boldsymbol{A})(\mathrm{d}\boldsymbol{B})\\
&\quad=\frac{2^{2r(k_1+k_2)}\pi^{2k_1q_1+2k_2q_2-2k_1r-2k_2r}}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)|\Delta_\Sigma|^{2k_1+2k_2}}\exp(\mathrm{Retr}(-2\Sigma^-(\boldsymbol{A}+\boldsymbol{B})))|\Delta_{\boldsymbol{A}}|^{2k_1-2m+1}\\
&\quad\times|\Delta_{\boldsymbol{B}}|^{2k_2-2m+1}(\mathrm{d}\boldsymbol{A})(\mathrm{d}(\boldsymbol{A}+\boldsymbol{B}))\\
&\quad=\frac{2^{2r(k_1+k_2)}\pi^{2k_1q_1+2k_2q_2-2k_1r-2k_2r}}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)|\Delta_\Sigma|^{2k_1+2k_2}}\exp(\mathrm{Retr}(-2\Sigma^-(\boldsymbol{A}+\boldsymbol{B})))|\Delta_{\boldsymbol{U}}|^{2k_1-2q_3+1}\\
&\quad\times|\Delta_{\boldsymbol{T}}|^{4k_1+4k_2+2-4m}|\boldsymbol{I}-\boldsymbol{U}|^{2k_2-2m+1}(\mathrm{d}\boldsymbol{U})(\mathrm{d}(\boldsymbol{A}+\boldsymbol{B})).\end{aligned}$$

对$(\mathrm{d}(\boldsymbol{A}+\boldsymbol{B}))$积分得$\boldsymbol{U}$的密度函数如下,

$$\frac{\pi^{2k_1q_1+2k_2q_2-2k_1q_3-2k_2q_3}\mathbb{H}\Gamma_{q_3}(2k_1+2k_2)}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)}|\Delta_{\boldsymbol{U}}|^{2k_1-2q_3+1}|\boldsymbol{I}-\boldsymbol{U}|^{2k_2-2m+1}.$$

注意到$\boldsymbol{A}+\boldsymbol{B}\sim\mathbb{H}W_m(n+p,\Sigma)$(引理1.3)和

$$(\mathrm{d}\boldsymbol{A})(\mathrm{d}\boldsymbol{U})=|\Delta_{\boldsymbol{U}}|^{2q_1-2q_3+1}|\Delta_{\boldsymbol{A}}|^{2m-2q_1-1}|\Delta_{\boldsymbol{T}}|^{4q_1}(\mathrm{d}\boldsymbol{U})(\mathrm{d}(\boldsymbol{A}+\boldsymbol{B}))\qquad(由引理1.2),$$

我们可以得到结论.

定理 2.2 令$\boldsymbol{A}$和$\boldsymbol{B}$是相互独立的, $\boldsymbol{A}\sim\mathbb{H}W_{q_2}^{q_1}(k_1,\boldsymbol{I}_{q_2}),\boldsymbol{B}\sim\mathbb{H}W_m^{q_2}(k_2,\boldsymbol{I}_m^{q_2})$, $\boldsymbol{I}_m^{q_2}=\mathrm{diag}(1,\cdots,1,0,\cdots,0)$, $\mathrm{rank}(\boldsymbol{I}_m^{q_2})=q_2$. 令$\boldsymbol{B}=\boldsymbol{T}^{\mathrm{T}}\boldsymbol{T}$, 其中$\boldsymbol{T}$是$q_2\times m$阶秩为$q_2$的四元数矩阵. 若$\boldsymbol{F}$是$m\times m$阶的自共轭的四元数矩阵,定义为$\boldsymbol{F}=\boldsymbol{T}^-\boldsymbol{A}(\boldsymbol{T}^-)^{\mathrm{T}}$. 则$\boldsymbol{F}$密度函数表达式如下.

$$\frac{\pi^{2k_1q_1-2k_1q_2}\mathbb{H}\Gamma_{q_2}(2k_1+2k_2)}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)}|\boldsymbol{I}+\boldsymbol{F}|^{-2k_1-2k_2}|\Delta_{\boldsymbol{F}}|^{2k_1+1-2m}.$$

其中$\boldsymbol{F}=\boldsymbol{H}_1\Delta_{\boldsymbol{F}}\boldsymbol{H}_1^{\mathrm{T}}$, $\boldsymbol{H}_1\in{}_qV_{q_1,m}$, $\Delta_{\boldsymbol{F}}=\mathrm{diag}(f_1,\cdots,f_{q_1}), f_1>f_2>\cdots>f_{q_1}>0$.

$$(\mathrm{d}\boldsymbol{F})=(2\pi^2)^{-q_1}\prod_{i=1}^{q_1}f_i^{4m-4q_1}\prod_{j<i}^{q_1}(f_j-f_i)^4\bigwedge_{i=1}^{q_1}\mathrm{d}f_i\wedge(\boldsymbol{H}_1^{\mathrm{T}}\mathrm{d}\boldsymbol{H}_1),$$

记作$\boldsymbol{F}\sim\mathbb{H}F(2k_1,2k_2)$.

