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A note on the lévy constant for continued fractions

Ting Zhong

Abstract

In this note, we study the lévy constant of continued fraction expansions. We show that for all $x \in [0,1)$, the upper lévy constant of x is finite except a set with Hausdorff dimension one-half.

Key Words: Continued fractions, lévy constant, Hausdorff dimension.

1. Introduction

It is well known that every irrational number $x \in [0,1)$ has a unique standard continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} =: [a_1, a_2, a_3, \cdots],$$

where each partial quotient $a_n(x) \in \mathbb{N}$ is uniquely defined by the number x.

For any $n \geq 1$ and $a_1, \cdots, a_n \in \mathbb{N}$, define a *CF-interval* of rank n as

$$I(a_1, a_2, \cdots, a_n) = \{x \in [0,1) : a_k(x) = a_k, 1 \leq k \leq n\}.$$

Therefore, (see [5], section 12), $I(a_1, \cdots, a_n)$ is the interval with endpoints $\frac{p_n}{q_n}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$, where p_n and q_n are defined by following recurrence relations

$$\begin{aligned} &p_{-1} = 1;\ p_0 = 0;\ p_n = a_n p_{n-1} + p_{n-2}, \quad n \geq 1. \\ &q_{-1} = 0;\ q_0 = 1;\ q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 1. \end{aligned} \tag{1}$$

Thus, the length of $I(a_1, a_2, \cdots, a_n)$ is

$$|I(a_1, a_2, \cdots, a_n)| = \left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{1}{q_n(q_n + q_{n-1})}. \tag{2}$$

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For an irrational number $x \in [0,1)$, we call

$$\beta^*(x) = \limsup_{n\to\infty} \frac{\log q_n(x)}{n} \quad \text{and} \quad \beta_*(x) = \liminf_{n\to\infty} \frac{\log q_n(x)}{n},$$

the upper lévy constant and lower lévy constant of x, respectively. If $\beta^*(x) = \beta_*(x)$, we say the lévy constant of x exists and denote the common value by $\beta(x)$. A famous result of P. Lévy [6] says that for almost all x, the lévy constant exists and

$$\beta(x) = \frac{\pi^2}{12\log 2} \approx 1.18657.$$

$\beta^*(x)$ and $\beta_*(x)$ describe the exponential growth rates of $q_n(x)$ in n. Faiver [2] showed that every quadratic number has a lévy constant. It is easy to see that for any irrational number $x \in [0,1)$, one has $\beta_*(x) \geq \log\frac{\sqrt{5}+1}{2}$, then Faiver [3] also established that for all $\lambda \geq \log\frac{\sqrt{5}+1}{2}$, there exists an $x \in I$ such that $\beta(x) = \lambda$ by employing an ergodic theorem. Later, Baxa [1] showed the following more general result by elementary means.

Theorem 1.1 *For any* $\log\frac{\sqrt{5}+1}{2} \leq \lambda_* \leq \lambda^* < \infty$*, there exist uncountably many* $x \in [0,1)$ *such that* $\beta_*(x) = \lambda_*$ *and* $\beta^*(x) = \lambda^*$*.*

In 2006, Wu [7] improved Baxa's result by showing the following theorem.

Theorem 1.2 *For any* $\log\frac{\sqrt{5}+1}{2} \leq \lambda_* \leq \lambda^* < \infty$*, let*

$$E(\lambda_*, \lambda^*) = \{x \in [0,1) : \beta_*(x) = \lambda_*,\ \beta^*(x) = \lambda^*\}.$$

Then

$$\dim_H E(\lambda_*, \lambda^*) \geq \frac{\lambda_* - \log\frac{\sqrt{5}+1}{2}}{\lambda^*}.$$

In this note, we consider the set of $x \in [0,1)$ whose upper lévy constant is infinite and obtain

Theorem 1.3 *Let*

$$E^\infty = \Big\{x \in [0,1) : \limsup_{n\to\infty} \frac{\log q_n(x)}{n} = \infty\Big\}.$$

Then

$$\dim_H E^\infty = \frac{1}{2}.$$

Here and in what follows, $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension of a subset of $[0,1)$, and $|\cdot|$ denotes the diameter. We sketch, very briefly, the definition and some basic properties of Hasdorff dimension. If $E \subset R$ and $\delta > 0$, define for each $s \geq 0$,

$$H^s(E) = \liminf_{\delta\to 0} \Big\{ \sum_{n=1}^{\infty} |I_n|^s : E \subset \bigcup_{n=1}^{\infty} I_n, |I_n| \leq \delta, n = 1, 2, \cdots \Big\},$$

$$\dim_H E = \inf\big\{s \geq 0 : H^s(E) = 0\big\} = \sup\big\{s \geq 0 : H^s(E) = \infty\big\}.$$

The following two facts are basic in calculating Hasdorff dimension of various sets.

