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Perturbation of Closed Range Operators

Mohammad Sal Moslehian and Ghadir Sadeghi

Abstract

Let T, A be operators with domains $\mathcal{D}(T) \subseteq \mathcal{D}(A)$ in a normed space X. The operator A is called T-bounded if $\|Ax\| \leq a\|x\| + b\|Tx\|$ for some $a, b \geq 0$ and all $x \in \mathcal{D}(T)$. If $\mathcal{A}$ has the Hyers–Ulam stability then under some suitable assumptions we show that both T and $S := A + T$ have the Hyers–Ulam stability. We also discuss the best constant of Hyers–Ulam stability for the operator S. Thus we establish a link between T-bounded operators and Hyers–Ulam stability.

Key Words: Hilbert space; perturbation; Hyers–Ulam stability; closed operator; semi-Fredholm operator.

1. Introduction and preliminaries

Let X, Y be normed linear spaces and T be a (not necessarily linear) mapping from X into Y. Following [5, 6] we say that T has the Hyers-Ulam stability if there exists a constant $K > 0$ with the property:

(i) For any y in the range $\mathcal{R}(T)$ of T, $\varepsilon > 0$ and $x \in X$ with $\|T(x) - y\| \leq \varepsilon$, there exists a $x_0 \in X$ such that $T(x_0) = y$ and $\|x - x_0\| \leq K\varepsilon$.

We call such $K > 0$ a Hyers-Ulam stability constant for T and denote by K_T the infimum of all Hyers-Ulam stability constants for T. If K_T is a Hyers-Ulam stability constant for T, then K_T called the Hyers-Ulam stability constant for T.

If T is linear then condition (i) is equivalent to:

(ii) For any $\varepsilon > 0$ and $x \in X$ with $\|Tx\| \leq \varepsilon$, there exists a $x_0 \in X$ such that $Tx_0 = 0$ and $\|x - x_0\| \leq K\varepsilon$.

If put $\mathcal{N}(T) := \{x \in X : Tx = 0\}$, condition (ii) is equivalent to

(iii) For any $x \in X$ there exists a $x_0 \in \mathcal{N}(T)$ such that $\|x - x_0\| \leq K\|Tx\|$.

We refer the interested reader for more results on the stability of various mappings to papers [10, 11, 12] and references therein, and for a comprehensive accounts of the Hyers-Ulam-Rassias stability of functional equations to the monographs [3, 8, 13].

In [6] the authors proved the following useful result.

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Theorem 1.1 *Let T be a closed operator from the subspace $\mathcal{D}(T)$ of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. The following assertions are equivalent:*

(i) T has the Hyers-Ulam stability;

(ii) T has closed range.

Moreover, if one of the conditions above is true, then $K_T = \gamma(T)^{-1}$, where

$$\gamma(T) = \sup\{\gamma > 0 : \|Tx\| \geq \gamma\|x\|, \qquad x \in \mathcal{D}(T) \cap (\mathcal{N}(T))^{\perp}\}.$$

(Here $\perp$ denotes the orthogonal complement in Hilbert spaces.)

Let X be a Banach space and let M, N be closed linear subspaces of X. Following [9] we define the quantity

$$\delta(M,N) := \inf\{\frac{\operatorname{dist}(x,N)}{\operatorname{dist}(x, M\cap N)} : \qquad x \in M, x \notin N\}(\leq 1)$$

If $M \subseteq N$, then we set $\delta(M,N) = 1$. Obviously $\delta(M,N) = 1$, if $M \supseteq N$. It is well know that $\delta(M,N)$ is not symmetric with respect to (M,N). If $\delta(M,N) = \delta(N,M)$, we say that the pair (M,N) is regular. It is known that any pair (M,N) is regular if X is a Hilbert space [9].

Let A and T be operators with their domains in a normed space X such that $\mathcal{D}(T) \subseteq \mathcal{D}(A)$, and

$$\|Ax\| \leq a\|x\| + b\|Tx\| \qquad (x \in \mathcal{D}(T)), \tag{1.1}$$

where a, b are nonnegative constants. Then we say that A is relatively bounded with respect to T or simply it is T-bounded [9].

A bounded operator A is clearly T-bounded for any T with $\mathcal{D}(T) \subseteq \mathcal{D}(A)$.

