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Solutions for $2n^{\rm th}$ order lidstone BVP on time scales

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Abstract

In this paper, we prove the existence of solutions for nonlinear Lidstone boundary value problems by using the monotone method on time scale and also we show the existence of at least one positive solution if f is either superlinear or sublinear by the fixed point theorem in a Banach space.

Key Words: Lidstone boundary value problem, upper and lower solutions, fixed point theorem, positive solution.

1. Introduction

Let T be any time scale (nonempty closed subset of $\mathbf{R}$) and $[0,1]$ is subset of T such that $[0,1]=\{t\in \mathrm{T}:\ 0\le t\le 1\}$.

In this paper, we shall consider the nonlinear Lidstone boundary value problem (LBVP),

$$(-1)^n y^{\triangle^{2n}}(t) = f(t, y^{\sigma}(t), y^{\triangle\triangle}(t), ..., y^{\triangle^{2(n-1)}}(t)), \quad t\in[0,1] \tag{1.1}$$

$$y^{\triangle^{2i}}(0) = y^{\triangle^{2i}}(\sigma(1))=0, \qquad 0\le i\le n-1, \tag{1.2}$$

where $n\ge 1$ and $f:[0,\sigma(1)]\times\mathbf{R^n}\to\mathbf{R}$ is continuous. We assume that $\sigma(1)$ is right dense so that $\sigma^j(1)=\sigma(1)$ for $j\ge 1$.

In this section we give some inequalities for certain Green's function which are proved in the reference [5]. In Section 2 we give the existence and uniqueness theorem for solution using the method of upper and lower solutions when they are given in the well order. This method is generally used to obtain the existence of solutions within specified bounds determined by the upper and lower solutions. Also we obtain a unique solution within the appropriate bounds. Then we develop the monotone method which yields the solution of the LBVP (1.1), (1.2). The method of upper and lower solutions have been applied by several authors in [4, 7, 10, 13] and the references therein. In [9] Ehme, Eloe and Henderson applied this method to $2n^{\rm th}$ order problems.

Cone theory techniques have been applied by several authors for ordinary differential equations and dynamic equations on time scales including two-point, three-point and Lidstone problems in [1, 5, 6] and the

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references therein. Lidstone boundary value problem (LBVP) has attracted considerable attention in recent years [12]. In Section 3 we discuss the existence of a positive solution for the LBVP (1.1), (1.2) under $f_0 = 0, f_\infty = \infty$ or $f_0 = \infty, f_\infty = 0$. Some preliminary definitions and theorems on time scales can be found in the books [2, 3].

To obtain a solution for the LBVP (1.1), (1.2) we need the $G_n(t,s)$ which is the Green's function of the boundary value problem,

$$y^{\triangle^{2n}}(t) = 0, \qquad t \in [0,1]$$
$$y^{\triangle^{2i}}(0) = y^{\triangle^{2i}}(\sigma(1)) = 0,$$

for $0 \le i \le n-1$.

The Green's function for the problem $y^{\triangle\triangle}(t) = 0, \quad y(0) = y(\sigma(1)) = 0$, is

$$G(t,s) = \frac{1}{\sigma(1)} \begin{cases} t(\sigma(s) - \sigma(1)), & t \le s \\ \sigma(s)(t - \sigma(1)), & t > \sigma(s). \end{cases} \tag{1.3}$$

If we let $G_1(t,s) := G(t,s)$, then for $2 \le j \le n$ we can recursively define

$$G_j(t,s) = \int_0^{\sigma(1)} G_{j-1}(t,r) G(r,s) \triangle r. \tag{1.4}$$

Further, it is easily seen that

$$(-1)^n G_n(t,s) \ge 0, \qquad (t,s) \in [0,\sigma(1)] \times [0,1]. \tag{1.5}$$

Lemma 1.1 [5] *For* $(t,s) \in [0,\sigma(1)] \times [0,1]$, *we have*

$$(-1)^n G_n(t,s) = |G_n(t,s)| \le \left(\frac{\sigma(1)}{4}\right)^{n-1} \frac{\sigma(s)(\sigma(1) - \sigma(s))}{\sigma(1)}, \tag{1.6}$$

and also

$$(-1)^n G_n(t,s) = |G_n(t,s)| \le \left(\frac{\sigma(1)}{4}\right)^{n}. \tag{1.7}$$

Lemma 1.2 [5] *Let* $\delta \in \left(0, \frac{\sigma(1)}{2}\right)$ *be given. For* $(t,s) \in [\delta, \sigma(1) - \delta] \times [0,1]$, *we have*

$$(-1)^n G_n(t,s) = |G_n(t,s)| \ge \theta_n(\delta) \frac{\sigma(s)(\sigma(1) - \sigma(s))}{\sigma(1)}, \tag{1.8}$$

where

$$\theta_n(\delta) = \left(\frac{\delta}{\sigma(1)}\right)^n (\delta^2(\sigma(1) - 2\delta))^{n-1}.$$

Using Lemma 1.1, we get

$$\begin{aligned}(-1)^n G_n(t,s) &\geq \theta_n(\delta)\frac{\sigma(s)(\sigma(1)-\sigma(s))}{\sigma(1)}\\ &\geq \theta_n(\delta)\left(\frac{4}{\sigma(1)}\right)^{n-1}\max_{t\in[0,\sigma(1)]}|G_n(t,s)|.\end{aligned}$$

For $\delta=\frac{\sigma(1)}{4}\in\left(0,\frac{\sigma(1)}{2}\right)$, we have $\theta_n\left(\frac{\sigma(1)}{4}\right)\left(\frac{4}{\sigma(1)}\right)^{n-1}=\frac{(\sigma(1))^{2n-2}}{2^{5n-3}}$. Let $\gamma_n=\frac{1}{2^{5n-3}}$, then $0<\gamma_n<1$ and $(-1)^n G_n(t,s)\geq\frac{1}{2^{5n-3}}\max_{t\in[0,\sigma(1)]}|G_n(t,s)|$.

