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Some inequalities concerning the rate of growth of polynomials

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Abstract

In this paper we consider a class of polynomials $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, not vanishing in $|z| < k$, $k \geq 1$ and investigate the dependence of $\max_{|z|=1} |p(Rz) - p(z)|$ on $\max_{|z|=1} |p(z)|$. Our result not only generalizes some polynomial inequalities, but also a variety of interesting results can be deduced from it by a fairly uniform procedure.

Key word and phrases: Polynomial, Zeros, Inequalities.

1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree atmost n, then according to a famous result known as Bernstein's inequality (for reference, see [12, p. 531] or [14]),

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{1}$$

whereas concerning the maximum modulus of $p(z)$ on a large circle $|z| = R > 1$, we have (for reference, see [12, p. 442])

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|\,. \tag{2}$$

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequalities (1) and (2) can be sharpened. In fact, if $p(z) \neq 0$ in $|z| < 1$, then (1) and (2) can respectively be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)| \tag{3}$$

and

$$\max_{|z|=R} |p(z)| \leq \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |p(z)|, \quad R > 1\,. \tag{4}$$

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Inequality (3) was conjectured by Erdös and later verified by Lax [10], whereas Ankeny and Rivlin [1] used (3) to prove (4).

As an extension of (3) Malik [11] verified that if $p(z)$ does not vanish in $|z|<k,\ k\geq 1$, then

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+k}\max_{|z|=1}|p(z)|\,. \tag{5}$$

Chan and Malik [7] generalized (5) in a different direction and proved that if $p(z)=a_0+\sum_{v=t}^{n}a_vz^v\,,\ t\geq 1$, is a polynomial of degree n which does not vanish in $|z|<k$, where $k\geq 1$, then

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+k^t}\max_{|z|=1}|p(z)|\,. \tag{6}$$

Inequality (6) was independently proved by Qazi [13, Lemma 1], who also under the same hypothesis proved that

$$\max_{|z|=1}|p'(z)|\leq n\left\{\frac{1+\dfrac{t}{n}\left|\dfrac{a_t}{a_0}\right|k^{t+1}}{1+k^{t+1}+\dfrac{t}{n}\left|\dfrac{a_t}{a_0}\right|(k^{t+1}+k^{2t})}\right\}\max_{|z|=1}|p(z)|. \tag{7}$$

The following result which is due to Gardner, Govil and Weems [8] is of independent interest, because it provides generalizations and refinements of inequalities (3), (5), (6) and (7).

Theorem A *If* $p(z)=a_0+\sum_{v=t}^{n}a_vz^v\,,\ t\geq 1$, *is a polynomial of degree* n *having no zeros in* $|z|<k$, *where* $k\geq 1$, *then*

$$\max_{|z|=1}|p'(z)|\leq n\left\{\frac{1+\left(\dfrac{t}{n}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}}{1+k^{t+1}+\left(\dfrac{t}{n}\right)\dfrac{|a_t|}{|a_0|-m}(k^{t+1}+k^{2t})}\right\}\left(\max_{|z|=1}|p(z)|-m\right), \tag{8}$$

where

$$m=\min_{|z|=k}|p(z)|\,.$$

Clearly for $m=0$, inequality (8) reduces to inequality (7).

Recently, Aziz and Shah [6] investigated the dependence of $\max_{|z|=1}|p(Rz)-p(z)|$ on $\max_{|z|=1}|p(z)|$, where $R>1$ and proved the following theorem.

Theorem B *Let* $p(z)=a_0+\sum_{v=t}^{n}a_vz^v\,,\ t\geq 1$, *be a polynomial of degree* n *which does not vanish in* $|z|<k$, *where* $k\geq 1$, *then for every* $R>1$ *and* $|z|=1$,

$$|p(Rz)-p(z)|\leq (R^n-1)\left\{\frac{1+\left\{\dfrac{R^t-1}{R^n-1}\right\}\left|\dfrac{a_t}{a_0}\right|k^{t+1}}{1+k^{t+1}+\left\{\dfrac{R^t-1}{R^n-1}\right\}\left|\dfrac{a_t}{a_0}\right|(k^{t+1}+k^{2t})}\right\}\max_{|z|=1}|p(z)|. \tag{9}$$

If we divide both sides of (9) by $R-1$ and make $R \to 1$, we get (7).