In this paper, we show that if a T-bounded operator A has the Hyers-Ulam stability then under some suitable assumptions the operator T and the perturbation $S := A + T$ have the Hyers-Ulam stability. We also discuss the best constant of Hyers-Ulam stability for the operator S. Thus we establish a link between T-bounded operators and the Hyers-Ulam stability.

2. Main Results

Throughout this section $\mathcal{H}$ and $\mathcal{K}$ denote Hilbert spaces and A and T are operators having their domains in $\mathcal{H}$ and their images in $\mathcal{K}$. We start our work with the following theorem.

Theorem 2.1 *Suppose that A is a T-bounded operator with a T-bound smaller than 1. If T is a closed operator and $S := T + A$, then the following assertions are equivalent:*

(i) S has the Hyers-Ulam stability;

(ii) S has closed range.

Moreover, if A is closed and the operators A and T have the Hyers-Ulam stability and $\mathcal{R}(S) = \mathcal{R}(A) + \mathcal{R}(T)$ then conditions (i) and (ii) are equivalent with the following assertions:

(iii)$\delta(M,N) > 0$, where $M = \mathcal{R}(A)$ and $N = \mathcal{R}(T)$;

(v) $\delta(M^{\perp}, N^{\perp}) > 0$, $M = \mathcal{R}(A)$ and $N = \mathcal{R}(T)$.

Proof. The operator S is closed since the operator A is T-bounded with a T-bound smaller than 1 and T is a closed operator (see [9, Theorem 1.1]). It follows from [6, Theorem 3.1] that operator S has the Hyers-Ulam stability if and only if S has closed range. Hence $(i) \Longleftrightarrow (ii)$.
Now, if $\mathcal{R}(S) = \mathcal{R}(A) + \mathcal{R}(T)$ and A and T have the Hyers-Ulam stability then $\mathcal{R}(A)$ and $\mathcal{R}(T)$ are closed and Theorems 4.2 and 4.8 of [9] show that $(ii) \Longleftrightarrow (iii)$ and $(iii) \Longleftrightarrow (v)$. □

Remark 2.2 *If A and T are closed operators as in the above theorem, the operators A and T have the Hyers-Ulam stability, $S := T + A$, $\mathcal{R}(S) = \mathcal{R}(A) + \mathcal{R}(T)$ and we have $\mathcal{R}(A) \subseteq \mathcal{R}(T)$ or $\mathcal{R}(T) \subseteq \mathcal{R}(A)$ then $\delta(\mathcal{R}(A), \mathcal{R}(T)) > 0$. Hence the operator S has the Hyers-Ulam stability and therefore its range is closed.*

Corollary 2.3 *Suppose that A is a T-bounded operator with a T-bound smaller than 1. Let A and T be closed, $S := A + T$ and let A and T have the Hyers-Ulam stability. Suppose that at least one of the spaces $\mathcal{R}(A)$ or $\mathcal{R}(T)$ is finite dimensional and assume that $\mathcal{R}(S) = \mathcal{R}(A) + \mathcal{R}(T)$. Then operator S has the Hyers-Ulam stability and so it has closed range.*

Proof. Without loss of generality assume that $\mathcal{R}(A)$ is finite dimensional. It is known that there exists $u \in \mathcal{R}(T)$ such that $dist(u, \mathcal{R}(A)) = \|u\|$ (see [2]). Hence

$$\delta(\mathcal{R}(A), \mathcal{R}(T)) = \delta(\mathcal{R}(T), \mathcal{R}(A)) > 0.$$

Therefore operator $S = T + A$ has the Hyers-Ualm stability. □

Corollary 2.4 *Suppose that A is a T-bounded operator with a T-bound smaller than 1. Let A and T be closed, $S := A + T$ and let A, T and S have the Hyers-Ulam stability. If $\mathcal{R}(A) \cap \mathcal{R}(T) = \{0\}$, then $\delta(\mathcal{R}(T), \mathcal{R}(A)) = 1$ and*

$$K_S \leq \min\{\frac{1}{\gamma(T)}, \frac{1}{\gamma(A)}\}.$$

Proof. Each $z \in \mathcal{R}(S)$ has a unique expression as $z = x + y$ in which $y \in \mathcal{R}(T)$ and $x \in \mathcal{R}(A)$. Consider the projection P of $\mathcal{R}(S)$ onto $\mathcal{R}(T)$ along $\mathcal{R}(A)$. Now we have

$$1 \;=\; \|P\| = \sup_{z \in \mathcal{R}(S)} \frac{\|Pz\|}{\|z\|} = \sup_{y \in \mathcal{R}(T), x \in \mathcal{R}(A)} \frac{\|y\|}{\|x + y\|} = \sup_{y \in \mathcal{R}(T)} \frac{\|y\|}{dist(y, \mathcal{R}(A))} = \delta(\mathcal{R}(A), \mathcal{R}(T))^{-1}.$$