Also in the reference [5], the positivity condition of the function $u(t)$ is given.

Lemma 1.3 [5] *Assume that* $u\in C^{\triangle^{2n}}[0,\sigma(1)]^{k^{2n}}$ *and* $u(t)$ *satisfies*

$$(-1)^n u^{\triangle^{2n}}(t)\geq 0, \qquad t\in[0,1] \tag{1.9}$$

$$(-1)^i u^{\triangle^{2i}}(0)\geq 0,\quad (-1)^i u^{\triangle^{2i}}(\sigma(1))\geq 0, \qquad 0\leq i\leq n-1. \tag{1.10}$$

Then u *is nonnegative on* $[0,\sigma(1)]$.

2. Existence and uniqueness

In this section, we give the existence and local uniqueness of a solution of the LBVP (1.1), (1.2) that lies between an upper and lower solution. We will use the norm

$$\|y\|:=\max_{t\in[0,\sigma(1)]}\{|y(t)|,|y^{\triangle\triangle}(t)|,...,|y^{\triangle^{2(n-1)}}(t)|\}$$

as the norm on $C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}}$.

We note that, for $t\in[0,\sigma(1)]$, $|y(t)|\leq\int_0^t|y^{\triangle}(s)|\triangle s\leq\sigma(1)\|y^{\triangle}\|_\infty$. Hence $\|y\|_\infty\leq\sigma(1)\|y^{\triangle}\|_\infty$. Similarly we get, for $t\in[0,\sigma(1)]$, $|y^{\triangle}(t)|\leq\int_0^t|y^{\triangle\triangle}(s)|\triangle s\leq\sigma(1)\|y^{\triangle\triangle}\|_\infty$. By continuing this process, we get

$$\|y\|_\infty\leq\sigma(1)\|y^{\triangle}\|_\infty\leq(\sigma(1))^2\|y^{\triangle\triangle}\|_\infty\leq...\leq(\sigma(1))^{2(n-1)}\|y^{\triangle^{2(n-1)}}(t)\|_\infty.$$

So, $\|y^{\triangle^{2(n-1)}}(t)\|_\infty\leq\|y\|\leq(\sigma(1))^{2(n-1)}\|y^{\triangle^{2(n-1)}}(t)\|_\infty$.

Definition 2.1 *Letting* $\alpha\in C^{\triangle^{2n}}[0,\sigma(1)]^{k^{2n}}$*, we say* α *is an upper solution for LBVP (1.1), (1.2) if* α *satisfies*

$$\begin{aligned}(-1)^n\alpha^{\triangle^{2n}}(t) &\geq f(t,\alpha^\sigma(t),\alpha^{\triangle\triangle}(t),...,\alpha^{\triangle^{2(n-1)}}(t)),\quad t\in[0,1],\\ (-1)^i\alpha^{\triangle^{2i}}(0) &\geq 0,\quad (-1)^i\alpha^{\triangle^{2i}}(\sigma(1))\geq 0,\qquad 0\leq i\leq n-1.\end{aligned}$$

Definition 2.2 *Letting* $\beta \in C^{\triangle^{2n}}[0,\sigma(1)]^{k^{2n}}$, *we say* β *is a lower solution for LBVP (1.1), (1.2) if* β *satisfies*

$$\begin{aligned}(-1)^n\beta^{\triangle^{2n}}(t) &\leq f(t,\beta^{\sigma}(t),\beta^{\triangle\triangle}(t),...,\beta^{\triangle^{2(n-1)}}(t)),\quad t\in[0,1],\\ (-1)^i\beta^{\triangle^{2i}}(0) &\leq 0,\quad (-1)^i\beta^{\triangle^{2i}}(\sigma(1))\leq 0,\qquad 0\leq i\leq n-1.\end{aligned}$$

The function $f(t,y_1,y_2,...,y_n)$ is said to be Lip$-\alpha\beta$ if there exist positive constants c_i such that for all $(x_1,x_2,...,x_n)$ *and* $(y_1,y_2,...,y_n)$ such that

$$(-1)^i\alpha^{\triangle^{2i}}(t)\geq(-1)^ix_{i+1}(t),\quad (-1)^iy_{i+1}(t)\geq(-1)^i\beta^{\triangle^{2i}}(t),\quad 0\leq i\leq n-1,$$

it follows that

$$|f(t,x_1,x_2,...,x_n)-f(t,y_1,y_2,...,y_n)|\leq\sum_{i=1}^{n}c_i|x_i-y_i|.$$

We note that if f is continuously differentiable on a suitable region, then f will be Lip$-\alpha\beta$.

