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On τ-lifting Modules and τ-semiperfect Modules

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Abstract

Motivated by [1], we study on τ-lifting modules (rings) and τ-semiperfect modules (rings) for a preradical τ and give some equivalent conditions. We prove that; *i)* if M is a projective τ-lifting module with $\tau(M) \subseteq \delta(M)$, then M has the finite exchange property; *ii)* if R is a left hereditary ring and τ is a left exact preradical, then every τ-semiperfect module is τ–lifting; *iii)* R is τ-lifting if and only if every finitely generated free module is τ-lifting if and only if every finitely generated projective module is τ-lifting; *iv)* if $\tau(R) \subseteq \delta(R)$, then R is τ-semiperfect if and only if every finitely generated module is τ-semiperfect if and only if every simple R–module is τ-semiperfect.

Key Words: τ-lifting modules, Projective τ-covers, τ-supplement submodules, τ-semiperfect modules.

1. Introduction

The concept of semiperfect rings was generalized to I-semiperfect ring for an ideal I of a ring by Yousif and Zhou in [16]. Then Nicholson and Zhou defined the concept of strongly lifting and gave some characterizations of I-semiperfect rings in [9]. A module theoretic version of I-semiperfect ring is studied in [10] and [11] by considering any fully invariant submodule of a module. Let M be an R-module. Following [10], M is said to be U–**semiperfect** if for any submodule N of M, there is a projective direct summand A of M such that $N = A \oplus B$ and $B \subseteq U$ for a fully invariant submodule U of M. Moreover, in [11], Özcan and Aydogdu generalized the concept of strongly lifting ideals and gave some characterization of U-semiperfect module. In [1], for a radical τ, Al-Takhman, Lomp and Wisbauer defined and studied the concept of τ-lifting, τ-supplement and τ-semiperfect modules. Following [1], M is τ**-lifting** if any submodule N of M has a decomposition $N = A \oplus (B \cap N)$ such that $M = A \oplus B$ and $B \cap N \subseteq \tau(B)$ and also they called that M is τ**-semiperfect** if for any submodule N of M, M/N has a projective τ-cover. It is clear that if M is projective, then the concepts of $\tau(M)$-semiperfect and τ-lifting are coincide and if N is a submodule of M with the decomposition in the definition of τ-lifting, then M/N has a projective τ-cover. Motivated by [1], we study on τ-lifting module and the relations between a projective τ-cover and the decomposition for a preradical τ. We also give some equivalent condition for a τ-semiperfect module and a τ-lifting module. The remainder of our paper is organized as follows.

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In Section 2, we define the concept of quasi-strongly lifting (QSL). We call submodule U is called **quasi strongly lifting** (QSL) in M if whenever $(A+U)/U$ is a direct summand of M/U, M has a direct summand P such that $P \subseteq A$ and $P+U = A+U$. Then we prove that $\tau(L)$ is QSL in L if L is direct summand of M and $\tau(M)$ is QSL in M. Also, we recall SDM submodule which is given in [3], and show that $\delta(M)$ is the sum of all SDM submodule of M if M is a projective module.

In Section 3, we concern with τ-lifting modules and consider certain preradicals Soc, Z and δ. We show that if M is τ-lifting, then M is refinable if and only if every submodule of $\tau(M)$ is DM in M if and only if every submodule of $\tau(M)$ is QSL in M and we prove that M is δ-lifting and M has the finite exchange property whenever M is a projective τ-lifting and $\tau(M) \subseteq \delta(M)$. For two preradicals τ, ρ, we also study the relation between a τ-lifting module and ρ-lifting module. We also prove that if M is a δ-lifting projective module, $M/Soc(M)$ is lifting, but we prove the converse if $M/Soc(M)$ is projective. Moreover, we show that if R is a left hereditary ring and τ is a left exact preradical, then every τ-semiperfect module is τ-lifting. Finally, we give some equivalent statements for τ-semiperfect modules (rings) and τ-lifting modules (rings) as well: *i)* R is τ-lifting if and only if every finitely generated free module is τ-lifting if and only if every finitely generated projective module is τ-lifting; *ii)* if M is a finitely generated projective module with $\tau(M) \subseteq \delta(M)$, then M is τ-semiperfect if and only if every simple factor module of M has a projective τ-semiperfect; *iii)* if $\tau({}_RR) \subseteq \delta({}_RR)$, then R is τ-semiperfect if and only if every finitely generated module is τ-semiperfect if and only if every simple R-module is τ-semiperfect.

A functor τ from the category of the left R-modules to itself is called a preradical if it satisfies the following properties:

i) $\tau(M)$ is a submodule of an R-module M,

ii) If $f: M' \to M$ is an R-module homomorphism, then $f(\tau(M')) \subseteq \tau(M)$ and $\tau(f)$ is the restriction of f to $\tau(M')$.

A preradical τ is called a left exact preradical if for any submodule K of M, $\tau(K) = \tau(M) \cap K$. But it is well known if K is a direct summand of M, then $\tau(K) = \tau(M) \cap K$ for a preradical. In this note, τ will be a preradical unless otherwise stated.

Throughout this paper, R denotes an associative ring with an identity and modules are an unital left R-modules. We write $Rad(M)$, $Soc(M)$ and $Z(M)$ for Jacobson radical, the socle, the singular submodule, respectively.

2. Strongly Lifting

Let U be a submodule of an R-module M. U is called **strongly lifting** in M if whenever $M/U = (A+U)/U \oplus (B+U)/U$, then M has a decomposition $M = P \oplus Q$ such that $P \subseteq A$, $(A+U)/U = (P+U)/U$ and $(B+U)/U = (Q+U)/U$ in [11]. By removing the condition on B, we may extend the definition; the submodule U is called **quasi strongly lifting** (QSL) in M if whenever $(A+U)/U$ is a direct summand of M/U, M has a direct summand P such that $P \subseteq A$ and $P+U = A+U$.