By the definition of $\gamma(T)$, we have $\|Tv\| \geq \gamma(T)\|v\|$. Hence $\|P\|\|Tv + Av\| \geq \|P(Tv + Av)\| \geq \gamma(T)\|v\|$. So $\|Sv\| \geq \gamma(T)\|v\|$. Since $\gamma(S) \geq \gamma(T)$, by [6, Theorem 3.1], we have $K_S \leq \frac{1}{\gamma(T)}$. We can analogously show that $K_S \leq \frac{1}{\gamma(A)}$. Thus $K_S \leq \min\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(T)}\}$. □

Recall that if x, y are elements of the Hilbert space $\mathcal{H}$, then the bounded operator $x \otimes y$ defined on $\mathcal{H}$ by $(x \otimes y)(z) = \langle z, y \rangle x$ is rank one if x, y are not zero. Let x_1, x_2, y be elements of $\mathcal{H}$ such that $\|x_1\| \leq \frac{\|x_2\|}{2}$. If $A = x_1 \otimes y, T = x_2 \otimes y$ and $S = A + T$, then $\mathcal{N}(A) = \mathcal{N}(T)$ and $\|Ax\| \leq \frac{\|Tx\|}{2}$. It is clear that A, T and

S have the Hyers-Ulam stability (note that they have closed range). This motivates us toward the following theorem.

Theorem 2.5 *Suppose that A is a T-bounded operator with a T-bound b and a constant a and A has the Hyers-Ulam stability.*
If $a = 0$ and $\mathcal{N}(A) = \mathcal{N}(T)$, then T has also the Hyers-Ulam stability.

Proof. There exists a constant $K_0 > 0$ such that for every $x \in \mathcal{D}(A)$ there exists $x_0 \in \mathcal{N}(A) = \mathcal{N}(T)$ such that $\|x - x_0\| \le K_0\|Ax\| \le K_0 b\|Tx\|$. Thus operator T has the Hyers-Ulam stability. □

Now we show that conditions $\mathcal{N}(A) = \mathcal{N}(T)$ and $a = 0$ in Theorem 2.5 are necessary.

Example 2.6 *Consider the operators $A, T : \ell^2 \longrightarrow \ell^2$ defined by*

$$A(x_1, x_2, \cdots) = (x_1, 0, 0, \cdots), \qquad (x_1, x_2, \cdots) \in \ell^2$$

and

$$T(x_1, x_2, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots), \qquad (x_1, x_2, \cdots) \in \ell^2.$$

It is clear that the operator A is T-bounded with constant $a = 0$. Then $\mathcal{R}(A)$ is of finite dimension. Hence the operator A has closed range. Hence A has the Hyers-Ulam stability and $\mathcal{N}(A) \neq \mathcal{N}(T)$. If we take a_n to be

$$a_n = \begin{cases} 1 & i \le n \\ 0 & i > n \end{cases}$$

then

$$(Ta_n)(i) = \begin{cases} 1/i & i \le n \\ 0 & i > n \end{cases}$$

and (Ta_n) converges to $b = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ which does not belong to the range of T. Therefore $\mathcal{R}(T)$ is not closed, i.e, operator T does not have the Hyers-Ulam stability.

Example 2.7 *Consider the operators $A, T : \ell^2 \longrightarrow \ell^2$ defined by*

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), \qquad (x_1, x_2, \cdots) \in \ell^2$$

and

$$T(x_1, x_2, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots), \qquad (x_1, x_2, \cdots) \in \ell^2.$$

The operator A is T-bounded with a nonzero constant a. Since $\gamma(A) > 0$, the operator A has closed range and $\mathcal{N}(A) = \mathcal{N}(T)$. The space $\mathcal{R}(T)$ is not closed, i.e, operator T does not have the Hyers-Ulam stability.

Let x_1, x_2, y be elements of $\mathcal{H}$ such that $x_1 \perp x_2$. If $A = x_1 \otimes y, T = x_2 \otimes y$ and $S = A + T$, then $\gamma(A) = \|x_1\|\|y\|, \gamma(T) = \|x_2\|\|y\|$ and $\gamma(S) = \gamma(A) + \gamma(T)$, therefore $K_S = \gamma(S)^{-1} = \frac{1}{\gamma(A)+\gamma(T)}$. This motivates us toward the following result.

