Turk J Math
33 (2009) , 205 – 213.

doi:10.3906/mat-0801-16

Some properties of gr-multiplication ideals

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Abstract

In this paper, we study some of the properties of gr-multiplication ideals in a graded ring R. We first characterize finitely generated gr-multiplication ideals and then give a characterization of gr-multiplication ideals by using the gr-localization of R. Finally we determine the set of gr- P-primary ideals of R when P is a gr-multiplication gr-prime ideal of R.

Key Words: Graded Rings, Graded Ideals, Gr-primary Ideals and Gr-multiplication Ideals.

1. Introduction

Let G be a group. A ring (R, G) is called a G-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for each g and h in G. For simplicity, we will denote the graded ring (R, G) by R. An element of a graded ring R is called homogeneous if it belongs to $\bigcup_{g \in G} R_g$ and this set of homogeneous elements is denoted by $h(R)$. If $x \in R_g$ for some $g \in G$, then we say that x is of degree g. A graded ideal I of a graded ring R is an ideal verifying $I = \bigoplus_{g \in G} (I \cap R_g) = \bigoplus_{g \in G} I_g$. Equivalently, I is graded in R if and only if I has a homogeneous set of generators. If $R = \bigoplus_{g \in G} R_g$ and $R' = \bigoplus_{g \in G} R'_g$ are two graded rings, then a mapping $\eta : R \to R'$ with $\eta(1_R) = 1_{R'}$ is called a gr-homomorphism if $\eta(R_g) \subseteq R'_g$ for all $g \in G$. A graded ideal P of a graded ring R is called gr-prime if whenever $x, y \in h(R)$ with $xy \in P$, then $x \in P$ or $y \in P$. A graded ideal M of a graded ring R is called gr-maximal if it is maximal in the lattice of graded ideals of R. A graded ring R is called a gr-local ring if it has unique gr-maximal ideal.

Let R be a graded ring and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fractions $S^{-1}R$ is a graded ring which is called the gr-ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g \in G} \left(S^{-1}R\right)_g$ where

$$\left(S^{-1}R\right)_g = \left\{\frac{r}{s} : r \in R \ , \ s \in S \ \text{ and } \ g = (deg \ \ s)^{-1} (deg \ \ r)\right\}.$$

2000 *AMS Mathematics Subject Classification:* 13 A 02.

Consider the ring gr-homomorphism $\eta : R \to S^{-1}R$ defined by $\eta(r) = \frac{r}{1}$. For any graded ideal I of R, the ideal of $S^{-1}R$ generated by $\eta(I)$ is denoted by $S^{-1}I$. Similar to non graded case, one can prove that

$$S^{-1}I = \left\{ \lambda \in S^{-1}R : \lambda = \frac{r}{s} \text{ for } r \in I \text{ and } s \in S \right\}$$

and that $S^{-1}I \neq S^{-1}R$ if and only if $S \cap I = \Phi$. Moreover, similar to the non graded case, we have the following properties for graded ideal I and J of R:

(1) $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$,

(2) $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$ and

(3) $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$ if J is finitely generated.

If $\mathcal{J}$ is a graded ideal in $S^{-1}R$, then $\mathcal{J} \cap R$ will denote the graded ideal $\eta^{-1}(\mathcal{J})$ of R. Moreover, similar to the non graded case one can prove that $S^{-1}(\mathcal{J} \cap R) = \mathcal{J}$.

Let P be any gr-prime ideal of a graded ring R and consider the multiplicatively closed subset $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_P^g and we call it the gr-localization of R. This ring is gr-local with the unique gr-maximal $S^{-1}P$ which will be denoted by PR_P^g. Moreover, for graded ideals I and J of R, if $IR_P^g = JR_P^g$ for every gr-prime (gr-maximal) ideal P of R, then $I = J$. For a positive integer n the graded ideal $(PR_P^g)^n \cap R$ of R is denoted by $P^{(n)g}$. For more definitions and theorems about gr-ring of fractions of graded rings, one can see [8].