Lemma 2.1 *Let U be a submodule of a projective module M. If U is QSL then U is strongly lifting in M.*

Proof. Let $M/U = (A+U)/U \oplus (B+U)/U$ for submodules A, B of M. Then there is a decomposition $M = P \oplus Q$ such that $P+U = A+U$ and $P \subseteq A$. Then $M = A+U+B = P+(U+B)$ and since M is projective, $M = P \oplus P'$ for a submodule $P' \subseteq U+B$. Then $M/U = (P+U)/U \oplus (P'+U)/U = (P+U)/U \oplus (B+U)/U$ and so $(P'+U)/U = (B+U)/U$. □

By using a similar proof of Theorem 2.3 in [11], we have the following lemma

Lemma 2.2 *Let M be a module and A be a direct summand of M such that M/A is projective then A is QSL in M.*

Proof. Let $M/A = (X_1+A)/A \oplus (X_2+A)/A$ for submodules X_1 and X_2. Assume that $M = A \oplus B$ and α is an isomorphism from B to M/A and so for submodules B_1 and B_2 of B, we have that $\alpha(B_i) = (X_i+A)/A$ and so $(B_i+A)/A = (X_i+A)/A$ for $i = 1, 2$. Then $B_1 \cap B_2 \subseteq (B_1+A) \cap (B_2+A) = A$ and so $B_1 \cap B_2 = 0$.

Now we claim that $B = B_1 + B_2$. Let $b \in B$ and so $b = b_1 + b_2 + a$ where $b_i \in B_i$ and $a \in A$ for $i = 1, 2$. Then since $A \cap B = 0$, it follows that $a = 0$. Then $M = A \oplus B_1 \oplus B_2$ and so B_i are projective. On the other hand, since $A \oplus B_i = A + X_i$, we have $A \oplus B_i = A \oplus Y_i$ where $Y_i \subseteq X_i$ by [7, 4.47]. Then A is QSL in M. □

Proposition 2.3 *Let M be a module such that $\tau(M)$ is QSL in M. If L is a direct summand of M, then $\tau(L)$ is QSL in L.*

Proof. Let $M = L \oplus K$ and $L/\tau(L) = [A+\tau(L)]/\tau(L) \oplus B/\tau(L)$ for submodules A, B of L. Then $[A+\tau(M)]/\tau(M) \oplus [B+K+\tau(M)]/\tau(M) = M/\tau(M)$ and so there is a decomposition $M = P \oplus Q$ such that $P \subseteq A$, $A+\tau(M) = P+\tau(M)$. Hence

$$A+\tau(M) = (A+\tau(L)) \oplus \tau(K) = (P+\tau(L)) \oplus \tau(K)$$

and so $A+\tau(L) = P+\tau(L)$. This completes the proof. □

Proposition 2.4 *Let M be projective. Then the following are equivalent:*

i) $\tau(M)$ is QSL in M,

ii) If $M/\tau(M) = (M_1+\tau(M)/\tau(M)) \oplus \ldots \oplus (M_t+\tau(M))/\tau(M))$ for any positive integer t, then $M = A_1 \oplus \ldots \oplus A_t$, where $A_1 \subseteq M_1$ and $A_i + \tau(M) = M_i + \tau(M)$ for all i.

Proof. It is enough to show that $i) \Longrightarrow ii)$.

Let $M/\tau(M) = ([M_1+\tau(M)]/\tau(M)) \oplus \ldots \oplus ([M_t+\tau(M)]/\tau(M))$ for any positive integer t then $M/\tau(M) = ([M_1+\tau(M)]/\tau(M)) \oplus ([M_2+\ldots+M_t+\tau(M)]/\tau(M))$. There is a direct summand A_1 of M such that $A_1 \subseteq M_1$ and $A_1+\tau(M) = M_1+\tau(M)$. Since M is projective, there is a decomposition $M = A_1 \oplus B$ such that $B \subseteq M_2+\ldots+M_t+\tau(M)$ and so $B+\tau(M) = M_2+\ldots+M_t+\tau(M)$. Then there are submodules N_i of B such that $N_i+\tau(B)+\tau(A) = M_i+\tau(M)$ and $B/\tau(B) = (N_2+\tau(B))/\tau(B) \oplus \ldots \oplus (N_t+\tau(B))/\tau(B)$; and since $\tau(B)$ is QSL in B, there is a decomposition $B = A_2 \oplus B_2$ such that $A_2 \subseteq N_2$, $B_2 \subseteq N_3+\ldots+N_t+\tau(B)$

and $A_2+\tau(B)=N_2+\tau(B)$ and so $A_2+\tau(M)=N_2+\tau(M)$. Then $M=A_1\oplus B=A_1\oplus A_2\oplus B_2$. And so after finite steps, we have the decomposition $M=A_1\oplus\ldots\oplus A_t$ where $A_1\subseteq M_1$ and $A_i+\tau(M)=M_i+\tau(M)$. □

Let K be a submodule of a module M. Following [15], K is called *δ-small* in M if $K+L\neq M$ for any proper submodule L of M with M/L singular. Zhou also defined the fully invariant submodule $\delta(M)=\cap\{K\leq M : M/K \text{ is singular simple in } R\text{-mod}\}=\sum\{K : K \text{ is } \delta\text{-small in } M\}$.

In [3], it is called that a proper submodule N of M is *SDM (resp., DM)* in M if there is a direct summand S of M such that $S\subseteq N$ and $M=S\oplus X$ (resp., $M=S+X$) whenever $N+X=M$ for a submodule X of M.

It is clear that a δ-small submodule of a module and any direct summand of a module is DM, but there is a SDM-submodule which is not δ-small (see Example 3.25).

We note the following lemma.

Lemma 2.5 *[15, Lemma 1.2] Let K be a submodule of a module M. Then K is δ-small if and only if $M=X\oplus Y$ for a projective semisimple submodule Y with $Y\leq K$ whenever $X+K=M$.*

Let $S(M)$ denote the sum of all SDM submodules of a module M. It is clear that $S(M)$ contains $Soc(M)$ and $\delta(M)$.

