Turk J Math
33 (2009) , 215 – 225.
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doi:10.3906/mat-0710-22

Q-modules

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Abstract

In this paper we characterize Q-modules and almost Q-modules. Next we estblish some equivalent conditions for an almost Q-module to be a Q-module. Using these results, some characterizations are given for Noetherian Q-modules.

Key Words: Q-module, almost Q-module, Q-ring, almost Q-ring, Laskerian ring, Laskerian module, Noetherian spectrum, multiplication ideal and quasi-principal ideal.

1. Introduction

Throughout this paper R denotes a commutative ring with identity and all modules are unital R-modules. $L(R)$ denotes the lattice of all ideals of R. Throughout this paper M denotes a unital R-module. In this paper we introduce and study the concepts of Q-modules and almost Q-modules which are generalizations of Q-rings [4] and almost Q-rings [14]. We prove that a faithful R-module M is a Q-module if and only if R is a Q-ring and M is a multiplication module (see Theorem 1). It is shown that a faithful R-module M is a Q-module if and only if M is a Laskerian multiplication module in which every non maximal prime submodule is a finitely generated multiplication submodule (see Theorem 2). Next we establish several characterizations for almost Q-modules (see Theorem 3, Theorem 4, Theorem 5, Theorem 6 and Theorem 7). Using these results, some equivalent conditions are established for an almost Q-module to be a Q-module (see Theorem 8). Finally Noetherian Q-modules are characterized (see Theorem 9).

2. Basic notions

For any $x \in R$, the principal ideal generated by x is denoted by (x). For any ideal I of R, $\sqrt{I}$ denotes the radical of I. Recall that an ideal I of R is called a *multiplication ideal* if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$. Multiplication ideals have been extensively studied; for example, see [1], [2] and [11]. If I is a multiplication ideal, then I is locally principal [1, Theorem 1 and Page 761]. An ideal I of R is called a *quasi-principal ideal* [15, Exercise 10, Page 147] (or a principal element of $L(R)$ [19]) if it satisfies the identities (i) $(A \cap (B : I))I = AI \cap B$ and (ii) $(A + BI) : I = (A : I) + B$, for all $A,\ B \in L(R)$. Obviously,

2000 *AMS Mathematics Subject Classification:* Primary 05C38, 15A15; Secondary 05A15, 15A18.

every quasi-principal ideal is a multiplication ideal. It should be mentioned that every quasi-principal ideal is finitely generated and also a finite product of quasi-principal ideals of R is again a quasi-principal ideal [15, Exercise 10, Page 147]. In fact, an ideal I of R is quasi-principal if and only if it is finitely generated and locally principal (see [7, Theorem 4]) or [19, Theorem 2]). R is a *π-ring* if every principal ideal is a finite product of prime ideals of R. π-rings have been extensively studied; for example, see [16]. R is said to be a *Q-ring* [4] if every ideal is a finite product of primary ideals. R is said to be an *almost Q-ring* if R_M is a Q-ring for every maximal ideal M of R. For more informations on Q-rings and almost Q-rings, the reader is referred to [4], [5], [13] and [14]. R is said to be a *Laskerian ring* [10] if every proper ideal is a finite intersection of primary ideals. We say that R has *Noetherian spectrum* if R satisfies the ascending chain condition for radical ideals [20]. It is well known that R has Noetherian spectrum if and only if every prime ideal is the radical of a finitely generated ideal [20, Corollary 2.4]. Also it is well known that if R has Noetherian spectrum, then every ideal has only finitely many minimal primes.

A submodule N of M is *proper* if $N \neq M$. For any two submodules N and K of M, the ideal $\{a \in R \mid aK \subseteq N\}$ will be denoted by $(N : K)$. Thus $(O : M)$ is the annihilator of M. M is said to be a *faithful module* if $(O : M)$ is the zero ideal of R. M is said to be a *multiplication module* [6] if every submodule of M is of the form IM, for some ideal I of R. A submodule N of M is said to be a *multiplication submodule* if for every submodule $N_1 \subseteq N$, there exists an ideal J of R such that $N_1 = JN$. An R-module M is said to be *locally cyclic* if M_P is a cyclic R_P-module for all maximal ideals P of R.