In this paper we shall prove the following more general result which includes not only Theorem A and Theorem B as special cases but also leads to a standard development of interesting generalizations of some well-known results.

Theorem. *Let $p(z) = a_0 + \sum\limits_{v=t}^{n} a_v z^v$, $t \geq 1$, be a polynomial of degree n which does not vanish in $|z| < k$, where $k \geq 1$, and $m = \min\limits_{|z|=k} |p(z)|$, then for every $R > 1$ and $|z| = 1$,*

$$|p(Rz) - p(z)| \leq (R^n - 1) \left\{ \frac{1 + \left(\frac{R^t - 1}{R^n - 1}\right) \frac{|a_t|}{|a_0| - m} k^{t+1}}{1 + k^{t+1} + \left(\frac{R^t - 1}{R^n - 1}\right) \frac{|a_t|}{|a_0| - m}(k^{t+1} + k^{2t})} \right\} \times \left\{ \max_{|z|=1} |p(z)| - m \right\}. \tag{10}$$

Remark 1 *If we divide the two sides of (10) by $R-1$ and make $R \to 1$, we immediately get (8). For $m = 0$, the above theorem reduces to Theorem B.*

If we use the fact that $|p(Rz)| \leq |p(Rz) - p(z)| + |p(z)|$, then the following corollary is an immediate consequence of the above theorem.

Corollary. *Let $p(z) = a_0 + \sum\limits_{v=t}^{n} a_v z^v$, $t \geq 1$, be a polynomial of degree n which does not vanish in $|z| < k$, where $k \geq 1$, and $m = \min\limits_{|z|=k} |p(z)|$, then for every $R > 1$,*

$$\begin{aligned}
\max_{|z|=R} |p(z)| \quad \leq \quad & \left[\frac{R^n + k^{t+1} \left\{ \dfrac{1 + \left(\frac{R^t - 1}{R^n - 1}\right) \frac{|a_t|}{|a_0| - m} k^{t-1}}{1 + \left(\frac{R^t - 1}{R^n - 1}\right) \frac{|a_t|}{|a_0| - m} k^{t+1}} \right\}}{1 + k^{t+1} \left\{ \dfrac{1 + \left(\frac{R^t - 1}{R^n - 1}\right) \frac{|a_t|}{|a_0| - m} k^{t-1}}{1 + \left(\frac{R^t - 1}{R^n - 1}\right) \frac{|a_t|}{|a_0| - m} k^{t+1}} \right\}} \right] \max_{|z|=1} |p(z)| \\
& - \left[\frac{(R^n - 1)m}{1 + k^{t+1} \left\{ \dfrac{1 + \left(\frac{R^t - 1}{R^n - 1}\right) \frac{|a_t|}{|a_0| - m} k^{t-1}}{1 + \left(\frac{R^t - 1}{R^n - 1}\right) \frac{|a_t|}{|a_0| - m} k^{t+1}} \right\}} \right].
\end{aligned} \tag{11}$$

It can be easily verified that for every n and $R > 1$, the function $\left(\frac{R^n + x}{1 + x}\right) \max\limits_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{1 + x}\right) m$, is a non-increasing function of x. If we combine this fact with Lemma

6 (stated in Section 2), according to which

$$k^{t+1}\left\{\frac{1+\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}}{1+\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}}\right\}\geq k^t, \quad t\geq 1,$$

we get

$$\max_{|z|=R}|p(z)|\leq\left(\frac{R^n+k^t}{1+k^t}\right)\max_{|z|=1}|p(z)|-\left(\frac{R^n-1}{1+k^t}\right)m, \tag{12}$$

which is a generalization of a result due to Aziz [3, Theorem 4]. Also for $k=t=1$, inequality (12) reduces to a result of Aziz and Dawood [4].

2. Lemmas

We need the following lemmas.