Lemma 2.6 *Let A,B be SDM submodule of a module M. Then $A+B$ is SDM in M.*

Proof. Let $A+B+K=M$ for a submodule K. Since A is SDM in M, there is a submodule S of A such that $S\oplus(B+K)=M$ and so $B+(S\oplus K)=M$. Then similarly $M=Q\oplus(S\oplus K)$ for a submodule Q of B. Then $A+B$ is *SDM* in M. □

Theorem 2.7 *Let M be a a projective module. Then*

i) Rx is SDM in M where $x\in S(M)$.

ii) $S(M)=\delta(M)$ and every finitely generated SDM submodule of M is δ-small.

Proof. *i)* Let $x\in S(M)$ and $Rx+K=M$ for a submodule K. Then $x\in\sum_{i=1}^{n}K_i$ where $n\in\mathbb{Z}$ and K_i is SDM in M and $\sum_{i=1}^{n}K_i$ is SDM in M. Then $(\sum_{i=1}^{n}K_i)+K=M$ and so for a submodule S, we have that $S\oplus K=M$. Then since M is projective and K is a direct summand, we have $M=A\oplus K$ for a submodule A of Rx. Hence Rx is SDM in M.

ii) Since M is projective, $\delta({}_RM)$ is the intersection of all essential maximal submodules of M. Take $x\in S(M)$ and assume that $x\notin L$ for an essential maximal submodule L. Since $x\in S(M)$, we get that $S\oplus L=M$ for a submodule S of Rx, a contradiction. Hence $S(M)=\delta(M)$. □

3. τ-lifting

We concern with τ-lifting modules and consider certain preradicals *Soc*, Z and δ. We state [1, Proposition 2.8] for a preradical τ.

Proposition 3.1 *For a submodule S of a module M, the following are equivalent:*

i) there is a decomposition $M = X \oplus X'$ such that $X \subseteq S$ and $X' \cap S \subseteq \tau(X')$,

ii) there is a decomposition $S = A \oplus T$ with $A \subseteq^{\oplus} M$ and $T \subseteq \tau(M)$,

iii) there exists a direct summand A of M such that $A \subseteq S$ and $S/A \subseteq \tau(M/A)$,

iv) there exists an idempotent homomorphism γ from M to M such that $(1-\gamma)(S) \subseteq \tau(M)$ and $\gamma(M) \subseteq S$.

For a submodule S of a module M, in [1], Al-Takhman, Lomp and Wisbauer say that S contains a *τ-dense direct summand* if S satisfies one of the conditions of Proposition 3.1 and also M is called *τ-lifting* if every submodule of M contains a τ-dense direct summand. In [11], τ-dense direct summand is named as $\tau(M)$ respects S.

Following [1], (i) a submodule $K \subseteq M$ is called *a τ-supplement* provided there exists some $U \subseteq M$ such that $U + K = M$ and $U \cap K \subseteq \tau(K)$; (ii) M is said to be *τ-supplemented* if every submodule $K \subseteq M$ has a τ-supplement in M; (iii) it is called *amply τ-supplemented* if for any submodules $K, V \subseteq M$ such that $M = K + V$, there is a τ-supplement U for K with $U \subseteq V$. It is clear that a τ-lifting module is τ-supplemented.

Lemma 3.2 *Let M be a projective τ-supplemented module and assume that every τ-supplement submodule is a direct summand of M. Then M is τ-lifting.*

Proof. Let U be a submodule of M. Then there is a submodule K of M such that $U \cap K \subseteq \tau(K)$ and $M = K + U$. Hence K is a direct summand of M and since $M = K + U$ and M is projective, it follows that $M = K \oplus A$ such that $A \subseteq U$. Then $U = A \oplus (K \cap U)$ and U is a τ-dense direct summand. □

Now we give relations between a τ-lifting module and an amply τ-supplemented module.

Lemma 3.3 *Let M be an amply τ-supplemented module and assume that every τ-supplement submodule is a direct summand of M. Then M is τ-lifting.*

Proof. By hypothesis, a submodule A of M has a τ-supplement B and so B has a τ-supplement submodule $B^{'}$ such that $B^{'} \subseteq A$ and $M = B' \oplus B''$ for some B''. Then $M = B' + B$ and so $A = B' + (A \cap B) = B' \oplus (A \cap B'')$. Let π denote the projection map from M to B''. Then $A \cap B'' = \pi(A) = \pi(A \cap B)$. Since B is a τ-supplement of A, it follows that $A \cap B \subseteq \tau(B)$ and so $A \cap B'' \subseteq \tau(B'')$. □

Lemma 3.4 *Let τ be a left exact preradical and M be a τ-lifting module. Then M is an amply τ-supplemented module.*

Proof. Let X and S be submodules of M such that $M = X + S$. We show that S contains a τ-supplement of X. By assumption, write $S = Y \oplus T$ where $M = Y \oplus Y^{'}$ for submodules $Y^{'}$, Y and $T = S \cap Y' \subseteq \tau(Y')$. Then $M = X + Y + T$ and also there is a decomposition $M = Y_1 \oplus Y_1'$ such that $(X + T) \cap Y = Y_1 \oplus T_1$ and $T_1 = (X + T) \cap Y \cap Y_1' \subseteq \tau(Y_1')$ and so $T_1 \subseteq \tau(Y_1') \cap Y = \tau(Y_1' \cap Y)$. Then $Y = Y_1 \oplus (Y_1' \cap Y)$ and so $M = X + T + (Y_1' \cap Y)$. Let $L = T + (Y_1' \cap Y)$ and so $L \subseteq S$.

$$X \cap L \subseteq [T \cap (X + (Y_1' \cap Y))] + [(Y_1' \cap Y) \cap (T + X)] \subseteq T + \tau(Y_1' \cap Y)$$