A proper submodule N of M is said to be a *maximal submodule*, if it is not properly contained in any other proper submodule of M. A proper submodule N of M is a *prime submodule*, if for any $r \in R$ and $m \in M$, $rm \in N$ implies either $m \in N$ or $r \in (N : M)$. A proper submodule N of M is a *primary submodule* if for any $r \in R$ and $m \in M$, $rm \in N$ implies either $m \in N$ or $r^n \in (N : M)$ for some positive integer n. By a *minimal prime submodule over* a submodule N of M (or a *prime submodule minimal over* N), we mean a prime submodule which is minimal in the collection of all prime submodules containing N. Minimal prime submodules over the zero submodule are simply called the minimal prime submodules. Let N be a proper submodule of M. Then *M-radical* of N, denoted by $\sqrt{N}$, is defined as the intersection of all prime submodules of M containg N. It is well known that maximal submodules and prime submodules exist in multiplication modules (for details, see [9]). Also if M is a multiplication module, then for every proper submodule N of M, $\sqrt{N} = \sqrt{(N : M)}M$ (see [9, Theorem 2.12]).

For general background and terminology, the reader is referred to [15].

3. Q-modules and almost Q-modules

In this section we obtain several characterizations for Q-modules and almost Q-modules. Using these results, Noetherian Q-modules are characterized.

We shall begin with the following definitions.

Definition 1 *An R-module M is said to be a Q-module if every proper submodule N of M is of the form IM, where I is a finite product of primary ideals of R.*

Definition 2 *An R-module M is said to be an almost Q-module if for any maximal ideal P of R, the R_P-module M_P is a Q-module.*

Note that by definition, every Q-module is a multiplication module. Also a faithful R-module M is a Q-module if and only if M is a multiplication module and M is a Q-module in the sense of [21]. Again note that every Q-module is an almost Q-module, but the converse need not be true. Observe that Q-rings are Q-modules and almost Q-rings are almost Q-modules. Also cyclic modules over Dedekind domains are examples of Q-modules.

We now prove some useful lemmas.

Lemma 1 *Suppose M is a faithful multiplication R-module. If R contains only a finite number of minimal prime ideals, then M is finitely generated.*

Proof. First observe from Corollary 2.11 of [9], that the minimal prime submodules of M will be of the form PM where P is a minimal prime ideal of R and hence the set of minimal prime submodules of M is finite. Now the proof of Lemma 1 follows on applying Theorem 3.7 of [9]. $\square$

Lemma 2 *Suppose M is a faithful multiplication R-module. If R is a Q-ring or M is a Q-module, then M is finitely generated.*

Proof. If R is a Q-ring, then R contains only a finite number of minimal prime ideals. Suppose M is a Q-module. Then the zero submodule is of the form $(0) = IM$, where I is a finite product of primary ideals of R. As M is faithful, it follows that I is the zero ideal of R and hence R contains only a finite number of minimal prime ideals, so by Lemma 1, M is finitely generated. $\square$

The following theorem gives a characterization for Q-modules.

Theorem 1 *Suppose M is a faithful R-module. Then R is a Q-ring and M is a multiplication R-module if and only if M is a Q-module.*

Proof. The proof of the theorem follows from Lemma 2 and [21, Theorem 3 and Theorem 4]. It should be mentioned that the proof of Theorem 1 also follows from Lemma 2 and Theorem 3.1 of [9]. $\square$

An R-module M is said to be a Laskerian module [10], if every proper submodule is a finite intersection of primary submodules.

Lemma 3 *Suppose M is a faithful multiplication R-module. If R is a Laskerian ring or M is a Laskerian module, then M is finitely generated.*

Proof. If R is a Laskerian ring or M is a Laskerian module, then the zero ideal of R is a finite intersection of primary ideals and hence R contains only a finite number of minimal prime ideals. So by Lemma 1, M is finitely generated. $\square$

Lemma 4 *Suppose M is a faithful multiplication R-module. Then R is a Laskerian ring if and only if M is a Laskerian module.*

Proof. Assume R is Laskerian and M is a faithful multiplication module. Now by Lemma 3, M is finitely generated. It is well known that any finitely generated module over a Laskerian ring is Laskerian. Hence we obtain M is Laskerian. Conversely, assume that M is a faithful multiplication module and is Laskerian. Again by Lemma 3, M is finitely generated. Thus the ring R admits a finitely generated, faithful Laskerian module and hence R is isomorphic to a submodule of a Laskerian module and so R is Laskerian as an R-module. Thus R is a Laskerian ring. □

It is well known that R is a Q-ring if and only if R is a Laskerian ring in which every non maximal prime ideal is a finitely generated multiplication ideal [4, Theorem 10 and Theorem 13]. We extend this result to multiplication modules.