Lemma 1 *If* $p(z)=a_0+\sum\limits_{v=t}^{n}a_vz^v$, $t\geq 1$, *is a polynomial of degree* n *having no zeros in* $|z|<k$, $k\geq 1$, *then for* $|z|=1$ *and* $R>1$,

$$|q(Rz)-q(z)|\geq k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\left|\dfrac{a_t}{a_0}\right|k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\left|\dfrac{a_t}{a_0}\right|k^{t+1}+1}\right\}|p(Rz)-p(z)|, \tag{13}$$

where $q(z)=z^n\overline{p\left(\dfrac{1}{\bar{z}}\right)}$.

The above lemma is due to Aziz and Shah [6].

The following lemma is due to Aziz and Rather [5].

Lemma 2 *If* $p(z)$ *is a polynomial of degree* n *having all its zero in* $|z|\leq t$, *where* $t\leq 1$, *then*

$$|p(Rz)-p(z)|\geq\left(\frac{R^n-1}{t^n}\right)\min_{|z|=t}|p(z)|, \text{ for } |z|=1 \text{ and } R\geq 1.$$

Lemma 3 *The function*

$$S(x)=k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\left(\dfrac{|a_t|}{x}\right)k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\left(\dfrac{|a_t|}{x}\right)k^{t+1}+1}\right\},$$

is a non-decreasing function of x.

Proof of Lemma 3. The proof follows by considering the first derivative test for $S(x)$.

Lemma 4 *If* $p(z)=\sum\limits_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n, $p(z)\neq 0$ *in* $|z|<k$, *then* $|p(z)|>m$ *for* $|z|<k$, *and in particular* $|a_0|>m$, *where* $m=\min\limits_{|z|=k}|p(z)|$.

The above lemma is due to Gardner, Govil and Musukula [9].

Lemma 5 *If* $p(z)=a_0+\sum\limits_{v=t}^{n} a_v z^v$, $t\geq 1$, *is a polynomial of degree* n *having no zeros in* $|z|<k$, $k\geq 1$ *and* $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$, *then for* $|z|=1$ *and* $R>1$,

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\}|p(Rz)-p(z)|\leq |q(Rz)-q(z)|-(R^n-1)m, \tag{14}$$

where $m=\min\limits_{|z|=k}|p(z)|$.

Proof of Lemma 5. Since $p(z)$ has all its zeros in $|z|\geq k\geq 1$ and $m=\min\limits_{|z|=k}|p(z)|$, therefore

$$m\leq |p(z)| \quad \text{for} \quad |z|=k\,.$$

Hence, it follows by Rouche's Theorem that for $m>0$ and for every complex number α with $|\alpha|\leq 1$, the polynomial $h(z)=p(z)-\alpha m$ does not vanish in $|z|<k$, $k\geq 1$.

Applying Lemma 1 to the polynomial $h(z)=p(z)-\alpha m$, we get for every complex number α with $|\alpha|\leq 1$

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0-m|}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0-m|}k^{t+1}+1}\right\}|p(Rz)-p(z)|\leq |q(Rz)-q(z)-m\bar\alpha(R^n-1)z^n|, \tag{15}$$

for $|z|=1$ and $R>1$.

Since for every α, $|\alpha|\leq 1$ we have

$$|a_0-\alpha m|\geq |a_0|-|\alpha|m\geq |a_0|-m \tag{16}$$

and $|a_0|>m$ by Lemma 4, we get on combining (15), (16) and Lemma 3 that for every α where $|\alpha|\leq 1$,

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\}|p(Rz)-p(z)|\leq |q(Rz)-q(z)-m\bar\alpha(R^n-1)z^n|, \tag{17}$$

for $|z| = 1$ and $R > 1$.

Also all the zeros of $q(z)$ lie in $|z| \leq \dfrac{1}{k} \leq 1$, it follows by Lemma 2 (with $p(z)$ replaced by $q(z)$ and t by $\dfrac{1}{k}$) that

$$|q(Rz) - q(z)| \geq (R^n - 1)k^n \min_{|z|=\frac{1}{k}} |q(z)|\,.$$

But

$$\min_{|z|=\frac{1}{k}} |q(z)| = \frac{1}{k^n} \min_{|z|=k} |p(z)|,$$

therefore, we have

$$|q(Rz) - q(z)| \geq (R^n - 1)m, \text{ for } |z| = 1 \text{ and } R > 1\,. \tag{18}$$

