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Modules With Unique Closure Relative to a Torsion Theory II

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Abstract

We study modules M over a general ring R such that every submodule has a unique closure with respect to a hereditary torsion theory τ on Mod-R using the fact that the module M satisfies a certain transitivity property on τ-closed submodules.

Key word and phrases: Closed submodule, UC-module, hereditary torsion theory.

1. Introduction

In this note all rings are associative with identity and all modules are unitary right modules. For any unexplained terms see [3], [4] and [5]. This paper is a continuation of [2] (for background material see [1]). Let R be a ring and let τ be a hereditary torsion theory on Mod-R, the category of all right R-modules. Given an R-module M, a submodule L of M is called τ-*essential* provided L is an essential submodule of M and M/L is a τ-torsion module. We begin with the following elementary result.

Lemma 1.1 *Let L be any τ-essential submodule of a module M.*

(i) Every τ-essential submodule K of L is a τ-essential submodule of M.

(ii) The submodule $L \cap N$ is a τ-essential submodule of N for every submodule N of M.

Proof. (i) Let K be any τ-essential submodule of L. Then L/K and M/L are both τ-torsion modules and hence so too is M/K. Moreover K is essential in L and L is essential in M so that K is essential in M. Thus K is a τ-essential submodule of M.

(ii) Note that $N/(L \cap N) \cong (N + L)/L$ so that $N/(L \cap N)$ is τ-torsion. Moreover $L \cap N$ is essential in N and hence τ-essential in N. □

Corollary 1.2 *Let $K \subseteq K'$ and $L \subseteq L'$ be submodules of a module M such that K is τ-essential in K' and L is τ-essential in L'. Then $K \cap L$ is τ-essential in $K' \cap L'$.*

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Proof. Note that K is essential in K' so that $K \cap L$ is essential in $K' \cap L$. Moreover,

$$(K' \cap L)/(K \cap L) \cong [(K' \cap L) + K]/K \subseteq K'/K,$$

so that $(K' \cap L)/(K \cap L)$ is τ-torsion. Thus $K \cap L$ is τ-essential in $K' \cap L$. Similarly, $K' \cap L$ is τ-essential in $K' \cap L'$. By Lemma 1.1, $K \cap L$ is τ-essential in $K' \cap L'$. $\square$

A submodule K of M is called *τ-closed in M* provided K has no proper τ-essential extension in M. Note that if K is a submodule of M such that either M/K is τ-torsion-free or K is a closed submodule of M then K is a τ-closed submodule of M. Given a submodule N of M, by Zorn's Lemma there exists a submodule K maximal with respect to the property that N is τ-essential in K. By Lemma 1.1 K is τ-closed in M. By a *τ-closure* of N in M we mean a τ-closed submodule K of M containing N such that N is τ-essential in K. The module M is called a *τ-UC-module* provided every submodule has a unique τ-closure in M. If τ_1 is the hereditary torsion theory on Mod-R for which every module is torsion, then τ_1-UC-modules are precisely the UC-modules discussed in [6].

In [2] we investigated when a given submodule N of a module M has a unique τ-closure and when the module M is τ-UC and this gave some generalizations of results in [6] and [7]. In this paper we shall continue our study of τ-UC-modules by showing that every module has a certain "transitivity" property which we shall denote by $T(\tau)$. We shall say that a module M *satisfies* $T(\tau)$ provided whenever L is a τ-closed submodule of M and K is a τ-closed submodule of L then K is a τ-closed submodule of M.

It is a well known fact that if $K \subseteq L$ are submodules of a module M such that K is a closed submodule of L and L is a closed submodule of M then K is a closed submodule of M (see, for example, [3, Section 1.10]). Thus if τ_1 is the torsion theory on Mod-R with the property that every R-module is torsion then every R-module satisfies $T(\tau_1)$. Now we shall show that every module satisfies $T(\tau)$ for every hereditary torsion theory τ.