Let I be a graded ideal in a graded ring R. The graded radical of I (denoted by g-$rad(I)$) is defined in [9] as the set of all $x \in R$ such that for each $g \in G$, there exists $n_g \gneqq 0$ such that $x_g^{n_g} \in I$.

A graded ideal Q of a graded ring R is called gr-primary if $Q \neq R$ and whenever $a, b \in h(R)$ with $ab \in Q$, then $a \in Q$ or $b \in$ g-$rad(Q)$. If Q is gr-primary ideal of R, then g-$rad(Q) = P$ is a gr-prime ideal of R and we say that Q is gr-P-primary. If I is a graded ideal of R with g-$rad(I) = M$, a gr-maximal ideal of R, then I is gr-M-primary; see [9].

Recall that a graded ring R is called gr-PIR if every graded ideal of R is gr-principal, where a gr-principal ideal of a graded ring R is generated by some homogeneous element in R. Also, recall that a graded ring R is called gr-$SPIR$ if R has unique gr-prime ideal P and every graded ideal of R is a power of P. Similar to the non graded case, one can prove that if R is a gr-$SPIR$, then R is a gr-PIR and the unique gr-prime ideal of R is nilpotent.

An ideal I of a ring R is called multiplication if whenever J is an ideal of R with $J \subseteq I$, then there is an ideal K of R such that $J = IK$. If every ideal in a ring R is multiplication, then R is called a multiplication ring. Multiplication ideals and rings have been studied in detail in [1], [2] and [7]. A generalization of multiplication graded ideals and rings to gr-multiplication ideals and rings have been studied in [3], [4] and [5].

In this paper, we study more properties of gr-multiplication ideals in a graded ring R and give a charaterization for finitely generated gr-multiplcation ideals. For an ideal I of a graded ring R, we define the graded ideal $\theta^g(I)$ and use it togother with the gr-localization of R to give a general characterization for gr-multiplication ideals. Finally, we determine the set of gr-P-primary ideals of a graded ring R when P is both gr-prime and gr-multiplication in R.

2. Properties for gr-multiplication ideals

Definition 2.1 *Let R be a graded ring graded by the group G. A graded ideal I of R is called a gr-multiplication ideal of R if whenever J is a graded ideal of R with $J \subseteq I$, then there is a graded ideal K of R such that $J = KI$. If every graded ideal in a graded ring R is gr-multiplication, then R is called a gr-multiplication ring.*

Clearly, any graded ideal which is multiplication is a gr-multiplication ideal. A graded ideal I of a graded ring R is called a gr-invertible ideal if there exists a graded ideal J of R such that $IJ = R$. Also, one can easily see that every gr-invertible ideal is gr-multiplication. In particular, the gr-principal ideals are gr-multiplication.

The class of gr-multiplication domains has been characterized in [5] as the class of gr-Dedekind domains which is the class of graded domains in which every graded ideal is gr-invertible. In [10], we can see an example of a gr-multiplication ring which is not multiplication. Indeed, the group ring $R[\mathbb{Z}]$, where R is a Dedekind domain is gr-Dedekind domain and so it is gr-multiplication domain. On the other hand, if R is not a field, then $R[\mathbb{Z}]$ is not a Dedekind domain and so it is not a multiplication domain, see [6].

If I and J are two graded ideals in a graded ring R, then the ideal $(J : I) = \{x \in R : xI \subseteq J\}$ is a graded ideal, see [4]. In the following theorem, we can see another equivalent definition of gr-multiplication ideals.