Theorem 2 *Suppose M is a faithful R-module. Then M is a Q-module if and only if M is a multiplication module and M is a Laskerian module in which every non maximal prime submodule is a finitely generated multiplication submodule.*

Proof. Suppose M is a Q-module. Then by Theorem 1, R is a Q-ring, so R is a Laskerian ring in which every non maximal prime ideal is a finitely generated multiplication ideal. By Lemma 4, M is a Laskerian module. Note that by Lemma 3, M is finitely generated. Let N be a non maximal prime submodule. Then $(N : M)$ is a non maximal prime ideal, so $(N : M)$ is a finitely generated multiplication ideal. Again by [17, Lemma 1.4] (or [9, Corollary 1.4]), N is a finitely generated multiplication submodule. The converse part follows from Theorem 1, Lemma 4 and [17, Lemma 1.4]. We also remark that the converse part can be easily verified with the help of Theorem 1, Lemma 4, [9, Corollary 2.11] and [9, Theorem 3.1]. □

Lemma 5 *Suppose M is a faithful and finitely generated multiplication R-module. Then R is an almost Q-ring if and only if M is an almost Q-module.*

Proof. Let P be a maximal ideal of R. Consider the R_P-module M_P. As M is a finitely generated faithful multiplication R-module, it follows that M_P is a faithful cyclic R_P-module. So by Theorem 1, M_P is a Q-module if and only if R_P is a Q-ring. Therefore R is an almost Q-ring if and only if M is an almost Q-module. □

Lemma 6 *Suppose M is a cyclic R-module. Then a submodule N of M is cyclic if and only if $N = rM$ for some $r \in R$.*

Proof. Let $M = Rx$ for some $x \in M$. Suppose $N = rM$ for some $r \in R$. Let $m = rx$. Then $Rm \subseteq N$. If $m' \in N$, then $m' = ry$ for some $y \in M$. But $y = r'x$ for some $r' \in R$. So $m' = ry = rr'x = r'm \in Rm$. Therefore $N = Rm$.

Conversely, assume that $N = Rm$ for some $m \in M$. Then $m = rx$ for some $r \in R$. So $N = Rm = R(rx) = r(Rx) = rM$. This completes the proof of the lemma. □

Lemma 7 *Let M be an R-module. Suppose every cyclic submodule of M is a finite intersection of primary submodules. Then for any submodule N of M, the following statements are equivalent:*

(i) *N is finitely generated and locally cyclic.*

(ii) *N is a multiplication module.*

(iii) *N is locally cyclic.*

Proof. It should be mentioned that Lemma 7 is Theorem 12 of [4] stated for modules. The proof of Lemma 7 is similar to the proof of [4, Theorem 12] except that (i)$\Rightarrow$(ii) follows from [9, Corollary 1.5] and (ii)$\Rightarrow$(iii) follows from [9, Theorem 1.2]. It is useful to remark that (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are true without the assumption that proper cyclic submodules of M admit primary decomposition. $\square$

Lemma 8 *Let R be an almost Q-ring. Let P be a non maximal prime ideal and let $Q \subseteq P$ be a primary ideal of R. Then Q is a multiple of P.*

Proof. Note that $P(Q :_R P) \subseteq Q$. We prove $Q = P(Q :_R P)$, by proving that for each maximal ideal M of R, $(P(Q :_R P))_M = Q_M$. First observe by [4, Lemma 11], that for any multiplicatively closed subset S of R, $(S^{-1}Q :_{S^{-1}R} S^{-1}P) = S^{-1}(Q :_R P)$. Let M be a maximal ideal of R. If P is not a subset of M, then $(P(Q :_R P))_M = P_M(Q_M :_{R_M} P_M) = Q_M$, since $P_M = R_M$. Let $P \subset M$. We have R_M is a Q-ring and P_M is a non-maximal prime ideal of R_M. Hence by [4, Lemma 5], P_M is a principal ideal of R_M. So $Q_M = P_M(Q_M :_{R_M} P_M) = (P(Q :_R P))_M$. $\square$

We now establish several characterizations for almost Q-modules.