Lemma 1.4 *Let* $E \subset R$ *and let* $s \geq 0$ *be given. Suppose for each* $\delta > 0$ *there is a sequence of intervals* $\{I_n\}$ *such as* $E \subset \bigcup I_n$, $|I_n| \leq \delta$ *for all* n, *and* $\sum_{n=1}^{\infty} |I_n|^s \leq 1$. *Then* $\dim_H E \leq s$.

Lemma 1.5 *Let* $E \subset R$ *be a Borel set and* μ *be a measure with* $\mu(E) > 0$. *If for any* $x \in E$

$$\liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s.$$

where $B(x,r)$ *denotes the open ball with center at* x *and radius* r. *Then* $\dim_H E \geq s$.

Lemma 1 is obvious; for Lemma 2, see ([4], Proposition 2.3).

2. Proof of Theorem 1.3

In this section, we show Theorem 1.3 in detail and divide the proof into two parts: upper bound and lower bound.

I. **Upper bound**. $\dim_{\mathrm{H}} E \leq \frac{1}{2}$.

Proof. By (1), we have

$$a_n q_{n-1} \leq q_n \leq 2 a_n q_{n-1}.$$

Successive application of this inequality gives

$$a_1 a_2 \cdots a_n \leq q_n \leq 2^n a_1 a_2 \cdots a_n. \tag{3}$$

Thus we get the following alternative description of E^{∞}:

$$E^{\infty} = \Big\{ x \in [0,1) : \limsup_{n \to \infty} \frac{\log a_1(x) + \log a_2(x) + \cdots + \log a_n(x)}{n} = \infty \Big\}.$$

Let

$$E^{(m)} = \Big\{ x \in [0,1) : \limsup_{n \to \infty} \frac{\log a_1(x) + \log a_2(x) + \cdots + \log a_n(x)}{n} > m \Big\}.$$

Then E^{∞} can be written

$$E^{\infty} = \bigcap_{m=1}^{\infty} E^{(m)} = \lim_{n \to \infty} E^{(m)},$$

and for every $x = [a_1, a_2, a_3, \cdots] \in E^{(m)}$, there exist infinitely many positive integers n_i such that

$$\frac{\log a_1(x) + \cdots + \log a_{n_i}(x)}{n_i} > m, \ \ i = 1, 2, 3, \cdots$$

So that, for any $\delta > 0$, the family of the *CF-intervals*

$$\mathcal{A}(m, \delta) = \Big\{ I(a_1, a_2, \cdots, a_{n_i}) : \frac{\log a_1(x) + \cdots + \log a_{n_i}(x)}{n_i} > m, |I(a_1, a_2, \cdots, a_{n_i})| \leq \delta, n_i \in \mathbb{N} \Big\}$$

is a $\delta-$cover of $E^{(m)}$.

Note that, for any two *CF-intervals*, say I and I', the following relation holds:

$I \cap I' \neq \phi \Longrightarrow I \subseteq I'$ or I' ⊆ I.

In fact, if $I = I(a_1, \cdots, a_n)$ and $I' = I(a'_1, \cdots, a'_{n+k})$ $(k \geq 0)$ have a common point $x = [x_1, x_2, \cdots]$, then $a_1 = x_1 = a'_1, a_2 = x_2 = a'_2, \cdots, a_n = x_n = a'_n$. It follows that $I' \subseteq I$.

We remove from $\mathcal{A}(m, \delta)$ all those the *CF-intervals* which are contained in other *CF-interval* in $\mathcal{A}(m, \delta)$, and denote the complement by $A(m, \delta)$. Then, $A(m, \delta)$ is a non-overlapping $\delta-$cover of $E^{(m)}$.

Now we define a family of measures $\{\mu_t : t > 1\}$ as

$$\mu_t(I(a_1, a_2, \cdots, a_n)) = e^{-np(t) - t\log \sum\limits_{i=1}^{n} \log a_i} \tag{4}$$

where $p(t) = \log \zeta(t) = \log \sum\limits_{n \geq 1} \frac{1}{n^t}, (t > 1)$.