Theorem 2.2 *Let I be a graded ideal in a graded ring R. Then I is gr-multiplication iff $I \cap J = I(J : I)$ for every graded ideal J of R.*

Proof. Suppose that $J \subseteq I$ for a graded ideal J of R. Then $J = I \cap J = I(J : I) = IJ$

Conversely, suppose that I is a gr-multiplication ideal in R. Let J be any graded ideal of R. Then $I \cap J \subseteq I$ and so there is a graded ideal K of R with $I \cap J = IK$. Therefore, $K \subseteq (I \cap J : I) \subseteq (J : I)$ and then $I \cap J = IK \subseteq I(J : I)$. On the other hand, clearly, $I(J : I) \subseteq I \cap J$ and therefore, $I(J : I) = I \cap J$. □

The following theorem is a characterization of gr-multiplication ideals in gr-local rings; see [3].

Theorem 2.3 *Let R be a gr-local ring with the unique gr-maximal ideal M. A graded ideal I of R is gr-multiplication iff I is gr-principal.*

Proof. If $I = \langle x \rangle$ for some $x \in h(R)$, then clearly I is a gr-multiplication ideal of R.

Conversely, suppose that I is gr-multiplication in R. Since I is graded, then it is generated by a set of homogeneous elements, say, $\{a_\alpha : \alpha \in \Lambda\}$. Now, for each $\alpha \in \Lambda$, $\langle a_\alpha \rangle \subseteq I$ and so there is a graded ideal B_α of R such that $\langle a_\alpha \rangle = IB_\alpha$. Therefore, $I = \sum_{\alpha\in\Lambda} \langle a_\alpha \rangle = \sum_{\alpha\in\Lambda} IB_\alpha = I \sum_{\alpha\in\Lambda} B_\alpha$. If $\sum_{\alpha\in\Lambda} B_\alpha = R$, then $B_{\alpha_0} = R$ for some $\alpha_0 \in \Lambda$, since otherwise if $B_\alpha \subset R$ for each $\alpha \in \Lambda$, then $B_\alpha \subseteq M$ for each $\alpha \in \Lambda$ and so $R = \sum_{\alpha\in\Lambda} B_\alpha \subseteq M$, a contradiction. Therefore, $\langle a_{\alpha_0} \rangle = IB_{\alpha_0} = I$ and I is gr-principal. If $\sum_{\alpha\in\Lambda} B_\alpha \neq R$, then $\sum_{\alpha\in\Lambda} B_\alpha \subseteq M$ and then $I = I \sum_{\alpha\in\Lambda} B_\alpha \subseteq IM \subseteq I$. Therefore, $I = IM$ and then $I = 0$ by proposition 2.4 in [4]. It follows that I is gr-principal. □

Theorem 2.4 *If I is a gr-multiplication ideal of a graded ring R and $S \subseteq h(R)$ is a multiplicatively closed subset of R, then $S^{-1}I$ is a gr-multiplication ideal of $S^{-1}R$.*

Proof. Let $\mathcal{J}$ be a graded ideal of $S^{-1}R$ such that $\mathcal{J} \subseteq S^{-1}I$. Then $\mathcal{J} = S^{-1}J$ for some graded ideal J of R. Now, $I \cap J \subseteq I$ and therefore, there is a graded ideal K of R such that $I \cap J = IK$. Thus

$$\mathcal{J} = S^{-1}I \cap S^{-1}J = S^{-1}(I \cap J) = S^{-1}(IK) = (S^{-1}I)(S^{-1}K).$$

Therefore, $S^{-1}I$ is a gr-multiplication ideal in $S^{-1}R$. □

Definition 2.5 *A graded ideal I of a graded ring R is called locally gr-principal if IR_P^g is gr-principal for any gr-prime ideal P of R.*

As a corollary of theorem 2.3, we have the following.

Corollary 2.6 *Any gr-multiplication ideal in a graded ring R is locally gr-principal.*

In [3], it has been proved that if I is a finitely generated graded ideal of R, then I is gr-multiplication if and only if I is locally gr-principal. In the following theorem, we can see another characterization of finitely generated gr-multiplication ideals. First, we have the following technical lemma.