Theorem 3 *Suppose M is a faithful and finitely generated multiplication R-module. Then M is an almost Q-module if and only if every non maximal prime submodule is locally cyclic.*

Proof. Let M be an almost Q-module. Then by Lemma 5, R is an almost Q-ring. So by [4, Lemma 5], every non maximal prime ideal is locally principal. Let N be a non maximal prime submodule. As M is a faithful and finitely generated multiplication R-module, it follows that $(N : M)$ is a non maximal prime ideal, so $(N : M)$ is locally principal. Let P be a maximal ideal of R. Then $N_P = ((N : M)M)_P = (N : M)_P M_P$, so by Lemma 6, N_P is a cyclic submodule of M_P. Therefore N is locally cyclic.

Conversely, assume that every non maximal prime submodule is locally cyclic. Let Q be a non maximal prime ideal of R. Then QM is a non maximal prime submodule of M. Let P be a maximal ideal of R. Suppose $Q \subseteq P$. Then by Lemma 6, $(QM)_P = Q_P M_P = I_P M_P$ for some principal ideal I_P of R_P. As M_P is a faithful cyclic R_P-module, by [9, Theorem 3.1], $Q_P = I_P$, so Q is locally principal and hence by [14, Theorem 1], R is an almost Q-ring. Consequently, M is an almost Q-module. $\square$

Theorem 4 *Suppose M is a faithful multiplication R-module in which every cyclic submodule of M is a finite intersection of primary submodules. Then M is an almost Q-module if and only if every non maximal prime submodule is a finitely generated multiplication submodule.*

Proof. Note that by hypothesis, the zero submodule is a finite intersection of primary submodules. So M contains only a finite number of minimal prime submodules and so by [9, Theorem 3.7], M is finitely generated. Now the result follows from Lemma 7 and Theorem 3. □

Theorem 5 *Suppose M is a faithful multiplication R-module. Suppose every principal ideal of R is a finite intersection of primary ideals. Then M is an almost Q-module if and only if every non maximal prime submodule is a finitely generated multiplication submodule.*

Proof. ($\Rightarrow$) We have by hypothesis and Lemma 1, M is finitely generated. Now by Lemma 5, R is an almost Q-ring. Let N be a non-maximal prime submodule of M. Then by [9, Corollary 2.11] and [9, Theorem 3.1], $N = PM$ for some non-maximal prime ideal P of R. It now follows from [4, Lemma 5] and [4, Theorem 12], that P is a finitely generated multiplication ideal and hence from [9, Corollary 1.4], we obtain that $N = PM$ is a multiplication submodule. It is clear that PM is finitely generated.

Let us prove the converse. As it is well known that any multiplication module is locally cyclic [9, Theorem 1.2], it follows from Theorem 3 that M is an almost Q-module. □

Theorem 6 *Suppose M is a faithful multiplication R-module. Suppose every principal ideal of R is a finite product of primary ideals. Then M is an almost Q-module if and only if every non maximal prime submodule is a multiplication submodule.*

Proof. Note that by hypothesis, M is finitely generated. Therefore every non maximal prime submodule is a multiplication submodule if and only if every non maximal prime ideal is a multiplication ideal. Now the result follows from [14, Corollary 1]. □

Theorem 7 *Suppose M is a faithful R-module in which every cyclic submodule is of the form IM, where I is a finite product of primary ideals. Then M is an almost Q-module if and only if every non maximal prime submodule is a multiplication submodule.*

Proof. Note that by hypothesis and by [9, Proposition 1.1], M is a finitely generated multiplication module. Suppose M is an almost Q-module. Then R is an almost Q-ring. Let N be a non maximal prime submodule. Then $N = PM$ for some non maximal prime ideal P of R. Let $x \in N$. By hypothesis, $Rx = IM$, where $I = Q_1Q_2 \cdots Q_n$ for some primary ideals $Q_1, Q_2, \cdots, Q_n$ of R. As $Rx = IM \subseteq N = PM$, it follows that $I \subseteq P$, so $Q_i \subseteq P$ for some primary ideal Q_i of R. Then by Lemma 8, Q_i is a multiple of P, so I is a multiple of P and hence $Rx = J_xN$ for some ideal J_x of R. Consequently, every submodule contained in N is of the form JN for some ideal J of R and hence N is a multiplication submodule. The converse part follows from Theorem 3. □