Since τ is left exact, we have $T+\tau(Y_1'\cap Y)\subseteq\tau(T)+\tau(Y_1'\cap Y)\subseteq\tau(T+(Y_1'\cap Y))$ and so L is a τ-supplement submodule of X in S. This completes the proof. □

Lemma 3.5 *Let M be a τ-lifting module. Then $\tau(M)$ is QSL in M.*

Proof. Let $M/\tau(M)=[K+\tau(M)/\tau(M)]\oplus L/\tau(M)$ for submodules K,L. Since M is τ-lifting, there is a decomposition $M=A\oplus B$ such that $K=A\oplus(B\cap K)$ and $B\cap K\subseteq\tau(M)$ and so $A+\tau(M)=K+\tau(M)$. □

Proposition 3.6 *Let M be a module. Then the following statements are equivalent:*

i) M is τ-lifting,

ii) M is τ-supplemented and $\tau(M)$ is QSL,

iii) $M/\tau(M)$ is semisimple and $\tau(M)$ is QSL.

Proof. *i)* $\Rightarrow$ *ii)* $\Rightarrow$ *iii)* Obvious.

iii) $\Rightarrow$ *i)* Let U be a submodule of M. Then we have that $M/\tau(M)=[U+\tau(M)/\tau(M)]\oplus[K/\tau(M)]$ for a submodule K and so there is a decomposition $M=A\oplus B$ such that $A\subseteq U$, $A+\tau(M)=U+\tau(M)$. Since $\tau(M)=\tau(A)\oplus\tau(B)$, it follows that $U\cap B\subseteq(U+\tau(M))\cap(B+\tau(M))=(A+\tau(M))\cap(B+\tau(M))=[(A+\tau(B))\cap B]+\tau(A)=\tau(M)$. Hence, $U\cap B\subseteq\tau(M)\cap B\subseteq\tau(B)$ and so U contains a τ-dense direct summand. □

A module M is called **refinable** if whenever $M=A+B$ for submodules A,B, there is a direct summand C of M such that $C\subseteq A$ and $M=C+B$ (see [6]). Then we have the following theorem

Theorem 3.7 *Let M be a module. Consider the following conditions:*

i) M is refinable,

ii) every submodule of $\tau(M)$ is QSL in M,

iii) every submodule of $\tau(M)$ is DM in M.

Then i) $\Longrightarrow$ ii) $\Longrightarrow$ iii). If M is τ-lifting then iii) $\Longrightarrow$ i).

Proof. *i)* $\Longrightarrow$ *ii)* Let N be a submodule of $\tau(M)$ and $(L+N)/N\oplus K/N=M/N$ for submodules L,K. Then $L+K=M$ and so there is a direct summand S of M such that $S+K=M$ and $S\subseteq L$. Hence $(S+N)/N\oplus K/N=(L+N)/N\oplus K/N$ and so $S+N=L+N$.

ii) $\Longrightarrow$ *iii)* Let K be a submodule of $\tau(M)$ such that $M=K+L$ for a submodule L and $N:=K\cap L$. Then K/N is a direct summand of M/N. Then there is a direct summand S of M such that $S\subseteq K$ and $S+N=K$. Then $S+L=M$ and so K is DM in M.

iii) $\Longrightarrow$ *i)* Assume every submodule of $\tau(M)$ is DM. Let $M=K+L$ for submodules L and K. Then $K=A\oplus(K\cap B)$ such that $M=A\oplus B$ and $K\cap B\subseteq\tau(B)$. It follows that $M=A+(K\cap B)+L$ and so $B=(K\cap B)+[(A+L)\cap B]$. Since every submodule of $\tau(B)$ is DM in B by [3, Lemma 3.2], there is a direct summand C of B such that $B=[(A+L)\cap B]+C$ and $C\subseteq K\cap B$ and so $A\oplus C$ is a direct summand of M and $M=(A+C)+L$. Then K is DM in M. □

A module M is said to have **the exchange property** if for any module X and a decomposition $X = M' \oplus Y = \oplus_{i \in I} A_i$ where $M' \cong M$, there exist submodules A_i' of A_i for each i such that $X = M' \oplus (\oplus A_i')$. The module M is said to have **the finite exchange property** whenever this condition holds for a finite set. Then in [8, Proposition 2.9], Nicholson proves that a projective module M has the finite exchange property if and only if whenever $M = A + B$ for a submodule A, B of M, there exists a direct summand P_1 of M such that $P_1 \subseteq A$ and $M = P_1 + B$. Then, we have the following lemma.

Lemma 3.8 *Let M be a projective module. If M has the finite exchange property, then $\delta(M)$ is δ-small.*

Proof. Let U be a submodule of M such that $U + \delta(M) = M$. Since M has the finite exchange property, there is a direct summand A of M such that $A \subseteq \delta(M)$ and $M = U + A$. Then by [10, Proposition 2.13], A is projective and semisimple and so $M = U \oplus S$ for a projective semisimple submodule S of A. Hence, $\delta(M)$ is a δ-small in M by Lemma 2.5. □

In [15], a module M is called **δ-semiperfect** if for any submodule N of M, there is a decomposition $M = A \oplus B$ such that $N = A \oplus (N \cap B)$, A is projective and $N \cap B$ is δ-small in B. By the definitions, a δ-semiperfect module is δ-lifting. In the following theorem, we prove that a projective δ-lifting module is δ-semiperfect and give a characterization for the finite exchange property if M is projective.

Theorem 3.9 *Let M be a projective τ-lifting module and $\tau(M) \subseteq \delta(M)$. Then we have:*

i) $\delta(M)$ is δ-small and M is δ-semiperfect.

ii)M has the finite exchange property.