Lemma 2.7 *Let R be a gr-local ring with gr-maximal ideal M and I be a gr-principal ideal in R. If $I = \langle a_1, a_2, ..., a_n \rangle$, then $I = \langle a_j \rangle$ for some $j \in \{1, 2, ..., n\}$.*

Proof. Suppose that $I = \langle a \rangle$ for some $a \in h(R)$ and suppose that $I = \langle a_1, a_2, ..., a_n \rangle$. Then $a = \sum_{i=1}^{n} a_i r_i$ where $r_i \in R$ for all i. Also for all i, $a_i = ax_i$ for some $x_i \in R$. Thus, $a(1 - \sum_{i=1}^{n} x_i r_i) = 0$. If $1 - \sum_{i=1}^{n} x_i r_i$ is a unit in R, then $a = 0$ and so $I = \langle 0 \rangle = \langle a_i \rangle$ for all $i = 1, 2, ..., n$ since $a_i = 0$ for all i. If $1 - \sum_{i=1}^{n} x_i r_i$ is not a unit, then $\sum_{i=1}^{n} x_i r_i \notin M$ and so $\sum_{i=1}^{n} x_i r_i$ is a unit. Therefore, there is some $j \in \{1, 2, ..., n\}$ such that $x_j r_j$ is a unit and then x_j is also a unit. Hence, $a = ax_j x_j^{-1} = a_j x_j^{-1}$ and $I = \langle a \rangle \subseteq \langle a_j \rangle$. Hence, $I = \langle a_j \rangle$ for some $j \in \{1, 2, ..., n\}$. □

Theorem 2.8 *Let $I = \langle a_1, a_2, ..., a_n \rangle$ be a finitely generated graded ideal of a graded ring R. Then the following are equivalent.*
(1) I is gr-multiplication.
(2) I is locallygr-principal.
(3) $\sum_{i=1}^{n} (\langle a_i \rangle : I_i) = R$, where $I_i = \langle a_1, a_2, ..., a_{i-1}, a_{i+1}, ..., a_n \rangle$.

Proof. (1) $\Leftrightarrow$ 2): see [3].

(2) $\Rightarrow$ 3): Suppose that I is locally gr-principal. Then for each gr-prime ideal P of R, we have $IR_P^g = \left\langle \frac{a_1}{1}, \frac{a_2}{1}, ..., \frac{a_n}{1} \right\rangle = \left\langle \frac{a_j}{1} \right\rangle = \langle a_j \rangle R_P^g$ for some $j \in \{1, 2, ..., n\}$ by lemma 2.7. Hence, for any gr-prime ideal P of R, $(\langle a_j \rangle R_P^g : I_j R_P^g) = R_P^g$, and then

$$(\sum_{i=1}^{n}(\langle a_i \rangle : I_i))R_P^g = \sum_{i=1}^{n}(\langle a_i \rangle R_P^g : I_i R_P^g) = R_P^g,$$

since I_i is finitely generated for each i. Therefore, $\sum_{i=1}^{n}(\langle a_i \rangle : I_i) = R$.

(3) $\Rightarrow$ 2): Suppose that $\sum_{i=1}^{n}(\langle a_i \rangle : I_i) = R$. Then for any gr-prime ideal P of R, we have

$$\sum_{i=1}^{n}(\langle a_i \rangle R_P^g : IR_P^g) = (\sum_{i=1}^{n}(\langle a_i \rangle : I))R_P^g = (\sum_{i=1}^{n}(\langle a_i \rangle : I_i))R_P^g = R_P^g.$$

Therefore, there is $j \in \{1, 2, ..., n\}$ such that $(\langle a_j \rangle R_P^g : IR_P^g) = R_P^g$ and then $IR_P^g \subseteq \langle a_j \rangle R_P^g = \left\langle \frac{a_j}{1} \right\rangle$. It follows that $IR_P^g = \left\langle \frac{a_j}{1} \right\rangle$ for each gr-prime ideal P of R and I is locally gr-principal. □

If I is a graded ideal in a graded ring R, then we define the subset $\theta^g(I)$ of R as $\theta^g(I) = \sum_{x \in I \cap h(R)} (\langle x \rangle : I)$. Clearly, $\theta^g(I)$ is a graded ideal of R.