Lemma 9 *Suppose M is a faithful multiplication R-module. Suppose dim $M \leq 2$ and every submodule generated by two elements has only finitely many minimal primes. Then R has Noetherian spectrum.*

Proof. First we show that every minimal prime submodule is the radical of a finitely generated submodule. By hypothesis, M has only finitely many minimal primes. Let $N_1, N_2, ..., N_n$ be the distinct minimal prime submodules. If $n = 1$, then N_1 is the radical of the zero submodule. Suppose $n > 1$. Then by [8, Theorem 10], $N_1 \not\subseteq \overset{n}{\underset{i=2}{\cup}} N_i$. Choose any $x \in N_1$ such that $x \notin \overset{n}{\underset{i=2}{\cup}} N_i$. Let $L_1, L_2, ..., L_m$ be the distinct primes minimal over Rx. Then $N_1 = L_j$ for some j, say $N_1 = L_1$. If $m = 1$, then N_1 is the radical of Rx. Suppose $m > 1$. Then $N_1 \not\subseteq \overset{m}{\underset{i=2}{\cup}} L_i$. Choose any $y \in N_1$ such that $y \notin \overset{m}{\underset{i=2}{\cup}} L_i$. By hypothesis, $Rx + Ry$ has only finitely many minimal primes. Let $L'_1, L'_2, ..., L'_k$ be the distinct primes minimal over $Rx + Ry$. Note that $N_1 = L'_j$ for some j, say $N_1 = L'_1$. If $k = 1$, then N_1 is the radical of $Rx + Ry$. Suppose $k > 1$. Observe that any L'_j different from N_1 contains L_i properly, for some $i \neq 1$, and each L_i different from N_1, is non-minimal. So each L'_j is maximal, for $j = 2, 3, ..., k$. Choose any element $z \in N_1$ such that $z \notin \overset{k}{\underset{i=2}{\cup}} L'_i$. Now it can be easily shown that N_1 is the radical of $Rx + Ry + Rz$. Thus we have shown that every minimal prime submodule is the radical of a finitely generated submodule.

Next we show that every non-minimal prime submodule is the radical of a finitely generated submodule. Let N be a non-minimal prime submodule. Then $N \not\subseteq \overset{n}{\underset{i=1}{\cup}} N_i$. Choose any $x \in N$ such that $x \notin \overset{n}{\underset{i=1}{\cup}} N_i$. Let $L_1, L_2, ..., L_m$ be the distinct primes minimal over Rx. Then $N \supseteq L_j$ for some j, say $N \supseteq L_1$. If $m = 1$ and $N = L_1$, then N is the radical of Rx and so we are through. Suppose $m \geq 1$ and $L_1 \subset N$. Then $N \not\subseteq \overset{m}{\underset{i=1}{\cup}} L_i$. Choose any $y \in N$ such that $y \notin \overset{m}{\underset{i=1}{\cup}} L_i$. Then $Rx + Ry$ has only finitely many minimal primes and every prime minimal over $Rx + Ry$ is a maximal submodule. Therefore there exists a finitely generated submodule K such that N is the radical of K. Finally assume that $m > 1$ and $N = L_1$. Then $N \not\subseteq \overset{m}{\underset{i=2}{\cup}} L_i$. Choose any $y \in N$ such that $y \notin \overset{m}{\underset{i=2}{\cup}} L_i$. Let $L'_1, L'_2, ..., L'_k$ be the distinct primes minimal over $Rx + Ry$. Note that $N \supseteq L'_j$ for some j, say $N \supseteq L'_1$. Since $x \in L'_1$ and $L_1 = N \supseteq L'_1$, it follows that $N = L_1 = L'_1$. If $k = 1$, then N is the radical of $Rx + Ry$. Suppose $k > 1$. Then $N \not\subseteq \overset{k}{\underset{i=2}{\cup}} L'_i$ and each L'_i different from N, is maximal. Choose any element $z \in N$ such that $z \notin \overset{k}{\underset{i=2}{\cup}} L'_i$. Then N is the radical of $Rx + Ry + Rz$. Thus every prime submodule is the radical of a finitely generated submodule.