Theorem 1.3 *Let R be any ring and let τ be any hereditary torsion theory on Mod-R. Then every R-module satisfies $T(\tau)$.*

Proof. Let $K \leq L$ be the submodules of M such that K is τ-closed in L and L is τ-closed in M. Suppose that K is a τ-essential submodule of a submodule N of M. Because K is τ-closed in L and K is τ-essential in $N \cap L$ we have $N \cap L = K$. Note that $(N + L)/L \cong N/K$ and hence it is τ-torsion. Thus L is τ-closed in M implies that L is closed in $N + L$. Let B be an essential closure of K in L. Because B is closed in L and L is closed in $N + L$, [3, pp.6] gives that B is closed in $N + L$ and hence also in $N + B$.

Suppose that $B \neq N + B$. There exists a non-zero submodule H of $N + B$ such that $B \cap H = 0$. Note that H embeds in the module $(N + B)/B \cong N/(N \cap B) = N/K$, so that H is τ-torsion. On the other hand, $B \cap H = 0$ implies that $K \cap H = 0$ and hence $N \cap H = 0$ because K is essential in N. thus H embeds in the module $(N + B)/N \cong B/(N \cap B) = B/K$. But K is essential in B and K is τ-closed in L together implies that B/K is τ-torsion free. Thus $H = 0$, a contradiction. Thus $B = N + B$ and hence $N \leq B$ and $N = K$. It follows that K is τ-closed in M. $\square$

Note that in the proof of Theorem 1.3 (and elsewhere in this paper) we use the fact that the torsion theory under consideration is hereditary. We do not know how far (if at all) the results in this paper can be extended to torsion theories which are not hereditary.

2. τ-UC-Modules

Let R be a ring and let τ be an hereditary torsion theory on Mod-R. Given an R-module M we define a relation ρ_τ on the lattice of submodules of M as follows: given submodules K and L of M, $K\rho_\tau L$ provided $K\cap L$ is τ-essential in both K and L.

Lemma 2.1 *With the above notation, for any module M, ρ_τ is an equivalence relation on the lattice of submodules of M.*

Proof. The relation ρ_τ is clearly reflexive and symmetric. Now let K, L and H be submodules of M such that $K\rho_\tau L$ and $L\rho_\tau H$. We will show that $K\rho_\tau H$. Since $K\rho_\tau L$, it follows that $K\cap L$ is τ-essential in both K and L. Similarly, $L\cap H$ is τ-essential in both L and H. By Corollary 1.2 $K\cap L\cap H=(K\cap L)\cap(L\cap H)$ is τ-essential in $K\cap L$. But this implies that $K\cap L\cap H$ is τ-essential in K by Lemma 1.1. Similarly $K\cap L\cap H$ is τ-essential in H. Thus $K\rho_\tau H$. □

Lemma 2.2 *Let M be a τ-UC-module and let $U, V and V'$ be submodules of M with V the τ-closure of U in M and U τ-essential in V'. Then $V'\subseteq V$.*

Proof. Clear. □

Lemma 2.3 *The following statements are equivalent for a module M.*

(i) M is a τ-UC-module.

(ii) Whenever K is a τ-essential submodule of a submodule K' of M and L is a τ-essential submodule of a submodule L' of M, then $K+L$ is a τ-essential submodule of $K'+L'$.

(iii) Whenever K_i is a τ-essential submodule of a submodule L_i of M for all i in an index set I, then $\sum_{i\in I}K_i$ is τ-essential in $\sum_{i\in I}L_i$.

(iv) Whenever $K\rho_\tau K'$ and $L\rho_\tau L'$, for submodules K, K', L, and L' of M, then $(K+L)\rho_\tau(K'+L')$.

(v) Whenever K and L are submodules of M such that $K\cap L$ is a τ-essential submodule of L then K is a τ-essential submodule of $K+L$.

Proof. $(i)\Rightarrow(ii)$ Let H denote the τ-closure of $K+L$ in M. Let G denote any τ-closure of K in H. By Theorem 1.3, G is τ-closed in M. By Lemma 2.2 $K'\subseteq G$ and so $K'\subseteq H$. Similarly $L'\subseteq H$. Hence $K+L$ is τ-essential in $K'+L'$.