Lemma 2.9 *Let I be a gr-multiplication ideal of a graded ring R. Then*
(1) $I = I\theta^g(I)$;
(2) $J = J\theta^g(I)$ for any graded ideal $J \subseteq I$.

Proof. (1) For $x \in I \cap h(R)$, $\langle x \rangle \subseteq I$ and so $\langle x \rangle = I(\langle x \rangle : I)$. Therefore

$$I = \sum_{x \in I \cap h(R)} \langle x \rangle = \sum_{x \in I \cap h(R)} I(\langle x \rangle : I) = I \sum_{x \in I \cap h(R)} (\langle x \rangle : I) = I\theta^g(I).$$

(2) Suppose that J is a graded ideal with $J \subseteq I$. Then $J = IK$ for some graded ideal K of R. Hence,

$$J = IK = I\theta^g(I)K = IK\theta^g(I) = J\theta^g(I).$$

□

Lemma 2.10 *Let I and J be graded ideals in a graded ring R and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then*
(1) $\theta^g(I)\theta^g(J) \subseteq \theta^g(IJ)$;
(2) $S^{-1}(\theta^g(I)) \subseteq \theta^g(S^{-1}I)$.

Proof. (1) Let $a \in I \cap h(R)$ and let $b \in J \cap h(R)$. It is enough to prove that $(\langle a\rangle : I)(\langle b\rangle : J) \subseteq (\langle ab\rangle : IJ)$. Let $\sum_{i=1}^{n} x_i y_i \in (\langle a\rangle : I)(\langle b\rangle : J)$ where $x_i \in (\langle a\rangle : I)$ and $y_i \in (\langle b\rangle : J)$ for $i = 1, 2, ..., n$. Then $x_i I \subseteq \langle a\rangle$ and $y_i J \subseteq \langle b\rangle$ for $i = 1, 2, ..., n$. Hence, $x_i y_i IJ \subseteq \langle ab\rangle$ and then $x_i y_i \in (\langle ab\rangle : IJ)$. Therefore, $\sum_{i=1}^{n} x_i y_i \in (\langle ab\rangle : IJ)$.

(2)
$$\begin{aligned} S^{-1}(\theta^g(I)) &= S^{-1}(\sum_{x\in I\cap h(R)} (\langle x\rangle : I)) = \sum_{x\in I\cap h(R)} S^{-1}(\langle x\rangle : I) \\ &\subseteq \sum_{x\in I\cap h(R)} (\left\langle \frac{x}{1}\right\rangle : S^{-1}I) \subseteq \theta^g(S^{-1}I). \end{aligned}$$

□

Recall that a graded ideal I in a graded ring R is called gr-finitely generated if I is generated by a finite set of homogeneous elements in R.

Theorem 2.11 *Let I be a graded ideal in a graded ring R. Then I is gr-finitely generated and locally gr-principal iff $\theta^g(I) = R$.*

Proof. Let M be a gr-maximal ideal in R. Then $IR_M^g = \langle x\rangle R_M^g$ for some $x \in I \cap h(R)$. Hence, $R_M^g = (\langle x\rangle R_M^g : IR_M^g) = (\langle x\rangle : I) R_M^g$ since I is gr-finitely generated. Therefore, $R_M^g = \theta^g(I)R_M^g$ and then $\theta^g(I) = R$.