Now we show that R has Noetherian spectrum. Since M has only finitely many minimal prime submodules, it follows that M is finitely generated. Let P be a prime ideal of R. Then by [9, Corollary 2.11 and Theorem 3.1], PM is a proper prime submodule of M. So $PM = \sqrt{N}$ for some finitely generated submodule N of M. As M is a multiplication module, by [9, Theorem 2.12], it follows that $PM = \sqrt{N} = \sqrt{(N:M)}M$, so by [9, Theorem 3.1], $P = \sqrt{(N:M)}$. Also by [17, Lemma 1.4], $(N:M)$ is a finitely generated ideal and hence R has Noetherian spectrum. $\square$

The following theorem gives some equivalent conditions for an almost Q-module to be a Q-module.

Theorem 8 *Let M be a faithful R-module. Then the following statements are equivalent:*

(i) *M is a Q-module.*

(ii) *M is a finitely generated almost Q-module in which every submodule generated by two elements is a finite intersection of primary submodules.*

(iii) *M is an almost Q-module in which every submodule generated by two elements is of the form IM, where I is a finite product of primary ideals of R.*

(iv) *M is a multiplication module in which every submodule generated by two elements has only finitely many minimal primes and every non maximal prime submodule is a multiplication submodule.*

Proof. (i)$\Rightarrow$(ii). Suppose (i) holds. By Lemma 2 and Theorem 1, M is a finitely generated multiplication module. Clearly, M is an almost Q-module. By Theorem 2, M is a Laskerian module. Therefore (ii) holds.

(ii)$\Rightarrow$(iii). Suppose (ii) holds. As M is an almost Q-module and every Q-module is a multiplication module, it follows that M is locally cyclic. As M is finitely generated, by [6, Lemma 2], M is a multiplication module. Let $x, y \in M$. Then $Rx + Ry = N_1 \cap N_2 \cap \cdots \cap N_k$, where each N_i is a primary submodule of M. As M is a multiplication module, it follows that $Rx + Ry = (N_1 : M)M \cap (N_2 : M)M \cap \cdots \cap (N_k : M)M = [(N_1 : M) \cap (N_2 : M) \cap \cdots \cap (N_k : M)]M$. Let $I = (N_1 : M) \cap (N_2 : M) \cap \cdots \cap (N_k : M)$. Note that each $(N_i : M)$ is a primary ideal, so $(N_1 : M) \cap (N_2 : M) \cap \cdots \cap (N_k : M)$ is a primary decomposition of I. Without loss of generality, assume that $(N_1 : M) \cap (N_2 : M) \cap \cdots \cap (N_k : M)$ is a normal primary decomposition of I. As M is a faithful finitely generated multiplication module, by Theorem 4, every non maximal prime ideal is a finitely generated multiplication ideal. So by [3, Lemma 2], I is a finite product of primary ideals. So $Rx + Ry = IM$, where I is a finite product of primary ideals and hence (iii) holds.

(iii)$\Rightarrow$(iv). By (iii) and [9, Proposition 1.1], M is a multiplication module. By (iii) and Theorem 7, every non maximal prime submodule is a multiplication submodule. Note that by (iii), M contains only finitely many minimal prime submodules, so M is finitely generated. Let $x, y \in M$. Then $Rx + Ry = Q_1 Q_2 \cdots Q_n M$, where each Q_i is a primary ideal of R. Let N be a prime submodule minimal over $Rx + Ry$. Then $N = PM$ for some prime ideal P of R. As $Q_1 Q_2 \cdots Q_n M \subseteq PM$, by [9, Theorm 3.1], $Q_i \subseteq P$ for some i. So $Rx + Ry \subseteq \sqrt{Q_i} M \subseteq PM$. Therefore $\sqrt{Q_i} M = PM$ and hence $Rx + Ry$ has only finitely many minimal prime submodules. Hence (iv) holds.

(iv)$\Rightarrow$(i). Suppose (iv) holds. By (iv), M is a finitely generated multiplication module. Also by Lemma 5 and Theorem 3, R is an almost Q-ring. So $\dim R \leq 2$ and so $\dim M \leq 2$. Again by Lemma 9, R has Noetherian spectrum and hence every ideal has only finitely many minimal primes. Therefore by [14, Theorem 2(v)], R is a Q-ring and hence M is a Q-module. This completes the proof of the theorem. □

For any $I \in L(R)$ and for any prime ideal P minimal over I, we denote $P_I = \cap\{Q \in L(R) \mid Q$ is a P-primary ideal containing $I\}$. It can be easily seen that P_I is the smallest P-primary ideal containing I. For any $I \in L(R)$, we denote $I^* = \cap\{P_I \mid P$ is a prime ideal minimal over $I\}$.