Theorem 2.1 *Assume there exist* α *and* β *such that* α *and* β *are upper and lower solutions of the LBVP* (1.1), (1.2), *respectively, which satisfy*

$$(-1)^i\beta^{\triangle^{2i}}(t)\leq(-1)^i\alpha^{\triangle^{2i}}(t),\quad 0\leq i\leq n-1 \tag{2.11}$$

for all $t\in[0,\sigma(1)]$ *and assume*

(1) $f(t,y_1,y_2,...,y_n):[0,\sigma(1)]\times\mathbf{R}^n\to\mathbf{R}$ *is continuous;*

(2) $f(t,y_1,y_2,...,y_n)$ *is Lip*$-\alpha\beta$; *and*

(3) $f(t,y_1,...,\overline{y}_k,...,y_n)-f(t,y_1,...,\underline{y}_k,...,y_n)\geq(-1)^{k-1}(\overline{y}_k-\underline{y}_k)$ (2.12)

for $(-1)^{k-1}\beta^{\triangle^{2(k-1)}}(t)\leq(-1)^{k-1}\underline{y}_k\leq(-1)^{k-1}\overline{y}_k\leq\ (-1)^{k-1}\alpha^{\triangle^{2(k-1)}}(t)$,
$1\leq k\leq n,\quad y_i\in\mathbf{R}$ *and* $t\in[0,\sigma(1)]$.

Then, if

$$\left(\frac{\sigma(1)}{4}\right)^n\left(\sum_{i=1}^{n}c_i+c_0\right)<1, \tag{2.13}$$

there exists a unique solution $y(t)$ *of the LBVP* (1.1), (1.2) *such that*

$$(-1)^i\beta^{\triangle^{2i}}(t)\leq(-1)^iy^{\triangle^{2i}}(t)\leq(-1)^i\alpha^{\triangle^{2i}}(t),\quad 0\leq i\leq n-1$$

for all $t\in[0,\sigma(1)]$ *and there exist two monotone sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$, *respectively,with* $\alpha_0=\alpha$ *and* $\beta_0=\beta$, *which converge uniformly to the extremal solutions in* $[\beta,\alpha]$ *of the problem LBVP (1.1), (1.2).*

Proof. For $0 \leq i \leq n-1$, we define

$$p_{2i}(t, y^{\triangle^{2i}}(t)) = \begin{cases} \max\{\beta^{\triangle^{2i}}(t), \min\{y^{\triangle^{2i}}(t), \alpha^{\triangle^{2i}}(t)\}\}, & \text{if } i \text{ is even;} \\ \max\{\alpha^{\triangle^{2i}}(t), \min\{y^{\triangle^{2i}}(t), \beta^{\triangle^{2i}}(t)\}\}, & \text{if } i \text{ is odd,} \end{cases}$$

where y is a function defined on $[0, \sigma(1)]$. If $y^{\triangle^{2i}}$ is continuous, then p_{2i} is continuous. Moreover,

$$(-1)^i \beta^{\triangle^{2i}}(t) \leq (-1)^i p_{2i}(t, y^{\triangle^{2i}}(t)) \leq (-1)^i \alpha^{\triangle^{2i}}(t)$$

and $0 \leq i \leq n-1$, and each p_{2i} is non-expansive function.

Define $F_1 : [0, \sigma(1)] \times C^{\triangle^{2(n-1)}}[0, \sigma(1)]^{k^{2(n-1)}} \rightarrow \mathbf{R}$ by

$$F_1(t, y^\sigma(t), y^{\triangle\triangle}(t), ..., y^{\triangle^{2(n-1)}}(t)) = f(t, p_0(\sigma(t), y^\sigma(t)), ..., p_{2(n-1)}(t, y^{\triangle^{2(n-1)}}(t))).$$

Thus,

$$|F_1(t, y^\sigma(t), y^{\triangle\triangle}(t), ..., y^{\triangle^{2(n-1)}}(t)) - F_1(t, z^\sigma(t), z^{\triangle\triangle}(t), ..., z^{\triangle^{2(n-1)}}(t))|$$
$$\leq \sum_{i=2}^{n} c_i |y^{\triangle^{2(i-1)}}(t) - z^{\triangle^{2(i-1)}}(t)| + c_1 |y^\sigma(t) - z^\sigma(t)|.$$

F_1 is also continuous. Choose $c_0 > 0$ such that

$$\left(\frac{\sigma(1)}{4}\right)^n \left(\sum_{i=1}^{n} c_i + c_0\right) < 1.$$

Now define $F_2 : [0, \sigma(1)] \times C^{\triangle^{2(n-1)}}[0, \sigma(1)]^{k^{2(n-1)}} \rightarrow \mathbf{R}$ by

$$F_2(t, y_1, y_2, ..., y_n) = \begin{cases} F_1(t, y_1, y_2, ..., y_n) - c_0(-1)^{n-1}(y_n - \alpha_n), & \text{if } (-1)^{n-1} y_n > (-1)^{n-1}\alpha_n; \\ F_1(t, y_1, y_2, ..., y_n), & \text{if } (-1)^{n-1}\beta_n \leq (-1)^{n-1} y_n \leq (-1)^{n-1}\alpha_n; \\ F_1(t, y_1, y_2, ..., y_n) + c_0(-1)^{n-1}(\beta_n - y_n), & \text{if } (-1)^{n-1} y_n < (-1)^{n-1}\beta_n, \end{cases}$$