$(ii)\Rightarrow(iii)$ Let $K=\sum_{i\in I}K_i$ and let $L=\sum_{i\in I}L_i$. For each $i\in I$, $(L_i+K)/K\cong L_i/(L_i\cap K)$ which is a homomorphic image of L_i/K_i and hence is τ-torsion. It follows that L/K is τ-torsion. Let $0\neq x\in\sum_{i\in I}L_i$. There exists a finite subset F of I such that $x\in\sum_{i\in F}L_i$. By (ii), $\sum_{i\in F}K_i$ is essential in $\sum_{i\in F}L_i$ so that $0\neq xr\in\sum_{i\in F}K_i\subseteq K$, for some $r\in R$. Thus K is essential in L and hence K is τ-essential in L.

$(iii) \Rightarrow (iv)$ Note that $K \cap K'$ is τ-essential in both K and K' and also $L \cap L'$ is τ-essential in both L and L'. By (iii), $N = (K \cap K') + (L \cap L')$ is τ-essential in both $K + L$ and $K' + L'$. Thus $(K + L)\rho_\tau(K' + L')$.

$(iv) \Rightarrow (v)$ Suppose that $K \cap L$ is τ-essential in the submodule L. Then clearly $(K \cap L)\rho_\tau L$. Since $K\rho_\tau K$, it follows by (iv) that $((K \cap L) + K)\rho_\tau(L + K)$. Hence $K\rho_\tau(K + L)$, so that K is τ-essential in $K + L$.

$(v) \Rightarrow (i)$ Let K and K' be τ-closures of a submodule N of M. Then $K \cap K'$ is τ-essential in K'. By (v), K is τ-essential in $K + K'$ so that $K = K + K'$. Thus $K' \subseteq K$. Similarly, $K \subseteq K'$. This gives that $K = K'$. It follows that M is a τ-UC-module □

Lemma 2.4 *The following statements are equivalent for a module M.*

(i) M is a τ-UC-module.

(ii) Whenever $K \subseteq L$ are submodules of M and K' is a τ-closure of K in M, then there exists a τ-closure L' of L in M such that $K' \subseteq L'$.

(iii) Whenever K is a τ-closed submodule of M, then $K \cap N$ is a τ-closed submodule of N, for every submodule N of M.

(iv) Whenever K and L are τ-closed submodules of M, then $K \cap L$ is a τ-closed submodule of M.

Proof. $(i) \Rightarrow (ii)$ By Lemma 2.3, $L = K + L$ is a τ-essential submodule of $K' + L$. Let L' be a τ-closure of $K' + L$ in M. By Lemma 1.1, L' is a τ-closure of L in M and $K' \subseteq L'$.

$(ii) \Rightarrow (iii)$ Let L be a τ-closure of $K \cap N$ in N and L' be a τ-closure of L in M. By Lemma 1.1, L' is a τ-closure of $K \cap N$ in M. Now (ii) gives that $L' \subseteq K$ because K is τ-closed in M. Hence $L \subseteq L' \subseteq K$ implies that $L \subseteq K \cap N$ and so $K \cap N = L$.

$(iii) \Rightarrow (iv)$ By (iii) $K \cap L$ is τ-closed in L and, by Theorem 1.3, $K \cap L$ is τ-closed in M.

$(iv) \Rightarrow (i)$ Let K and L be τ-closures of a submodule N of M. Then N is τ-essential in both K and L and so $K \cap L$ is τ-essential in both K and L. By (iv), $K \cap L$ is τ-closed in M. Hence $K \cap L = K = L$. Therefore M is a τ-UC-module. □

Corollary 2.5 *The following statements are equivalent for a module M.*

(i) M is a τ-UC-module.

(ii) Whenever K_i is a τ-closed submodule of a submodule L_i of M, for all i in some index set I, then $\cap_{i \in I} K_i$ is τ-closed in $\cap_{i \in I} L_i$.

(iii) Whenever K_i is a τ-closed submodule of M, for all i in some index set I, then $\cap_{i \in I} K_i$ is τ-closed in M.

Proof. $(i) \Rightarrow (ii)$ Let K be a τ-essential extension of $\cap_{i \in I} K_i$ in $\cap_{i \in I} L_i$. Then $K_i \cap K$ is a τ-essential submodule of K. By Lemma 2.3 K_i is a τ-essential submodule of $K_i + K$ and hence $K_i = K + K_i$. Thus $K \subseteq K_i$ for all $i \in I$ and $K = \cap_{i \in I} K_i$.