Proof. *i)* Let U be a submodule of M such that $U + \delta(M) = M$. Then $U = A \oplus (B \cap U)$ such that $M = A \oplus B$ and $B \cap U \subseteq \tau(M)$. Then $M = A + \delta(M)$ and so $M = A \oplus C$ for a submodule C of $\delta(M)$. Then by [10, Proposition 2.13], C is projective and semisimple. Since $M = U + C$, we get that $M = U \oplus K$ for a projective semisimple submodule K of C. Hence, $\delta(M)$ is a δ-small in M by Lemma 2.5 and so $B \cap U$ is δ-small in B. Hence, M is δ-semiperfect.

ii) Let $X \subseteq \tau(M)$. Then by *i)*, X is δ-small in M and so X is DM in M. Therefore M has the finite exchange property by Theorem 3.7. □

Corollary 3.10 *Let M be a projective module and $\tau = Rad$ or $\tau = Z$. Then M is τ-lifting if and only if $\tau(M) = Rad(M)$ is small and M is lifting.*

Proof. Let M be τ-lifting. If $L + Z(M) = M$, then $L = A \oplus (B \cap L)$ where $M = A \oplus B$ and $B \cap L \subseteq Z(M)$. Since M/A is singular, it follows that A is essential and so $A = M = L$. Hence $Z(M)$ is small and since N is a projective τ-lifting module with $Rad(M) \subseteq \tau(M)$, it follows that M is lifting and $\tau(M) = Rad(M)$. □

If M is a τ-lifting module, then by the same argument of [1, 2.2], there is a decomposition $M = L \oplus B$ such that L is semisimple and $\tau(M)$ is an essential submodule of B. Then we have

Lemma 3.11 *Let M be a module. Then we have*

i) If M is Soc-lifting, then $Soc(M)$ is essential in M.

ii) If M is projective δ-lifting module, then $Z(M) \subseteq Rad(M) \subseteq \delta(M)$ and $\delta(M)$ is essential in M.

Proof. *i)* If M is Soc-lifting then by [1, 2.2], $Soc(M)$ is essential in M.

ii) If M is a projective, then $Soc(M) \subseteq \delta(M)$ and so by [1, 2.2], $\delta(M)$ is essential in M.

Let $x \in Z(M)$ and so $Rx = A \oplus (B \cap Rx)$ where $M = A \oplus B$ and $B \cap Rx \subseteq \delta(M)$. Then A is singular and projective and so $A = 0$. Hence $Z(M) \subseteq \delta(M)$.

Let $x \in Z(M)$ and let L be a submodule with $Rx+L = M$. Since Rx is δ-small in M, there is a semisimple projective submodule $S \subseteq Rx \subseteq Z(M)$ such that $S \oplus L = M$. Hence $L = M$ and so $Z(M) \subseteq Rad(M)$. □

Proposition 3.12 *Let τ and ρ be preradicals and M be a τ-lifting module such that $\tau(M) + L = M$ and $\tau(M) \cap L \subseteq \rho(L)$ for a submodule L of M. Then there is a decomposition $M = A \oplus B$ such that A is ρ-lifting and $B \subseteq \tau(M)$.*

Proof. Let M be τ-lifting. Then there is a decomposition $M = A \oplus B$ such that $L = A \oplus (B \cap L)$ and $B \cap L \subseteq \tau(B)$ and so $B \cap L \subseteq \tau(M) \cap L \subseteq \rho(L)$.

Now we show that A is ρ-lifting and $B \subseteq \tau(M)$. Let K be a submodule of A. Since A is a direct summand of M, it also τ-lifting. Then there is a decomposition $A = X \oplus Y$ such that $K = X \oplus Y \cap K$ and $Y \cap K \subseteq \tau(Y)$. Also $Y \cap K \subseteq \tau(Y) \cap L \subseteq \rho(M) \cap Y = \rho(Y)$ since Y is a direct summand of M. Then A is ρ-lifting.

Since $\tau(M) = \tau(A) \oplus \tau(B)$, we get $M = \tau(M) + L = \tau(A) + \tau(B) + A + B \cap L = A \oplus \tau(B)$ and so $\tau(B) = B \subseteq \tau(M)$. □

Corollary 3.13 *Let M be a τ-lifting projective module such that $\tau(M) + L = M$ and $\tau(M) \cap L \subseteq \rho(L)$ for a submodule L of M where τ and ρ are elements of the set $P = \{\delta, Soc, Z, Rad\}$. Then M is ρ-lifting.*

Proof. By Proposition 3.12, there is a decomposition $M = A \oplus T$ such that A is ρ-lifting and $T \subseteq \tau(M)$. If $\tau = Z$, then $T = 0$. If $\tau \in \{\delta, Soc, Rad\}$, then by Proposition 3.9, $\tau(M)$ is δ-small in M and so does T. Then T is semisimple and so T is ρ-lifting. Hence, by [10, Proposition 2.13], M is ρ-lifting. □

Let τ, ρ and σ be preradicals and M be a module. Then we say that M has $*$**-property** for $\{\tau, \rho, \sigma\}$ if $\sigma(N/\rho(N)) = \tau(N)/\rho(N)$ for any direct summand N of M. For example, if M is a projective module, then by [10, Proposition 2.13], $Rad(M/Soc(M)) = \delta(M)/Soc(M)$. Then we have the following proposition, which is a generalization of [16, Theorem 1.4].

Proposition 3.14 *Let M be a module with $*$-property for $\{\tau, \rho, \sigma\}$. If M is τ-lifting, then $M/\rho(M)$ is σ-lifting.*

In particular, the converse holds whenever $\rho(M)$ is QSL in M and M is projective.

Proof. Let $\overline{M}$ denote $M/\rho(M)$ and $\overline{L}$ denote $L/\rho(M)$ for a submodule L of M.