Conversely, suppose $\theta^g(I) = R$. Then there exist $x_1, x_2, ..., x_n \in I \cap h(R)$ such that $R = \theta^g(I) = (\langle x_1\rangle : I) + (\langle x_2\rangle : I) + ... + (\langle x_n\rangle : I)$. Thus,

$$\begin{aligned} I &= I\theta^g(I) = I(\langle x_1\rangle : I) + I(\langle x_2\rangle : I) + ... + I(\langle x_n\rangle : I) \\ &\subseteq \langle x_1\rangle + \langle x_2\rangle + ... + \langle x_n\rangle \subseteq I. \end{aligned}$$

So, $I = \langle x_1, x_2, ..., x_n\rangle$ is gr-finitely generated. Now, let M be a gr-maximal ideal of R. Since $\theta^g(I) = R$, there is $x \in I \cap h(R)$ with $(\langle x\rangle : I) \nsubseteq M$. Therefore, there exists $r \in R - M$ with $rI \subseteq \langle x\rangle$ and then $rIR_M^g = \langle r\rangle R_M^g\, IR_M^g = IR_M^g \subseteq \langle x\rangle R_M^g$. Hence, $IR_M^g = \langle x\rangle R_M^g$ for any gr-maximal ideal M of R and so I is locally gr-principal. □

Definition 2.12 *A graded ideal I of a graded ring R is called meet-gr-principal if $JI \cap K = (J \cap (K : I))I$ for all graded ideals J and K of R.*

We are ready now for the following characterization of gr-multiplication ideals similar to that in the non graded case; see [1].

Theorem 2.13 *Let I be a graded ideal in a graded ring R. Then the following are equivalent:*
(1) I is meet-gr-principal.
(2) I is gr-multiplication.
(3) $IR_M^g = \langle 0\rangle R_M^g$ for any gr-maximal ideal M of R with $M \supseteq \theta^g(I)$.

Proof. (1) $\Rightarrow$ 2): Let J be a graded ideal of R with $J \subseteq I$. Then $J = RI \cap J = (R \cap (J:I))\, I = (J:I)\, I$ and then I is a gr-multiplication ideal by theorem 2.2.

(2) $\Rightarrow$ 3): Suppose that I is a gr-multiplication ideal. Let M be any gr-maximal ideal of R such that $\theta^g(I) \subseteq M$. Let $x \in I \cap h(R)$. Then $\langle x \rangle$ is a graded ideal and $\langle x \rangle = \langle x \rangle\, \theta^g(I)$ by lemma 2.9 and so, $\langle x \rangle R^g_M = \langle x \rangle R^g_M\ \theta^g(I) R^g_M$. By proposition 2.4 in [4], we see that $\langle x \rangle R^g_M = \langle 0 \rangle R^g_M$ and so $IR^g_M = \langle 0 \rangle R^g_M$.

(3) $\Rightarrow$ 1): Let J and K be graded ideals of R. We prove that $JI \cap K = (J \cap (K:I))\, I$. Clearly, $JI \cap K \supseteq (J \cap (K:I))\, I$ is always true. We prove the other containment locally. Let M be a gr-maximal ideal of R. If $\theta^g(I) \subseteq M$, then $IR^g_M = \langle 0 \rangle R^g_M$ by assumption and so $((J \cap (K:I))\, I)\, R^g_M = \langle 0 \rangle R^g_M = (JI \cap K)\, R^g_M$. Suppose that $\theta^g(I) \nsubseteq M$. Then $(\langle x \rangle : I) \nsubseteq M$ for some $x \in I \cap h(R)$ and so there is $r \in R$ such that $rI \subseteq \langle x \rangle$ and $r \notin M$. Let $b = y_1 z_1 + y_2 z_2 + ... + y_n z_n \in JI \cap K$ where $y_k \in J$ and $z_k \in I$ for $k = 1, 2, ..., n$. Then there exist $r_1, r_2, ..., r_n \in R$ such that

$$\begin{aligned} rb &= r(y_1 z_1 + y_2 z_2 + ... + y_n z_n) = y_1(rz_1) + y_2(rz_2) + ... + y_n(rz_n) \\ &= y_1(r_1 x) + y_2(r_2 x) + ... + y_n(r_n x) = (y_1 r_1 + y_2 r_2 + ... + y_n r_n)\, x. \end{aligned}$$

where the third equality holds since $rI \subseteq \langle x \rangle$. Now,

$$(y_1 r_1 + y_2 r_2 + ... + y_n r_n)\, rI \subseteq (y_1 r_1 + y_2 r_2 + ... + y_n r_n)\, \langle x \rangle = \langle rb \rangle \subseteq K.$$