where $\beta_1 = \beta^\sigma$ and $\alpha_1 = \alpha^\sigma$, for $n = 1$ and $\beta_n = \beta^{\triangle^{2(n-1)}}$ and $\alpha_n = \alpha^{\triangle^{2(n-1)}}$, for $n \geq 2$. Then F_2 is continuous. By considering various cases, it can be shown that F_2 satisfies

$$|F_2(t, y^\sigma(t), y^{\triangle\triangle}(t), ..., y^{\triangle^{2(n-1)}}(t)) - F_2(t, z^\sigma(t), z^{\triangle\triangle}(t), ..., z^{\triangle^{2(n-1)}}(t))|$$
$$\leq c_1 |y^\sigma(t) - z^\sigma(t)| + \sum_{i=2}^{n-1} c_i |y^{\triangle^{2(i-1)}}(t) - z^{\triangle^{2(i-1)}}(t)|$$
$$+ (c_n - c_0)|y^{\triangle^{2(n-1)}}(t) - z^{\triangle^{2(n-1)}}(t)|.$$

This shows F_2 is also Lipschitz.

Define $T : C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}} \to C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}}$ by

$$Ty(t) := \int_0^{\sigma(1)} (-1)^n G_n(t,s) F_2(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s.$$

For $y, z \in C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}}$, it follows that

$$\begin{aligned}
|Ty(t) - Tz(t)| &\leq \int_0^{\sigma(1)} |G_n(t,s)||F_2(t, y^\sigma(t), y^{\triangle\triangle}(t), ..., y^{\triangle^{2(n-1)}}(t)) \\
&\quad - F_2(t, z^\sigma(t), z^{\triangle\triangle}(t), ..., z^{\triangle^{2(n-1)}}(t))|\triangle s \\
&\leq \left(\frac{\sigma(1)}{4}\right)^n \left\{ \sum_{i=1}^n c_i\|y-z\| + c_0\|y-z\| \right\} \\
&= \left(\frac{\sigma(1)}{4}\right)^n \left(\sum_{i=1}^n c_i + c_0 \right) \|y-z\|.
\end{aligned}$$

This shows that T is a contraction from the choice of c_0 guarantees. Hence T has a unique fixed point $y(t)$ which is the solution of LBVP (1.1), (1.2).

We now demonstrate $p_{2i}(t, y^{\triangle^{2i}}(t)) = y^{\triangle^{2i}}(t)$, for $0 \leq i \leq n-1$ on $t \in [0,\sigma(1)]$

Suppose that $(-1)^i y^{\triangle^{2i}}(t) > (-1)^i \alpha^{\triangle^{2i}}(t)$, $0 \leq i \leq n-1$ for all $t \in [0,\sigma(1)]$, this implies for $t \in [0,1]$,

$$\begin{aligned}
(-1)^n y^{\triangle^{2n}}(t) &= F_2(t, y^\sigma(t), y^{\triangle\triangle}(t), ..., y^{\triangle^{2(n-1)}}(t)) \\
&= F_1(t, y^\sigma(t), y^{\triangle\triangle}(t), ..., y^{\triangle^{2(n-1)}}(t)) \\
&\quad - c_0(-1)^{n-1}(y^{\triangle^{2(n-1)}} - \alpha^{\triangle^{2(n-1)}})(t) \\
&\leq f(t, p_0(\sigma(t), y^\sigma(t)), p_2(t, y^{\triangle\triangle}(t)), ..., p_{2(n-1)}(t, y^{\triangle^{2(n-1)}}(t))) \\
&\leq f(t, \alpha^\sigma(t), \alpha^{\triangle\triangle}(t), ..., \alpha^{\triangle^{2(n-1)}}(t)) \\
&\leq (-1)^n \alpha^{\triangle^{2n}}(t).
\end{aligned}$$

Hence, we have $(-1)^n(y-\alpha)^{\triangle^{2n}}(t) \leq 0$ for $t \in [0,1]$ and from the boundary conditions $(-1)^i(y-\alpha)^{\triangle^{2i}}(0) \leq 0$ and $(-1)^i(y-\alpha)^{\triangle^{2i}}(\sigma(1)) \leq 0$, for $0 \leq i \leq n-1$.

Thus we get

$$-[(-1)^{n-1}(y-\alpha)^{\triangle^{2(n-1)}}]^{\triangle\triangle}(t) \leq 0, \quad t \in [0,1]$$

$$(-1)^{n-1}(y-\alpha)^{\triangle^{2(n-1)}}(0) \leq 0 \;\; and \;\; (-1)^{n-1}(y-\alpha)^{\triangle^{2(n-1)}}(\sigma(1)) \leq 0.$$

This shows, from Lemma 1.3, $v(t) = (-1)^{n-1}(y-\alpha)^{\triangle^{2(n-1)}}(t) \leq 0$ for $t \in [0,\sigma(1)]$, which is a contradiction.

It follows that $(-1)^i y^{\triangle^{2i}}(t) \leq (-1)^i \alpha^{\triangle^{2i}}(t)$ on $t \in [0,\sigma(1)]$.

Similarly, $(-1)^i \beta^{\triangle^{2i}}(t) \leq (-1)^i y^{\triangle^{2i}}(t)$ on $t \in [0,\sigma(1)]$. Thus $y(t)$ is a local unique solution of LBVP

(1.1), (1.2) which lies between an upper and lower solution.