Lemma 10 *Suppose every non-maximal prime ideal of R is a multiplication ideal and the maximal ideals of R are finitely generated. Let I be a quasi-principal ideal which has only finitely many minimal primes. Then I is a finite intersection of primary ideals.*

Proof. Observe that by hypothesis, [9, Theorem 1.2] and by Cohen's theorem, R is a locally Noetherian ring. Note that by hypothesis and by [14, Theorem 1], R is an almost Q-ring. So by [4, Corollary 6], $\dim R \leq 2$. By hypothesis, I^* is a finite intersection of primary ideals. Suppose I is not contained in any minimal prime. We show that $I = I^*$. Let M be a maximal ideal. If I is not contained in M, then $I_M = {I^*}_M$. Suppose $I \subseteq M$. If M is minimal over I, then $I_M = {I^*}_M$. Suppose M is not minimal over I. Then rank $M = 2$, so by [4, Corollary 6], R_M is a π-domain. As I is locally principal and R_M is a π-domain, it follows that $I_M = {I^*}_M$ (see the proof of [16, Theorem 1.2] or [5, Theorem 3]). This shows that $I_M = {I^*}_M$ for all maximal ideals containing I and hence $I = I^*$.

Now assume that $P_1, P_2, ..., P_m$ be the primes minimal over I. Let $P_1, P_2, ..., P_t$ be the non-maximal minimal primes and let $P_{t+1}, P_{t+2}, ..., P_m$ be the primes which are either maximal or rank one non-maximal primes. We show that, for each $i \in \{1, 2, ..., t\}$, the ideal $((I + P_i(P_i)_I) : (P_i)_I)$ has only finitely many minimal primes. Let $i \in \{1, 2, ..., t\}$. Since P_i is a multiplication ideal, by Lemma 9(i) of [14], P_i is properly contained in the ideal $((I+P_i(P_i)_I) : (P_i)_I)$. Since $\dim R \leq 2$, it follows that every prime minimal over $((I+P_i(P_i)_I) : (P_i)_I)$ is either a non-minimal maximal ideal or a rank one non-maximal prime. As every non-maximal prime is a multiplication ideal, by [2, Theorem 3], the rank one non-maximal primes are quasi-principal ideals. By hypothesis, the minimal primes over $((I + P_i(P_i)_I) : (P_i)_I)$ are finitely generated and so by [14, Lemma 5], the ideal $((I + P_i(P_i)_I) : (P_i)_I)$ has only finitely many minimal primes. Thus the ideals $((I + P_i(P_i)_I) : (P_i)_I)$ for $i = 1, 2, ..., t$ have only finitely many minimal primes, say $M_1, M_2, ..., M_n$. Again note that these are either non-minimal maximal ideals or rank one non-maximal prime ideals. Without loss of generality, assume that $M_1, M_2, ..., M_k$ are rank one maximal prime ideals and $M_{k+1}, M_{k+2}, ..., M_n$ are either rank two maximal ideals or rank one non-maximal prime ideals. Let M be any maximal ideal different from $M_1, M_2, ..., M_k$. We claim that $I_M = {I^*}_M$. Obviously, if I is not contained in M, then $I_M = {I^*}_M$. Suppose $I \subseteq M$. If either M is minimal over I or rank $M = 2$, then $I_M = {I^*}_M$. Suppose M is not minimal over I and rank $M = 1$. Then M is different from $M_1, M_2, ..., M_n$, so $((I + P_i(P_i)_I) : (P_i)_I) \nsubseteq M$ for $i = 1, 2, ..., t$ and hence by Nakayama's lemma $((P_i)_I)_M = I_M$ for all $P_i \subseteq M$ (see also the proof of [14, Lemma 10(i)]). Therefore $((P_i)_I)_M = I_M$ or $((P_i)_I)_M = R_M$ for $i = 1, 2, ..., t$. Consequently, $I_M = {I^*}_M$. If $I_{M_i} = {I^*}_{M_i}$ for $i = 1, 2, ..., k$, then $I_M = {I^*}_M$ for all maximal ideals, so $I = I^*$. Suppose $I_{M_i} \neq {I^*}_{M_i}$ for $i = 1, 2, ..., l$ $(1 \leq l \leq k)$. As R_{M_i} is a Laskerian ring, it follows that there exist M_i-primary Q_i such that $I_{M_i} = (I^*)_{M_i} \cap (Q_i)_{M_i}$ for $i = 1, 2, ..., l$. Then $I_M = (I^* \cap Q_1 \cap Q_2 \cap ... \cap Q_l)_M$ for all maximal ideals M of R. Therefore $I = I^* \cap Q_1 \cap Q_2 \cap ... \cap Q_l$ and hence I is a finite intersection of primary ideals. This completes the proof of the lemma. □