Hence,

$$(y_1 r_1 + y_2 r_2 + ... + y_n r_n)\, r \in J \cap (K:I).$$

and then

$$\frac{y_1 r_1 + y_2 r_2 + ... + y_n r_n}{1} \in (J \cap (K:I))\, R^g_M.$$

Now,

$$\frac{b}{1} = (\frac{r}{1})^{-1}(\frac{rb}{1}) = (\frac{r}{1})^{-1}(\frac{y_1 r_1 + y_2 r_2 + ... + y_n r_n}{1})(\frac{x}{1}) \in (J \cap (K:I))\, R^g_M I R^g_M.$$

Therefore, $(JI \cap K) R^g_M = (J \cap (K:I))\, R^g_M\ IR^g_M$ for any gr-maximal ideal M of R and so $JI \cap K = (J \cap (K:I))\, I$. □

We have the following as a corollary of the previous theorem and lemma 2.10.

Corollary 2.14 *If I and J are gr-multiplication ideals of a graded ring R, then IJ is gr-multiplication.*

Proof. Let M be a gr-maximal ideal of R such that $\theta^g(IJ) \subseteq M$. Then $\theta^g(I)\theta^g(J) \subseteq \theta^g(IJ) \subseteq M$ and so either $\theta^g(I) \subseteq M$ or $\theta^g(J) \subseteq M$. Hence, by theorem 2.13, either $IR^g_M = 0R^g_M$ or $JR^g_M = 0R^g_M$. In both cases, $(IJ)R^g_M = 0R^g_M$ and then IJ is a gr-multiplication ideal of R again by theorem 2.13. □

3. Gr-primary ideals with gr-multiplication gr-radicals

Definition 3.1 *Let P be a gr-prime ideal in a graded ring R. Then we define the graded rank of P (denoted by gr-rank(P)) as the supremum of the lengths of all chains of distinct proper gr-prime ideals of R having P*

as last term. The gr-dimension of a graded ring R is defined as the supremum of the lengths of all chains of distinct gr-prime ideals of R and is denoted by gr-dim(R).

Now, any gr-prime ideal in the graded ring R_P^g is of the form $P'R_P^g$ where P' is a gr-prime ideal of R with $P' \subseteq P$. Therefore we conclude that gr-$dim(R_P^g) =$ gr-$rank(P)$. Recall that a gr-prime ideal P of R is called minimal gr-prime over a graded ideal I if there is no gr-prime ideal Q of R such that $I \subseteq Q \subset P$.

Definition 3.2 *Let I be a graded ideal in a graded ring R. Then the graded rank of I (denoted by gr-$rank(I)$) is defined as the infimum of the values of gr-$rank(P)$ as P runs over all of the minimal gr-prime ideals of I.*

Theorem 3.3 *Let I be a gr-multiplication ideal in a graded ring R. If gr-$rank(I) \gneqq 0$, then I is gr-finitely generated.*

Proof. Suppose that I is not gr-finitely generated, then by theorem 2.11, $\theta^g(I) \neq R$ and, therefore, $\theta^g(I) \subseteq M$ for some gr-maximal ideal M of R. Hence, $IR_M^g = 0R_M^g$ by theorem 2.13 and so gr-$rank(I) \leq$ gr-$rank(IR_M^g) = 0$, a contradiction. □

Now, in the following main theorem, we determine the set of all gr-P-primary ideals of a graded ring R where P is any gr-prime ideal of R that is gr-multiplication . First, we have the following lemma.