By (2) and (3), we have for any $\epsilon > 0$,

$$\log |I(a_1, a_2, \cdots, a_n)|^{\frac{\epsilon+t}{2}} \leq -(\epsilon + t)\log(a_1 a_2 \cdots a_n) = -\epsilon \sum_{i=1}^{n} \log a_i - t \sum_{i=1}^{n} \log a_i. \tag{5}$$

By the definition of $A(m, \delta)$, for every $I = I(a_1, a_2, \cdots, a_n) \in A(m, \delta)$ with $m \geq \frac{p(t)}{\epsilon}$, we have

$$-\epsilon \sum_{i=1}^{n} \log a_i \leq -\epsilon \cdot mn \leq -np(t). \tag{6}$$

Combining (4), (5) and (6), we get for any $\epsilon > 0$, and $I = I(a_1, a_2, \cdots, a_n) \in A(m, \delta)$ with $m \geq \frac{p(t)}{\epsilon}$,

$$|I(a_1, a_2, \cdots, a_n)|^{\frac{\epsilon+t}{2}} \leq e^{-np(t) - t\log \sum\limits_{i=1}^{n} \log a_i} = \mu_t(I(a_1, a_2, \cdots, a_n)).$$

Since $A(m, \delta)$ is a non-overlapping $\delta-$cover of $E^{(m)}$, we sum the above inequality to have

$$\sum_{I \in A(m,\delta)} |I|^{\frac{t+\epsilon}{2}} \leq \sum_{I \in A(m,\delta)} \mu_t(I) = \mu_t\Big(\bigcup_{I \in A(m,\delta)} I\Big) \leq 1.$$

By Lemma 1.4, we get for any $t > 1, \epsilon > 0$ and $m \geq \frac{p(t)}{\epsilon}$,

$$\dim_H E^{(m)} \leq \frac{t + \epsilon}{2}.$$

Letting $\epsilon \to 0$ and since $t > 1$ is arbitrary, we obtain

$$\dim_H E^{\infty} \leq \frac{1}{2}.$$

□

II. **Lower bound**. $\dim_H E^\infty \geq \frac{1}{2}$.

Proof. Put

$$F = \Big\{x \in [0,1) : 2^n \leq a_n(x) < 2^{n+1},\ \text{for all } n \geq 1\Big\}. \tag{7}$$

It is easy to check that $F \subset E^\infty$. So it is enough to prove $\dim_H F \geq \frac{1}{2}$. To give a precise view on the structure of F, we shall make use of a kind of symbolic space defined as follows.

$$\mathcal{D}_n = \Big\{(a_1, \cdots, a_n) \in \mathbb{N}^n : 2^k \leq a_k < 2^{k+1},\ \text{for all } \ 1 \leq k \leq n\Big\}.$$

For any $(a_1, \cdots, a_n) \in \mathcal{D}_n$, call

$$J(a_1, \cdots, a_n) = cl\Big\{x \in [0,1) : a_k(x) = a_k, 1 \leq k \leq n\Big\}$$

an admissible *CF-intervals* of rank n, where "cl" denotes the closure of a set in $[0,1)$. It is observable that

$$F = \bigcap_{n=1}^{\infty} \bigcup_{(a_1,\cdots,a_n)\in\mathcal{D}_n} J(a_1, \cdots, a_n).$$

Let μ be a probability measure supported on F such that for every admissible intervals $J(a_1, \cdots, a_n)$,

$$\mu(J(a_1, \cdots, a_n)) = \frac{1}{\sharp \mathcal{D}_n} = \frac{1}{2^{1+2+\cdots+n}}, \tag{8}$$

where $\sharp$ denotes the cardinality.

Now we estimate the μ-measure of arbitrary ball $B(x,r)$ with center $x \in F$ and radius r small enough. Choose $n \geq 1$ such that

$$|J(a_1, \cdots, a_{n+1})| \leq r < |J(a_1, \cdots, a_n)|.$$

Calculations show

$$|J(a_1, \cdots, a_n)| \leq \sum_{1\leq i\leq 4} |J(a_1, \cdots, a_{n-1}, a_n + i)|.$$

So that, from $a_n \geq 2$ and $r < |J(a_1, \cdots, a_n)|$ we have

$$B(x,r) \subset J(a_1, \cdots, a_{n-1}). \tag{9}$$

On the other hand, from (2), (3) and (7), We have

$$r \geq |J(a_1, \cdots, a_{n+1})| > \frac{1}{2q_{n+1}^2} > \frac{1}{2^{2n+3}a_1^2a_2^2\cdots a_{n+1}^2} > \frac{1}{2^{2n+3+(n+1)(n+4)}}. \tag{10}$$

Combining (8), (9) and (10), we get

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq \liminf_{r\to 0} \frac{\frac{(n-1)n}{2}}{2n+3+(n+1)(n+4)} = \frac{1}{2}.$$

By using lemma 1.5, we obtain $\dim_H F \geq \frac{1}{2}$, which shows $\dim_H E^{\infty} \geq \frac{1}{2}$ since $F \subset E^{\infty}$. This completes the proof. □

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Ting ZHONG
Department of Mathematics, Jishou University
Zhangjiajie, Hunan, 427000, P.R. CHINA
e-mail: zhongting_2005@126.com

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