证明 由引理1.1, 有

$$f_{\boldsymbol{A}}(\boldsymbol{A})=\frac{2^{2k_1q_2}\pi^{2k_1q_1-2k_1q_2}}{\mathbb{H}\Gamma_{q_1}(2k_1)}\exp(\mathrm{Retr}(-2\boldsymbol{A}))|\Delta_{\boldsymbol{A}}|^{2k_1-2q_2+1},$$
$$f_{\boldsymbol{B}}(\boldsymbol{B})=\frac{2^{2k_2q_2}}{\mathbb{H}\Gamma_{q_2}(2k_2)}\exp(\mathrm{Retr}(-2\boldsymbol{B}))|\Delta_{\boldsymbol{B}}|^{2k_2-2m+1}.$$

得到$\boldsymbol{A},\boldsymbol{B}$的联合密度函数表达式如下,

$$\begin{aligned}
&\frac{2^{2q_2(k_1+k_2)}\pi^{2k_1(q_1-q_2)}}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)}\exp(\mathrm{Retr}(-2\boldsymbol{A}))\exp(\mathrm{Retr}(-2\boldsymbol{B}))|\Delta_{\boldsymbol{A}}|^{2k_1-2q_2+1}\\
&\quad\times|\Delta_{\boldsymbol{B}}|^{2k_2-2m+1}(\mathrm{d}\boldsymbol{A})(\mathrm{d}\boldsymbol{B})\\
&=\frac{2^{2q_2(k_1+k_2)}\pi^{2k_1(q_1-q_2)}}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)}\exp(\mathrm{Retr}(-2(\boldsymbol{I}+\boldsymbol{F})\boldsymbol{B}))|\Delta_{\boldsymbol{F}}|^{2k_1+1-2m}\\
&\quad\times|\Delta_{\boldsymbol{B}}|^{2k_1+2k_2-2m+1}(\mathrm{d}\boldsymbol{F})(\mathrm{d}\boldsymbol{B})\\
&=\frac{2^{2q_2(k_1+k_2)}\pi^{2k_1(q_1-q_2)}}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)}\exp(\mathrm{Retr}(-2(\boldsymbol{I}+\boldsymbol{F})\boldsymbol{B}))|\Delta_{\boldsymbol{F}}|^{2k_1+1-2m}\\
&\quad\times|\Delta_{\boldsymbol{B}}|^{2k_1+2k_2-2m+1}|\boldsymbol{I}+\boldsymbol{F}|^{1-2m}(\mathrm{d}((\boldsymbol{I}+\boldsymbol{F})^{1/2}\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{F})^{1/2}))(\mathrm{d}\boldsymbol{F}).
\end{aligned}$$

注意到$|\Delta_{\boldsymbol{A}}|=|\Delta_{\boldsymbol{F}}||\Delta_{\boldsymbol{B}}|$和

$$(\mathrm{d}\boldsymbol{F})(\mathrm{d}\boldsymbol{B})=|\Delta_{\boldsymbol{T}}|^{-4q_1}|\Delta_{\boldsymbol{F}}|^{-2q_1-1+2m}|\Delta_{\boldsymbol{A}}|^{-2q_2+2q_1+1}(\mathrm{d}\boldsymbol{A})(\mathrm{d}\boldsymbol{B}),$$

对上式关于$\mathrm{d}((\boldsymbol{I}+\boldsymbol{F})^{1/2}\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{F})^{1/2})$ 积分可得$\boldsymbol{F}$的密度函数如下,

$$\frac{\pi^{2k_1q_1-2k_1q_2}\mathbb{H}\Gamma_{q_2}(2k_1+2k_2)}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)}|\boldsymbol{I}+\boldsymbol{F}|^{-2k_1-2k_2}|\Delta_{\boldsymbol{F}}|^{2k_1+1-2m}.$$

定理 2.3 设$\boldsymbol{U}$, $\boldsymbol{F}$定义如定理 2.1, 定理 2.2 所示. 我们有 $\boldsymbol{U}$ 和 $\boldsymbol{F}$ 的非零特征值的联合密度函数如下:

(1) 假定$\boldsymbol{U}\sim\mathbb{H}B_{q_3}(2k_1,2k_2)$, 则

$$\begin{aligned}
f(\lambda_1,\cdots,\lambda_{q_1})=&\frac{\pi^{2k_1q_1+2k_2q_2-2k_1q_3-2k_2q_3+2q_1q_3-2q_1}\mathbb{H}\Gamma_{q_3}(2k_1+2k_2)}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)\mathbb{H}\Gamma_{q_1}(2q_3)}\\
&\times\prod_{i=1}^{q_1}|1-\lambda_i|^{2k_2-2m+1}\prod_{i=1}^{q_1}\lambda_i^{2q_3-4q_1+2k_1+1}\prod_{j<i}^{q_1}(\lambda_j-\lambda_i)^4;
\end{aligned}$$

(2) 假定$\boldsymbol{F}\sim\mathbb{H}F(2k_1,2k_2)$, 则

$$\begin{aligned}
f(f_1,\cdots,f_{q_1})=&\frac{\pi^{2k_1q_1-2k_1q_2+2q_1m-2q_1}\mathbb{H}\Gamma_{q_2}(2k_1+2k_2)}{\mathbb{H}\Gamma_{q_1}(2k_1)\mathbb{H}\Gamma_{q_2}(2k_2)\mathbb{H}\Gamma_{q_1}(2m)}\\
&\times\prod_{i=1}^{q_1}|1+f_i|^{-2k_1-2k_2}\prod_{i=1}^{q_1}f_i^{2m-4q_1+1+2k_1}\prod_{j<i}^{q_1}(f_j-f_i)^4,
\end{aligned}$$

其中$\lambda_i,f_i,(i=1,\cdots,q_1)$分别是$\boldsymbol{U}$和$\boldsymbol{F}$的非零特征值.

证明 注意到$|\boldsymbol{I}_{q_3}-\boldsymbol{U}|=|\boldsymbol{I}_{q_1}-\Delta_{\boldsymbol{U}}|$, 且由文献[14]定理 2, 可得到结论. 类似地, 可以得到该定理的第二部分.

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