Lemma 3.4 *If I is a graded ideal of a graded ring R such that g-$rad(I)$ is gr-finitely generated, then there exists a positive integer t such that $(g\text{-}rad(I))^t \subseteq I$.*

Proof. Suppose that g-$rad(I) = \langle a_1, a_2, ..., a_n \rangle$ for $a_1, a_2, ..., a_n \in h(R)$. Then there exist $t_1, t_2, ..., t_n \in \mathbb{N}$ such that $a_i^{t_i} \in I$. Let $t = 1 + \sum_{i=1}^{n} (t_i - 1)$. Then $(\text{g-}rad(I))^t$ is a graded ideal generated by

$$L = \left\{ a_1^{k_1} a_2^{k_2} ... a_n^{k_n} : k_1, k_2, ..., k_n \in \mathbb{N} \ , \ \sum_{i=1}^{n} k_i = t \right\}.$$

If $k_i \lneqq t_i$ for all $i = 1, 2, ..., n$, then $\sum_{i=1}^{n} k_i \leq \sum_{i=1}^{n} (t_i - 1) \lneqq t$, a contradiction. Therefore, there exists j, $1 \leq j \leq n$ such that $k_j \geq t_j$ and then $a_1^{k_1} ... a_j^{k_j} ... a_n^{k_n} \in I$. Hence, $L \subseteq I$ and then $(\text{g-}rad(I))^t \subseteq I$. □

Theorem 3.5 *Let P be a gr-prime ideal of a graded ring R that is gr-multiplication. If gr-$rank(P) \gneqq 0$, then $\{P^n\}_{n=1}^{\infty}$ is the set of gr-P-primary ideals of R. If gr-$rank(P) = 0$, then there is a least positive integer m with $(PR_P^g)^m = 0R_P^g$ and in this case $\{P^n\}_{n=1}^{m}$ is the set of gr-P-primary ideals of R.*

Proof. Suppose that gr-$rank(P) \gneqq 0$, then P is gr-finitely generated by theorem 3.3. Let Q be a gr-P-primary ideal in R. Then $P^t \subseteq Q$ for some positive integer t by lemma 3.4. By passing to the graded ring R/P^t, we have, Q/P^t is gr-P/P^t-primary ideal and since clearly, gr-$rank(P/P^t) = 0$, it is enough to consider the case where gr-$rank(P) = 0$. Since P is gr-multiplication, then PR_P^g is gr-principal by theorem 2.3 and since gr-$rank(P) = 0$, then PR_P^g is the only gr-prime ideal of R_P^g and each graded ideal of R_P^g is a power of PR_P^g. Hence, R_P^g is a gr-$SPIR$ and so there is a least positive integer m such that $(PR_P^g)^m = \langle 0 \rangle R_P^g$ and the only graded ideals (which are gr-PR_P^g-primary) of R_P^g are $PR_P^g, (PR_P^g)^2, ..., (PR_P^g)^m$. Therefore, the only gr-P-primary ideals of R are $P^{(1)g}, P^{(2)g}, ..., P^{(m)g}$. Now, for a fixed i, $1 \leq i \leq m$, we have $P^i \subseteq P^{(i)g}$. Suppose k is the largest integer with $P^{(i)g} \subseteq P^k$. Since by corollary 2.14, P^k is gr-multiplication, there is a graded ideal A of R such that $P^{(i)g} = AP^k$ where $A \nsubseteq P$. Since $P^{(i)g}$ is gr-P-primary, $P^k \subseteq P^{(i)g}$ and so, $P^k = P^{(i)g}$. Now, $\left(P^{(k)g}\right) R_P^g = P^k R_P^g = P^{(i)g} R_P^g = P^i R_P^g$ and therefore, $i = k$ and $P^{(i)g} = P^i$. It follows that $P, P^2, ..., P^m$ are the only gr-P-primary ideals of R. □

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Received 15.01.2008