Consider the associated problem

$$\begin{aligned}(-1)^n y^{\triangle^{2n}}(t) &= f(t,\eta^{\sigma}(t),\eta^{\triangle\triangle}(t),...,\eta^{\triangle^{2(n-1)}}(t)), \quad t\in[0,1] &(2.14)\\ y^{\triangle^{2i}}(0) &= y^{\triangle^{2i}}(\sigma(1))=0, \qquad 0\le i\le n-1 &(2.15)\end{aligned}$$

with $\eta\in C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}}$. Since the conditions on the function f are satisfied for $\eta^{\triangle^{2i}}(t),\ \ 0\le i\le n-1$, the problem (2.14)–(2.15) has a unique solution y. Define $T: C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}}\to C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}}$ by $T\eta=y$.

Now, we divide the proof into three steps.

Step 1. We show

$$TD\subseteq D. \tag{2.16}$$

Here, $D=\{\eta\in C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}}:(-1)^k\beta^{\triangle^{2k}}\le(-1)^k\eta^{\triangle^{2k}}\le(-1)^k\alpha^{\triangle^{2k}},0\le k\le n-1\}$ is a nonempty bounded closed subset in $C^{\triangle^{2(n-1)}}[0,\sigma(1)]^{k^{2(n-1)}}$.

In fact, for $\xi\in D$, set $w=T\xi$. By the definition of α,β and D, and by (2.12), we have that for $t\in[0,1]$

$$\begin{aligned}(-1)^n(\alpha-\omega)^{\triangle^{2n}}(t) \ \ge\ & f(t,\alpha^{\sigma}(t),\alpha^{\triangle\triangle}(t),...,\alpha^{\triangle^{2(n-1)}}(t))\\ &-f(t,\xi^{\sigma}(t),\xi^{\triangle\triangle}(t),...,\xi^{\triangle^{2(n-1)}}(t))\ge 0,\end{aligned} \tag{2.17}$$

and

$$(-1)^k(\alpha-\omega)^{\triangle^{2k}}(0)\ge 0,\quad (-1)^k(\alpha-\omega)^{\triangle^{2k}}(\sigma(1))\ge 0,\quad 0\le k\le n-1. \tag{2.18}$$

By the technique of the proof of Lemma 1.3, combining (2.17) and (2.18), we have that

$$(-1)^k(\alpha-\omega)^{\triangle^{2k}}(t)\ge 0$$

for $t\in[0,\sigma(1)],\quad k=0,1,...,n-1$. Thus

$$(-1)^k\omega^{\triangle^{2k}}(t)\le(-1)^k\alpha^{\triangle^{2k}}(t),\quad for\quad t\in[0,\sigma(1)],\quad 0\le k\le n-1.$$

Analogously,

$$(-1)^k\omega^{\triangle^{2k}}(t)\ge(-1)^k\beta^{\triangle^{2k}}(t),\quad for\quad t\in[0,\sigma(1)],\quad 0\le k\le n-1.$$

Thus, (2.16) holds.

Step 2. Let $\upsilon_1=T\eta_1,\quad \upsilon_2=T\eta_2$, where $\upsilon_1,\upsilon_2\in D$ satisfy $(-1)^k{\eta_1}^{\triangle^{2k}}(t)\ge(-1)^k{\eta_2}^{\triangle^{2k}}(t),\quad 0\le k\le n-1$. We show that

$$(-1)^k{\upsilon_1}^{\triangle^{2k}}(t)\ge(-1)^k{\upsilon_2}^{\triangle^{2k}}(t),\quad 0\le k\le n-1. \tag{2.19}$$

In fact, by (2.12) and the definition of υ_1, υ_2, for $t \in [0,1]$

$$\begin{aligned}(-1)^n(\upsilon_2-\upsilon_1)^{\triangle^{2n}}(t) \quad &\geq \quad f(t,\eta_2^\sigma(t),\eta_2^{\triangle\triangle}(t),...,\eta_2^{\triangle^{2(n-1)}}(t)\\ &\quad -f(t,\eta_1^\sigma(t),\ \eta_1^{\triangle\triangle}(t),...,\eta_1^{\triangle^{2(n-1)}}(t))\geq 0,\end{aligned}$$

$$(-1)^k(\upsilon_2-\upsilon_1)^{\triangle^{2k}}(0)=(-1)^k(\upsilon_2-\upsilon_1)^{\triangle^{2k}}(\sigma(1)),\quad k=0,1,...,n-1.$$

With the use of Lemma 1.3, similar to Step 1, for $t\in[0,\sigma(1)]$ we can easily prove

$$(-1)^k{\upsilon_1}^{\triangle^{2k}}(t)\geq(-1)^k{\upsilon_2}^{\triangle^{2k}}(t),\quad 0\leq k\leq n-1.$$

Thus, (2.19) holds.