Lemma 11 *Suppose M is a faithful cyclic R-module. Let I be an ideal of R and $N = IM$ be a submodule of M. Then $N = Rx + Ry$ for some $x, y \in M$ if and only if $I = (a) + (b)$ for some $a, b \in R$.*

Proof. The proof of the lemma follows from Lemma 6 and [9, Theorem 3.1]. □

We now establish some equivalent conditions for Noetherian Q-modules.

Theorem 9 *Let M be a faithful R-module. Then the following statements are equivalent:*

(i) *M is a Noetherian Q-module.*

(ii) M *is a locally Noetherian* Q*-module.*

(iii) *The maximal submodules are locally finitely generated and every submodule generated by two elements is of the form* IM*, where* I *is a finite product of primary ideals of* R*.*

(iv) M *is an almost* Q*-module in which the maximal submodules are finitely generated and every cyclic submodule is of the form* IM*, where* I *is a finite product of primary ideals of* R*.*

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(iv). Suppose (iii) holds. By (iii), M is a finitely generated multiplication module. Let P be a maximal ideal of R. Consider the R_P-module M_P. Then PM is a maximal submodule of M. By (iii), $(PM)_P$ is finitely generated in M_P, so by [17, Lemma 1.4], $((PM)_P : M_P) = (PM : M)_P = P_P$ is finitely generated in R_P. Therefore P is a locally finitely generated ideal of R. Let I' be an ideal of R_P generated by two elements of R_P. Then I' is of the form $(x)_P + (y)_P$ for some $x, y \in R$. Let $I = (x) + (y)$. Then by Lemma 11, $(IM)_P$ is of the form N_P, where N is a submodule generated by two elements of M. So by hypothesis, $N = JM$, where J is a finite product of primary ideals of R. Therefore $(IM)_P = (JM)_P = J_P M_P$, so $I_P = J_P$ and hence I_P is a finite product of primary ideals of R_P. So by [14, Lemma 13], R_P is a Noetherian Q-ring and hence by Theorem 8, M is a Q-module and so R is a Q-ring. As the maximal ideals are locally finitely generated, by [14, Lemma 14], the maximal ideals are finitely generated and hence the maximal submodules are finitely generated. Therefore (iv) holds.

(iv)$\Rightarrow$(i). Suppose (iv) holds. By (iv), M is a finitely generated multiplication module. Also by hypothesis, the maximal ideals are finitely generated. Again by Theorem 7, non maximal prime submodules are multiplication submodules, so non maximal prime ideals are multiplication ideals. Let $x \in M$. Then by (iv) and [9, Theorem 3.1], $(Rx : M)$ is a finite product of primary ideals, so $(Rx : M)$ has only finitely many minimal primes. Note that it follows from [9, Theorem 3.1] that the ideal $(Rx : M)$ is finitely generated. As M is a faithful locally cyclic module, we have from Lemma 6 and [9, Theorem 3.1], that $(Rx : M)$ is locally cyclic. Thus $(Rx : M)$ is a quasi-principal ideal. So by Lemma 10, $(Rx : M)$ is a finite intersection of primary ideals and hence $Rx = (Rx : M)M$ is a finite intersection of primary submodules. Now by Theorem 4, non maximal prime submodules are finitely generated and so every non-maximal prime ideal of R is finitely generated. Therefore every prime ideal is finitely generated and hence by Cohen's Theorem, R is Noetherian. As any finitely generated module over any Noetherian ring is Noetherian, it follows that M is Noetherian. Hence by Theorem 8, M is a Noetherian Q-module. This completes the proof of the theorem. □

The authors wish to thank the referee for his helpful comments and suggestions.

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Received 23.10.2007