$(ii) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (i)$ By Lemma 2.4. □

Lemma 2.6 *The following statements are equivalent for a module M.*

(i) *M is a τ-UC-module.*

(ii) *For every τ-closed submodule K of M, $K \cap mR$ is not τ-essential in mR for all $m \in M \backslash K$.*

(iii) *$N^+ = \{m \in M : N \cap mR$ is τ-essential in $mR\}$ is a submodule of M, for every submodule N of M.*

Proof. $(i) \Rightarrow (ii)$ Let K be any τ-closed submodule of M. Assume that $K \cap mR$ is τ-essential in mR for some $m \in M$. By Lemma 2.3, K is τ-essential in $K + mR$ and hence $K = K + mR$ so that $m \in K$. Thus (ii) holds.

$(ii) \Rightarrow (iii)$ Let N be any submodule of M and let K be any τ-closure of N in M. Let $m \in N^+$. Then $N \cap mR \subseteq K \cap mR \subseteq mR$. Thus $K \cap mR$ is τ-essential in mR, so that $m \in K$. It follows that $N^+ \subseteq K$. On the other hand, if $x \in K$ then $N \cap xR$ is τ-essential in xR and hence $x \in N^+$. Thus $K \subseteq N^+$ and we have proved that $N^+ = K$.

$(iii) \Rightarrow (i)$ Let N be any submodule of M. Note that N is τ-essential in N^+. Suppose that L is a submodule of M such that N^+ is τ-essential in L. Let $x \in L$. Then $N \cap xR$ is τ-essential in xR so that $x \in N^+$. Thus N^+ is the unique τ-closure of N in M. □

The next result partially generalizes [6] and [7].

Theorem 2.7 *Let R be a ring and let τ be any hereditary torsion theory on Mod-R. Then the following statements are equivalent for an R-module M.*

(1) M is a τ-UC-module.

(2) Whenever K is a τ-essential submodule of a submodule K' of M and L is a τ-essential submodule of a submodule L' of M, then $K + L$ is a τ-essential submodule of $K' + L'$.

(3) Whenever K_i is a τ-essential submodule of a submodule L_i of M, for all i in an index set I, then $\sum_{i \in I} K_i$ is τ-essential in $\sum_{i \in I} L_i$.

(4) Whenever $K \rho_\tau K'$ and $L \rho_\tau L'$, for submodules K, K', L, and L' of M, then $(K + L) \rho_\tau (K' + L')$.

(5) Whenever K and L are submodules of M such that $K \cap L$ is a τ-essential submodule of L, then K is a τ-essential submodule of $K + L$.

(6) Whenever $K \subseteq L$ are submodules of M and K' is a τ-closure of K in M, then there exists a τ-closure L' of L in M such that $K' \subseteq L'$.

(7) Whenever K is a τ-closed submodule of M, then $K \cap N$ is a τ-closed submodule of N, for every submodule N of M.

(8) Whenever K and L are τ-closed submodules of M, then $K \cap L$ is a τ-closed submodule of M.

(9) Whenever K_i is a τ-closed submodule of a submodule L_i of M, for all i in some index set I, then $\cap_{i \in I} K_i$ is τ-closed in $\cap_{i \in I} L_i$.

(10) Whenever K_i is a τ-closed submodule of M, for all i in some index set I, then $\cap_{i \in I} K_i$ is τ-closed in M.

(11) For every τ-closed submodule K of M, $K \cap mR$ is not τ-essential in mR for all $m \in M \backslash K$.

(12) $N^+ = \{m \in M : N \cap mR$ is τ-essential in $mR\}$ is a submodule of M, for every submodule N of M.

Moreover, in this case N^+ is the unique τ-closure of N in M, for every submodule N of M.

Proof. For the equivalence of (1) - (12) see Lemmas 2.3, 2.4 and 2.6 and Corollary 2.5.

Now suppose that M is a τ-UC-module. Let N be any submodule of M. Then N^+ is the unique τ-closure of N in M by the proof of Lemma 2.6. □

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