Assume that M is τ-lifting and $\overline{N}$ is a submodule of $\overline{M}$. Then $N = A \oplus (B \cap N)$ where $M = A \oplus B$ and $B \cap N \subseteq \tau(B)$. On the other hand, we have

$$\begin{aligned}(A+\rho(M)) \cap (B+\rho(M)) &= (A+\rho(B)) \cap (B+\rho(A)) \\ &= \rho(B) + [A \cap (B+\rho(A))] \\ &= \rho(A)+\rho(B)\end{aligned}$$

Thus $\overline{A} \oplus \overline{B} = \overline{M}$ and it is enough to show that $\overline{B} \cap \overline{N} \subseteq \sigma(\overline{B})$. Then we get that $\overline{B \cap N} = \overline{B} \cap \overline{N}$ and by $\sigma(B/\rho(B)) = \tau(B)/\rho(B)$ $[B \cap N + \rho(M)]/\rho(M) \subseteq [\tau(B)+\rho(M)]/\rho(M) \subseteq \sigma([B+\rho(M)]/\rho(M))$. Hence $M/\rho(M)$ is σ-lifting.

For the converse, assume that L is a submodule of M. Then there is a decomposition $\overline{M} = \overline{C} \oplus \overline{D}$ such that $\overline{L} = \overline{C} \oplus \overline{D} \cap \overline{L}$ and $\overline{D} \cap \overline{L} \subseteq \sigma(\overline{D})$. Since $\rho(M)$ is QSL in M, there is a decomposition $M = A \oplus B$ such that $A \subseteq L$, $\overline{A} = \overline{C}$ and $\overline{B} = \overline{D}$. Then it is enough to show that $L \cap B \subseteq \tau(B)$ since $L = A \oplus (B \cap L)$. Then $\overline{B \cap L} = \overline{B} \cap \overline{L} = \overline{D} \cap \overline{L} \subseteq \sigma(\overline{D}) = \sigma(\overline{B})$ and so $B \cap L \subseteq \rho(M) \subseteq \tau(M)$ and $B \cap L \subseteq \tau(M) \cap B = \tau(B)$ since B is direct summand. □

Proposition 3.15 *Let M be a projective module. If M is δ-semiperfect, then $M/Soc(M)$ is lifting.*

In particular, the converse holds whenever $M/Soc(M)$ is projective.

Proof. Let $\overline{M}$ denote $M/Soc(M)$ and $\overline{L}$ denote $L/Soc(M)$ for a submodule L of M.

Assume that M is δ-lifting and $\overline{N}$ is a submodule of $\overline{M}$. Then $N = A \oplus (B \cap N)$ where $M = A \oplus B$ and $B \cap N$ is δ-small in B. Then it follows that $B \cap N$ is δ-small in $B + Soc(M)$. On the other hand, we have $\overline{A} \oplus \overline{B} = \overline{M}$ and we get that $\overline{B \cap N} = \overline{B} \cap \overline{N}$.

Now it is enough to show that $\overline{B \cap N}$ is small in $\overline{B}$. Let $\overline{B \cap N} + \overline{T} = \overline{B}$ for a submodule $T/Soc(M)$. Then $B + Soc(M) = T + B \cap N$ and so there is a projective semisimple submodule S such that $S \oplus T = B + Soc(M)$ and so $T = B + Soc(M)$. Then $\overline{B \cap N}$ is small in $\overline{B}$.

For the converse, assume that $\overline{M}$ and M are projective and L is a submodule of M. Then $M/(L + SocM)$ has a projective cover and so there is a decomposition $M = A \oplus B$ such that $L + Soc(M) = A \oplus [(L + Soc(M)) \cap B]$ and $(L + Soc(M)) \cap B$ is small in B. Then
$M = Soc(M) + L + B = C \oplus (L + B)$ for a submodule C of $Soc(M)$. Since $L + B$ is projective and B is a direct summand of M, it follows that $L + B = B \oplus D$ for a submodule D of L and so we get that $L = D \oplus (L \cap (C + B))$. Therefore, M is δ-semiperfect. □

In [1], it is said that a module M has **a projective τ-cover** if there is an epimorphism f from a projective module P to M such that $Kerf \subseteq \tau(P)$ and an R-module M is called **τ-semiperfect** if every factor module of M has a projective τ-cover. Now we give some properties of a τ-semiperfect module.

Lemma 3.16 *Let M be a τ-semiperfect module. Then we have that* *i)* $Rad(M) \subseteq \tau(M)$ *and* $M/\tau(M)$ *is semisimple.*

ii) if $\tau(M)$ *contains all projective semisimple direct summands, then* $\delta(M) \subseteq \tau(M)$.

iii) if $\tau = Soc$, *then* $\delta(M) \subseteq Soc(M)$.

iv) if M *is projective and* $\tau = Soc$, *then* $Z(M) \subseteq Rad(M) \subseteq Soc(M) = \delta(M)$.

v) if M *is projective and* U *is DM in* M, *then* U *is a* τ*-dense direct summand submodule.*

Proof. First, we observe the following for an element x of M. Let f be an epimorphism from a projective module P to M/Rx such that $Kerf \subseteq \tau(P)$. Let $\pi : M \to M/Rx$ be a canonical epimorphism. Since P is projective, it follows that there is a homomorphism α from P to M such that $\pi\alpha = f$. Hence $M = \alpha(P) + Rx$. Let $K := \alpha(P)$ and take $y \in K \cap Rx$. Then $y = \alpha(t)$ for some $t \in P$ and $f(t) = \pi\alpha(t) = \pi(y) = 0$ and so $t \in Kerf \subseteq \tau(P)$. Hence $y \in \tau(M) \cap K$.

i) If $x \in Rad(M)$ then $K = M$ and so $K \cap Rx = Rx$ and $\tau(M) \cap K = \tau(M)$. This means $x \in \tau(M)$.