Step3. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are obtained by recurrence
$\alpha_0=\alpha,\quad \beta_0=\beta,$
$\alpha_n=T\alpha_{n-1},\quad \beta_n=T\beta_{n-1},\quad n=1,2,....$

From the result of Step 1 and Step 2, we have that

$$\beta\leq\beta_1\leq...\leq\beta_n\leq...\leq\alpha_n\leq...\leq\alpha_1\leq\alpha \tag{2.20}$$

$$(-1)^k\beta^{\triangle^{2k}}\leq(-1)^k{\alpha_n}^{\triangle^{2k}},\quad (-1)^k{\beta_n}^{\triangle^{2k}}\leq(-1)^k\alpha^{\triangle^{2k}} \tag{2.21}$$

$k=1,2,...,n-1$.
Moreover, from the definition of T, we get

$$(-1)^n{\alpha_n}^{\triangle^{2n}}(t)=f(t,\alpha_{n-1}^\sigma,{\alpha_{n-1}}^{\triangle\triangle},...,{\alpha_{n-1}}^{\triangle^{2(n-1)}}), \tag{2.22}$$

$$(-1)^k{\alpha_n}^{\triangle^{2k}}(0)=(-1)^k{\alpha_n}^{\triangle^{2k}}(\sigma(1))=0,\qquad k=0,1,2,...,n-1. \tag{2.23}$$

Analogously,

$$(-1)^n{\beta_n}^{\triangle^{2n}}(t)=f(t,\beta_{n-1}^\sigma,{\beta_{n-1}}^{\triangle\triangle},...,{\beta_{n-1}}^{\triangle^{2(n-1)}}), \tag{2.24}$$

$$(-1)^k{\beta_n}^{\triangle^{2k}}(0)=(-1)^k{\beta_n}^{\triangle^{2k}}(\sigma(1))=0,\qquad k=0,1,2,...,n-1. \tag{2.25}$$

From (2.20)–(2.22) and the continuity of f, we have that there exists $M_{\alpha,\beta}>0$ depending only on α and β (but not on n or t) such that

$$|{\alpha_n}^{\triangle^{2n}}(t)|\leq M_{\alpha,\beta},\quad \text{for all}\quad t\in[0,\sigma(1)]. \tag{2.26}$$

From the mean value theorem on Time scale, using the boundary condition (2.23), we get that for each $n\in N_x$, there exists $\varsigma_n,\xi_n\in(0,\sigma(1))$ such that

$${\alpha_n}^{\triangle^{2n-1}}(\xi_n)\leq 0\leq{\alpha_n}^{\triangle^{2n-1}}(\varsigma_n). \tag{2.27}$$

This, together with (2.26), implies

$$|{\alpha_n}^{\triangle^{2n-1}}(t)| = \left|{\alpha_n}^{\triangle^{2n-1}}(\xi_n) + \int_{\xi_n}^{t} {\alpha_n}^{\triangle^{2n}}(s)\triangle s\right| \leq \sigma(1)M_{\alpha,\beta}. \tag{2.28}$$

By combining (2.21) and (2.23), we can get that there is $C_{\alpha,\beta} > 0$ depending only on α and β (but not on n and t), such that for $k = 1, 2, ..., n-1$

$$|{\alpha_n}^{\triangle^{2n-2k}}(t)| \leq C_{\alpha,\beta}, \quad for \quad all \quad t \in [0, \sigma(1)], \tag{2.29}$$

$$|{\alpha_n}^{\triangle^{2n-2k-1}}(t)| \leq C_{\alpha,\beta}, \quad for \quad all \quad t \in [0, \sigma(1)]. \tag{2.30}$$

Thus from (2.20) and (2.28)-(2.30), we know that $\{\alpha_n\}$ is bounded in $C^{\triangle^{2n}}[0, \sigma(1)]^{k^{2n}}$. Similarly, β_n is bounded in $C^{\triangle^{2n}}[0, \sigma(1)]^{k^{2n}}$.

Now, by using the fact that α_n and β_n is bounded in $C^{\triangle^{2n}}[0, \sigma(1)]^{k^{2n}}$, we can conclude that α_n, β_n converge uniformly to the extremal solutions in $[0, \sigma(1)]$ of the solution (1.1), (1.2). □

3. Existence of positive solutions

We assume throughout this section that

$$f : [0, \sigma(1)] \times (\mathbf{R}^+)^{\mathbf{n}} \to \mathbf{R}^+,$$

and

$$f_0 := \lim_{y_n \to 0} \frac{f(t, y_1, y_2, ..., y_n)}{y_n}, \quad f_\infty := \lim_{y_n \to \infty} \frac{f(t, y_1, y_2, ..., y_n)}{y_n},$$

exist uniformly in the extended reals. The case

$$f_0 = 0, \quad f_\infty = \infty$$

is called the superlinear case and the case

$$f_0 = \infty, \quad f_\infty = 0$$

is called the sublinear case. To prove our result, we will use the following theorem which can be found in Krasnoselskii's book [11] and in Deimling's book [8].

Theorem 3.1 *(Guo-Krasnosel'skii Fixed Point Theorem) Let B be a Banach space, $P \subseteq B$ a cone, and suppose that Ω_1, Ω_2 are open subsets of P with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that*

$(i) \|Tu\| \leq \|u\|, \quad u \in P \cap \partial\Omega_1 \quad and \quad \|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_2, \ or$

$(ii) \|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_1 \quad and \quad \|Tu\| \leq \|u\|, \quad u \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.2 *If either the superlinear case* $f_0 = 0, \quad f_\infty = \infty$ *or sublinear case* $f_0 = \infty, \quad f_\infty = 0$ *holds, then the LBVP* $(1.1), (1.2)$ *has a positive solution.*

Proof. We consider the Banach space

$$B = \{y : y \in C^{\triangle^{2(n-1)}}[0, \sigma(1)]^{k^{2(n-1)}}\}$$

equipped with a norm $\|.\|$ defined by

$$\|y\| := \max_{t\in[0,\sigma(1)]}\{|y(t)|, |y^{\triangle\triangle}(t)|, ..., |y^{\triangle^{2(n-1)}}(t)|\}.$$