Now choosing the argument of α with $|\alpha| = 1$ on the right hand side of (17) such that for $|z| = 1$ and $R > 1$,

$$|q(Rz) - q(z) - m\bar{\alpha}(R^n - 1)z^n| = |q(Rz) - q(z)| - (R^n - 1)m\,,$$

which is possible by (18), we conclude that

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\}|p(Rz) - p(z)| \leq |q(Rz) - q(z)| - (R^n - 1)m, \quad \text{for } |z| = 1$$

and $R > 1$, which is inequality (14) and that proves Lemma 5 completely. □

Lemma 6 *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k \geq 1$ *and* $m = \min_{|z|=k} |p(z)|$, *then*

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\} \geq k^t\,, \quad t \geq 1\,.$$

Proof of Lemma 6. We will first show that

$$\frac{R^t - 1}{R^n - 1} \leq \frac{t}{n} \tag{19}$$

holds for all $R > 1$ and $1 \leq t \leq n$.

To establish (19), it suffices to consider the case $1 \leq t \leq n-1$ and $R > 1$. For $R > 1$ and $1 \leq t \leq n-1$, we have

$$\begin{aligned} tR^n - nR^t + (n-t) &= tR^t(R^{n-t}-1) - (n-t)(R^t-1) \\ &= (R-1)\{tR^t(R^{n-t-1} + R^{n-t-2} + \ldots + 1) - (n-t)(R^{t-1} + R^{t-2} + \ldots + R + 1)\} \\ &\geq (R-1)\{t(n-t)R^t - (n-t)tR^{t-1}\} \\ &= t(n-t)(R-1)^2 R^{t-1} \\ &> 0\,. \end{aligned}$$

This implies $t(R^n-1) > n(R^t-1)$, for all $R > 1$ and $1 \leq t \leq n-1$, which is equivalent to (19).

Also, we have by an inequality (see [8, Proof of Lemma 3]),

$$\frac{|a_t|k^t}{|a_0| - m} \leq \frac{n}{t}\,, \quad t \geq 1\,. \tag{20}$$

Combining (19) and (20), we get

$$\frac{|a_t|k^t}{|a_0| - m} \leq \frac{R^n - 1}{R^t - 1}\,.$$

The above inequality is clearly equivalent to

$$\left(\frac{R^t-1}{R^n-1}\right) \frac{|a_t|k^t}{|a_0| - m}(k-1) \leq (k-1),$$

which implies

$$\left(\frac{R^t-1}{R^n-1}\right) \frac{|a_t|k^{t+1}}{|a_0| - m} + 1 \leq \left(\frac{R^t-1}{R^n-1}\right) \frac{|a_t|k^t}{|a_0| - m} + k,$$

from which Lemma 6 follows. □

Lemma 7 *If $p(z)$ is a polynomial of degree n, then for every $R > 1$,*

$$|p(Rz) - p(z)| + |q(Rz) - q(z)| \leq (R^n - 1) \max_{|z|=1} |p(z)|$$

The above lemma is due to Aziz [2].

3. Proof of the theorem

Since $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geq 1$, does not vanish in $|z| < k$, $k \geq 1$, by Lemma 5, we have

$$k^{t+1} \left\{ \frac{\left(\frac{R^t-1}{R^n-1}\right) \frac{|a_t|}{|a_0| - m} k^{t-1} + 1}{\left(\frac{R^t-1}{R^n-1}\right) \frac{|a_t|}{|a_0| - m} k^{t+1} + 1} \right\} |p(Rz) - p(z)| \leq |q(Rz) - q(z)| - (R^n - 1)m\,. \tag{21}$$

Inequality, (21) when combined with Lemma 7, gives

$$\left\{1+k^{t+1}\left(\frac{\left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|}{|a_0|-m}k^{t+1}+1}\right)\right\}|p(Rz)-p(z)| \leq |p(Rz)-p(z)|+|q(Rz)-q(z)|-(R^n-1)m$$

$$\leq (R^n-1)\left\{\max_{|z|=1}|p(z)|-m\right\},$$

from which the theorem follows. □

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