Take a submodule $U/\tau(M)$ of $M/\tau(M)$ to show that $M/\tau(M)$ is semisimple. Then M/U has a projective τ-cover f from P to M/U such that $Kerf \subseteq \tau(P)$. Let π be a canonical epimorphism from M to M/U. Then $\pi\alpha = f$ for some $\alpha \in Hom(P, M)$ since P is projective and so $M = U + \alpha(P)$. Let $u = \alpha(p) \in U \cap \alpha(P)$. Then $f(p) = \pi\alpha(p) = 0$ and so $p \in Kerf \subseteq \tau(P)$. Hence $u = \alpha(p) \in \tau(M)$ and so, $U \cap \alpha(P) \subseteq \tau(M)$. Then $U/\tau(M)$ is a direct summand of $M/\tau(M)$.

ii) If $x \in \delta(M)$, then Rx is δ-small and so there is a semisimple projective submodule S of Rx such that $M = K \oplus S$ and so $Rx = (K \cap Rx) \oplus S$. If $S \subseteq \tau(M)$, then $Rx \subseteq \tau(M)$.

iii) Clear.

iv) If M is projective, then $Soc(M) \subseteq \delta(M)$ and so by *ii)*, $Soc(M) = \delta(M)$.

If M is *Soc*-semiperfect, then M is *Soc*-lifting and so by [2, Corollary 4.7], $Z(M) \subseteq Rad(M) \subseteq Soc(M) = \delta(M)$

v) Let U be DM in M and f be an epimorphism from a projective module P to M/U such that $Kerf \subseteq \tau(P)$ and there is an homomorphism α from P to M such that $\pi\alpha = f$ where π is the canonical epimorphism from M to M/U. Then $M = U + \alpha(P)$ and so $M = S + \alpha(P)$ for a direct summand S of M in U. Since M is projective, $M = S \oplus Q$ for a submodule Q of $\alpha(P)$. Take $x \in \alpha(P) \cap U$ and so $x = \alpha(t)$ for some $t \in P$. Since $f(t) = \pi\alpha(t) = 0$, it follows that $t \in Kerf \subseteq \tau(P)$ and $x \in \tau(M)$. Therefore, U is a τ-dense direct summand. □

By the argument of the proof of Lemma 3.16, we have the following corollary.

Corollary 3.17 *Let* M *be a finitely generated module and assume that every simple factor module of* M *has projective* τ*-cover. Then* $M/\tau(M)$ *is semisimple.*

Observe that a projective τ-lifting module is τ-semiperfect. If $\tau = Soc$, then a projective τ-semiperfect is τ-lifting by [10, Lemma 2.22]. However, we don't know whether or not a projective τ-semiperfect module is τ-lifting. Now, under some conditions which are given below, we prove that a projective τ-semiperfect module is τ-lifting.

Theorem 3.18 *Let* τ *be a left exact preradical and* R *be a left hereditary ring. Then a projective* τ*-semiperfect module is* τ*-lifting.*

Proof. Let M be a projective τ-semiperfect module and U be a submodule of M. Assume f is an epimorphism from a projective module Q to M/U such that $Kerf \subseteq \tau(Q)$. Let π be the canonical epimorphism from M to M/U. Since M is projective, there is a homomorphism h from M to Q such that $fh = \pi$. Let $H := h(M)$ and so since R is a left hereditary ring, it follows that H is projective. Then there is a homomorphism α from H to M such that $h\alpha = 1_H$ and so $M = Kerh \oplus \alpha(H)$. Let $a \in Kerh$ and so $fh(a) = \pi(a) = 0$ and so $Kerh \subseteq U$. On the other hand, if $x \in \alpha(H) \cap U$ then $x = \alpha(t)$ for $t \in H$ and so $f(t) = fh\alpha(t) = \pi\alpha(t) = 0$. Then $t \in Kerf \subseteq \tau(Q)$ and so $t \in \tau(Q) \cap H = \tau(H)$ and so $\alpha(t) \in \tau(\alpha(H))$. Therefore, U is a τ-dense direct summand and so M is τ-lifting. □

Theorem 3.19 *Let M be a finitely generated module. Consider the following statements:*

i) M is τ-semiperfect and $\tau(M)$ is QSL.

ii) Every simple factor module of M has a projective τ-cover and $\tau(M)$ is QSL.

iii) M is τ-lifting.

Then we have i) $\Longrightarrow$ ii) $\Longrightarrow$ iii). If M is projective then iii) $\Longrightarrow$ i.

Proof. *i)* $\Rightarrow$ *ii)* Obvious.

ii) $\Rightarrow$ *iii)* Let L be a submodule of M. Since $M/\tau(M)$ is semisimple by Corollary 3.17, it follows that $M/[\tau(M)+L] = \oplus_{i\in K} S_i$ where S_i is simple. Let $f_i : P_i \to S_i$ be a projective τ-cover of S_i. Put $P := \oplus_{i\in K} P_i$ and $f := \oplus_{i\in K} f_i$. Then $f : P \to M/[\tau(M)+L]$ is a projective τ-cover of $M/[\tau(M)+L]$ by [1, 2.13]. Let π be a canonical epimorphism from M to $M/[\tau(M)+L]$. Then there is a homomorphism α from P to M such that $\pi\alpha = f$ and so $M = \alpha(P) + [\tau(M)+L]$. Let $X := \alpha(P)$.

Let $x = \alpha(p) \in [L+\tau(M)] \cap X$ for $p \in P$. Since $f(p) = \pi\alpha(p) = \pi(x) = 0$ and $Kerf \subseteq \tau(P)$, we have $x \in \tau(M)$ and so $(L+\tau(M)) \cap X \subseteq \tau(M)$. Then

$$\begin{aligned} [X+\tau(M)] \cap [L+\tau(M)] &= ([X+\tau(M)] \cap L) + \tau(M) \\ &\subseteq [(X+L) \cap \tau(M)] + [(\tau(M)+L) \cap X] + \tau(M) \subseteq \tau(M) \end{aligned}$$

Hence $M/\tau(M) = [X+\tau(M)]/\tau(M) \oplus [L+\tau(M)]/\tau(M)$ and by hypothesis, there is a decomposition $M = A \oplus B$ such that $A \subseteq L$ and $A+\tau(M) = L+\tau(M)$. Then $M/\tau(M) = [A+\tau(M)]/\tau(M) \oplus [B+\tau(M)]/\tau(M) = [L+\tau(M)]/\tau(M) \oplus [B+\tau(M)]/\tau(M)$ and so
$(L+\tau(M)) \cap (B+\tau(M)) = \tau(M)$. It follows that $B \cap L \subseteq \tau(M)$ and so $B \cap L \subseteq \tau(B)$. Therefore, L contains a τ-dense direct summand.

iii) $\Longrightarrow$ *i)* If M is projective, then M is τ-semiperfect. Also by Lemma 3.5, $\tau(M)$ is QSL. □

Theorem 3.20 *The following statements are equivalent for a ring R:*

i) ${}_RR$ is τ-lifting,

ii) Every finitely generated free R-module is τ-lifting,

iii) Every finitely generated projective R-module is τ-lifting.

iv) If F is a finitely generated free R-module and N is a fully invariant submodule, then F/N is τ-lifting.