Let

$$\gamma_n^* := \min\left\{\gamma_n, \left(\frac{4}{\sigma(1)}\right)^n \min_{s\in[\xi,\omega]}(-1)^n G_n(\sigma(\omega), s)\right\},$$

where γ_n is the constant defined in Lemma 1.2. Then define a cone P in B by

$$P := \{y \in B : \min_{t\in[0,\sigma(1)]} y(t) \geq 0 \ \ and \ \ \min_{t\in[\xi,\sigma(\omega)]} y^{\triangle^{2(n-1)}}(t) \geq \gamma_n^*\|y\|\}.$$

It is easy to check that P is a cone of nonnegative functions in $C^{\triangle^{2(n-1)}}[0, \sigma(1)]^{k^{2(n-1)}}$. Define an operator T as:

$$Ty(t) := \int_0^{\sigma(1)} (-1)^n G_n(t, s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s$$

for $t \in [0, \sigma(1)]$. We now show that

$$T : P \to P.$$

First note that $y \in P$ implies that $Ty(t) \geq 0$ on $[0, \sigma(1)]$ and

$$\begin{aligned}
\min_{t\in[\xi,\omega]} Ty(t) &= \int_0^{\sigma(1)} \min_{t\in[\xi,\omega]} (-1)^n G_n(t, s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\
&\geq \gamma_n \int_0^{\sigma(1)} \max_{t\in[0,\sigma(1)]} |G_n(t, s)| f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s
\end{aligned}$$

by Lemma 1.2. It follows that

$$\min_{t\in[\xi,\omega]} Ty(t) \geq \gamma_n\|Ty\| \geq \gamma_n^*\|Ty\|.$$

Also, using Lemma 1.1 we have

$$\begin{aligned}
Ty(\sigma(\omega)) &= \int_0^{\sigma(1)} (-1)^n G_n(\sigma(\omega), s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\
&\geq \gamma_n^* \int_0^{\sigma(1)} \left(\frac{\sigma(1)}{4}\right)^n f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\
&\geq \gamma_n^*\|Ty\|.
\end{aligned}$$

Hence $Ty \in P$ and so $T: P \to P$ which is what we want to prove. Therefore T is completely continuous.

Assume now that we are in the superlinear case

$$f_0 = 0, \ f_\infty = \infty.$$

Since

$$\lim_{y \to 0^+} \frac{f(t, y_1, y_2, ..., y_n)}{y_n} = 0$$

uniformly on $[0, \sigma(1)]$, we may choose an $r > 0$ such that

$$f(t, y_1, y_2, ..., y_n) \le \eta y_n, \quad 0 \le y_n \le r, \quad 0 \le t \le \sigma(1),$$

where

$$\eta := \left(\frac{4}{\sigma(1)}\right)^n.$$

Then if Ω_1 is the ball in B centered at the origin with radius r and if $y_n \in P \cap \partial\Omega_1$, then we have

$$\begin{aligned}
Ty(t) &= \int_0^{\sigma(1)} (-1)^n G_n(t,s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\
&\le \int_0^{\sigma(1)} \left(\frac{\sigma(1)}{4}\right)^{n-1} \frac{\sigma(s)(\sigma(1)-\sigma(s))}{\sigma(1)} f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\
&\le \eta \left(\frac{\sigma(1)}{4}\right)^{n-1} \int_0^{\sigma(1)} \frac{\sigma(s)(\sigma(1)-\sigma(s))}{\sigma(1)} y^{\triangle^{2(n-1)}}(s)\triangle s \\
&\le \eta \left(\frac{\sigma(1)}{4}\right)^{n-1} r \int_0^{\sigma(1)} \frac{\sigma(s)(\sigma(1)-\sigma(s))}{\sigma(1)} \triangle s \\
&\le \eta \left(\frac{\sigma(1)}{4}\right)^{n} r = r = \|y^{\triangle^{2(n-1)}}\|_\infty \le \|y\|,
\end{aligned}$$

and so $\|Ty\| \le \|y\|$ for all $y \in P \cap \partial\Omega_1$.

Next we use the assumption

$$\lim_{y \to \infty} \frac{f(t, y_1, y_2, ..., y_n)}{y_n} = \infty$$

uniformly on $[0, \sigma(1)]$. Let $t_0 \in [\xi, \omega]$ and let

$$\mu := \left(\gamma_n^* \int_\xi^\omega (-1)^n G_n(t_0, s)\triangle s\right)^{-1}.$$

Then there is an $\overline{R}$ such that

$$f(t, y_1, y_2, ..., y_n) \ge \mu y_n, \quad y_n \ge \overline{R}.$$

If we define

$$R := \max\left\{2r, \ \frac{\overline{R}}{\gamma_n^*}\right\}$$

and $\Omega_2 = \{y \in B : \ \ \|y\| < R\}$, $y \in P \cap \partial\Omega_2$, we have

$$\min_{t\in[\xi,\sigma(\omega)]} y^{\triangle^{2(n-1)}}(t) \geq \gamma_n^* \|y\| = \gamma_n^* R \geq \overline{R}.$$

Therefore, for all $t \in [\xi, \sigma(\omega)]$,

$$f(t, y^\sigma(t), y^{\triangle\triangle}(t), ..., y^{\triangle^{2(n-1)}}(t)) \geq \mu y^{\triangle^{2(n-1)}}(t) \geq \mu\gamma_n^* R.$$

Hence,

$$\begin{aligned} Ty(t_0) &= \int_0^{\sigma(1)} (-1)^n G_n(t_0, s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\ &\geq \int_\xi^{\omega} (-1)^n G_n(t_0, s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\ &\geq \mu\gamma_n^* \|y\| \int_\xi^{\omega} (-1)^n G_n(t_0, s)\triangle s \\ &= \|y\| = R \end{aligned}$$

and so $\|Ty\| \geq \|y\|$ for all $y \in P \cap \partial\Omega_2$. Consequently, by Part (i) of Theorem 3.1, it is follows that T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ and this implies that our given LBVP (1.1), (1.2) has a positive solution.

Next, assume we are in the sublinear case

$$f_0 = \infty, \quad f_\infty = 0.$$

Choose $r_1 > 0$ such that

$$f(t, y_1, y_2, ..., y_n) \geq \overline{\eta} y_n,$$

for $0 < y \leq r_1$, $t \in [0, \sigma(1)]$, where

$$\overline{\eta} \geq \mu$$

where μ is given in the first part of proof. Then for $y \in P$ and $\|y\| = r_1$, we have

$$\begin{aligned} Ty(t_0) &= \int_0^{\sigma(1)} (-1)^n G_n(t_0, s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\ &\geq \int_\xi^{\omega} (-1)^n G_n(t_0, s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s))\triangle s \\ &\geq \overline{\eta} \int_\xi^{\omega} (-1)^n G_n(t_0, s) y^{\triangle^{2(n-1)}}(s)\triangle s \\ &\geq \overline{\eta}\gamma_n^* \int_\xi^{\omega} (-1)^n G_n(t_0, s) \|y\|\triangle s \\ &= \|y\| \overline{\eta}\gamma_n^* \frac{1}{\mu\gamma_n^*} \\ &= \|y\| \frac{\overline{\eta}}{\mu} \geq \|y\| = r_1. \end{aligned}$$

Therefore, if $\Omega_1 \subset B$ is a ball of radius r_1 centered at the origin, then for $y \in P \cap \partial\Omega_1$, we have

$$\|Ty\| \geq \|y\|.$$

Next, since $f_\infty = 0$, there exists $\overline{r_2} > 0$ such that

$$f(t, y_1, y_2, ..., y_n) \leq \eta y_n,$$

for $y_n \geq \overline{r_2}$, $t \in [0, \sigma(1)]$, where η is defined by

$$\frac{1}{\eta} \geq \left(\frac{\sigma(1)}{4}\right)^n.$$

We consider two cases.

Case I. Suppose $f(t, y_1, y_2, ..., y_n)$ is bounded on $[0, \sigma(1)] \times (0, \infty)$. In this case, there is an $N > 0$ such that

$$f(t, y_1, y_2, ..., y_n) \leq N,$$

for $t \in [0, \sigma(1)], \quad y_n \in (0, \infty)$. In this case, choose

$$r_2 = \max\left\{2r_1, \frac{N}{\eta}\right\}.$$

Then for $y \in P$ with $\|y\| = r_2$, we have for all $t \in [0, \sigma(1)]$,

$$\begin{aligned} Ty(t) &= \int_0^{\sigma(1)} (-1)^n G_n(t,s) f(s, y^\sigma(s), y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s)) \triangle s \\ &\leq N \int_0^{\sigma(1)} (-1)^n G_n(t,s) \triangle s \\ &\leq N \left(\frac{\sigma(1)}{4}\right)^n \\ &\leq \frac{N}{\eta} \leq r_2, \end{aligned}$$

so that $\|Ty\| \leq \|y\|$.

Case II. Assume $f(t, y_1, y_2, ..., y_n)$ is unbounded on $[0, \sigma(1)] \times (0, \infty)$. In this case

$$g(r) := \max\{f(t, y_1, y_2, ..., y_n) : t \in [0, \sigma(1)], 0 \leq y_n \leq r\}$$

satisfies

$$\lim_{r \to \infty} g(r) = \infty.$$

We can therefore choose

$$r_2 > \max\{2r, \overline{r_2}\}$$

such that

$$g(r_2) \geq g(r)$$

for $0 \leq r \leq r_2$ and hence, for $y \in P$ and $\|y\| = r_2$, we have

$$\begin{aligned} Ty(t) &= \int_0^{\sigma(1)} (-1)^n G_n(t,s) f(s, y^\sigma(s) y^{\triangle\triangle}(s), ..., y^{\triangle^{2(n-1)}}(s)) \triangle s \\ &\leq \int_0^{\sigma(1)} (-1)^n G_n(t,s) g(r_2) \triangle s \\ &\leq \eta r_2 \int_0^{\sigma(1)} (-1)^n G_n(t,s) \triangle s \\ &\leq \eta r_2 \left(\frac{\sigma(1)}{4}\right)^n \\ &\leq r_2 = \|y\|, \end{aligned}$$

and again we hence $\|Ty\| \leq \|y\|$ for $y \in P \cap \partial\Omega_2$, where $\Omega_2 = \{y \in B : \|y\| \leq r_2\}$ in both cases. It follows from part (ii) of Theorem 3.1 that T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ and this implies that our given LBVP (1.1), (1.2) has a positive solution. □

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