Proof. $i) \Rightarrow ii)$ Let R be τ-lifting. Then by [10, Theorem 2.10], a finitely generated free module is τ-lifting.

$ii) \Rightarrow iii) \Rightarrow i), iv) \Rightarrow i)$ It is clear.

$ii) \Rightarrow iv)$ Let K/N be a submodule of F/N. Then there is a decomposition $F = A \oplus B$ such that $K = A \oplus (B \cap K)$ and $B \cap K \subseteq \tau(B)$. Then $F/N = (A+N)/N \oplus (B+N)/N$ and $(A+N)/N \subseteq K/N$. Moreover, $(B+N)/N \cap K/N = (B \cap K + N)/N \subseteq \tau(B+N/N)$. Hence M is τ-lifting. □

Corollary 3.21 *Let a ring R be τ-lifting. Then for a finitely generated projective module M, $\tau(M)$ is QSL.*

Theorem 3.22 *Let M be a finitely generated module with a δ-small submodule $\tau(M)$. Then M is τ-semiperfect if and only if every simple factor module of M has a projective τ-cover.*

Proof. Let every simple factor module of M have projective τ-cover. Then $M/\tau(M)$ is semisimple by Corollary 3.17. Let U be a submodule of M and so $M/(U+\tau(M))$ is semisimple. Then there is a homomorphism f from a projective module P to $M/(U+\tau(M))$ such that $Kerf \subseteq \tau(P)$. Let π be a map from M/U to $M/(U+\tau(M))$ such that $\pi(m+U) = m + (U+\tau(M))$. Then there is a homomorphism α from P to M/U such that $\pi\alpha = f$ and so $M/U = \alpha(P) + (U+\tau(M))/U$ and $Ker\alpha \subseteq \tau(P)$. On the other hand, $(U+\tau(M))/U$ is δ-small in M/U as $\tau(M)$ is δ-small. Hence, $M/U = \alpha(P) \oplus S$ for a semisimple projective submodule S of $(U+\tau(M)/U)$. Then $P \oplus S$ is projective and also we define and epimorphism h from $P \oplus S$ to M/U such that $h(p,s) = \alpha(p) + s$. Take an element $(p,s) \in Kerh$ and so $h(p,s) = \alpha(p) + s = 0$. Then $(p,s) \in Ker\alpha \oplus 0 \subseteq \tau(P) \oplus 0 \subseteq \tau(P \oplus S)$. Therefore, M/U has a projective τ-cover. □

Theorem 3.23 *Let $\tau(_RR) \subseteq \delta(_RR)$ then the following statements are equivalent for a ring R:*

i) $_RR$ is τ-semiperfect,

ii) Every finitely generated R-module M is τ-semiperfect,

iii) Every simple R-module has a projective τ-cover.

Proof. $i) \Rightarrow ii)$ Let M be a finitely generated module and L be a submodule of M. Then $M/(L + \tau(_RR)M)$ is a finitely generated $R/\tau(_RR)$-module. Since $R/\tau(_RR)$ is semisimple by Lemma 3.16, we get that $M/(L+\tau(_RR)M)$ is a semisimple $R/\tau(_RR)$-module and so it is a semisimple R-module. Hence there are simple R-modules S_i such that $M/(L+\tau(_RR)M) = S_1 \oplus ... \oplus S_n$ and so $S_i = Ra_i$ is isomorphic to R/I for some left ideal I. Then S_i has a projective τ-cover and so does $M/(L+\tau(_RR)M)$. Let f be an epimorphism from a projective module P to $M/(L+\tau(_RR)M)$ with $Kerf \subseteq \tau(P)$ and π be a natural map from M/L to $M/(L+\tau(_RR)M)$. Since P is projective, there is an homomorphism g from P to M/L such that $g\pi = f$ and so $M/L = g(P) + [(L+\tau(_RR)M)/L]$. Then since $(L+\tau(_RR)M)/L$ is δ-small in M/L and by Lemma 2.5, it follows that $M/L = g(P) \oplus K$ for a semisimple projective submodule K of M/L. Since g is a projective τ-cover from P to $g(P)$, we get that M/L has a projective τ-cover. Hence M is τ-semiperfect.

$ii) \Rightarrow iii)$ Clear.

$iii) \Rightarrow i)$ By Corollary 3.17, $R/\tau(_RR)$ is semisimple and so by the argument of $i) \Rightarrow ii)$, R is τ-semiperfect. □

Since $Soc_R R$ is strongly lifting, we have the following corollary.

Corollary 3.24 *The following statements are equivalent for a ring R;*
(1) R is Soc-lifting,
(2) R is Soc-semiperfect,
(3) $R/Soc({}_RR)$ is semisimple,
(4) R is Soc-supplemented.

Example 3.25 *[3] Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ be the ring of upper triangular matrices over a field F. Then $N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is a projective left ideal, $L = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is a maximal left ideal and $I = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ is an ideal of R. Consider the R-module $M = N \oplus R/L$. Then $Soc({}_RM) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \oplus R/L$ is SDM but not δ-small because $0 \oplus R/L$ is not δ-small in M.*

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