Corollary 2.8 *Suppose that A is a T-bounded operator with a T-bound b smaller than 1 and constant $a = 0$, $\mathcal{N}(A) = \mathcal{N}(T)$ and A has the Hyers-Ulam stability. Then $S := T + A$ has the Hyers-Ulam stability, if $\mathcal{R}(A) \perp \mathcal{R}(T)$. Moreover, if T is a closed operator then $\mathcal{R}(S)$ is closed and $K_S = \frac{1}{\gamma(T)+\gamma(A)}$.*

Proof. Suppose that K is a Hyers-Ulam stability constant for A. By Theorem 2.5, $K' = Kb$ is a Hyers-Ulam stability constant for T. In fact, for each $v \in \mathcal{D}(T)$ there exists $v_0 \in \mathcal{N}(T)$ such that

$$\|v - v_0\| \leq (Kb)\|Tv\| \leq K\|Tv\|$$

since b is smaller than 1.
Hence for $x \in \mathcal{D}(S) = \mathcal{D}(T)$ there exists $x_0 \in \mathcal{N}(T) = \mathcal{N}(A)$ such that

$$\|x - x_0\| \leq K(\|Ax\| + \|Tx\|) = K\|Ax + Tx\|.$$

Now we show that $\mathcal{N}(S) = \mathcal{N}(T)$. If $x \in \mathcal{N}(S) - \mathcal{N}(T)$, then $-Ax = Tx$ and so $\|Tx\| = \|Ax\| \leq b\|Tx\|$. Hence $b \geq 1$ which is a contradiction. Thus $\mathcal{N}(S) \subseteq \mathcal{N}(T)$ since $\mathcal{N}(A) = \mathcal{N}(T)$ and $\mathcal{N}(T) \subseteq \mathcal{N}(S)$. Therefore $\mathcal{N}(S) = \mathcal{N}(T)$. Thus S has the Hyers-Ulam stability.
Assume that T is a closed operator. Then so is S. Hence $\mathcal{R}(S)$ is closed. Since $\frac{\|Sx\|}{\|x\|} = \frac{\|Tx+Ax\|}{\|x\|} = \frac{\|Tx\|}{\|x\|} + \frac{\|Ax\|}{\|x\|}$ and $\mathcal{N}(T) = \mathcal{N}(S)$ we have $\gamma(S) = \gamma(T) + \gamma(A)$. Hence, by [6, Theorem 3.1], $K_S = \frac{1}{\gamma(T)+\gamma(A)}$. □

The following result can be regarded as a special case of [1, Theorem 2.2] with a Hyers-Ulam stability approach.

Theorem 2.9 *Suppose that A is a T-bounded operator with a T-bound b smaller than 1 and constant $a = 0$, and $\mathcal{N}(A) = \mathcal{N}(T)$. Assume that A has the Hyers-Ulam stability and that T is a closed operator. Then $S := T + A$ is a closed operator, S has the Hyers-Ulam stability and*

$$\frac{1}{\gamma(A) + \gamma(T)} \leq K_S \leq \frac{1}{(1 - b)\gamma(T)}.$$

Proof. By Theorem 2.5 the operator T has the Hyers-Ulam stability. Hence it has closed range and so $\gamma(T) > 0$. Since the operator A is T-bounded with a T-bound smaller than 1 and since by [9, Theorem 1.1] T is a closed operator, we deduce that the operator S is closed. In view of $\|Ax\| \leq b\|Tx\|$, we get

$$\|Tx\| - \|Sx\| \leq \|Ax + Tx - Tx\| \leq b\|Tx\| \qquad (x \in \mathcal{D}(T)).$$

Hence $(1 - b)\|Tx\| \leq \|Sx\|$. Thus

$$(1 - b)\frac{\|Tx\|}{\|x\|} \leq \frac{\|Sx\|}{\|x\|} \qquad x \in (\mathcal{D}(T) - \{0\}).$$

Since $\mathcal{N}(T)=\mathcal{N}(S)$ we have $0<(1-b)\gamma(T)\leq\gamma(S)$, therefore S has closed range [9, Theorem 5.2]. Thus S has the Hyers-Ulam stability and $K_S=\gamma(S)^{-1}\leq\frac{1}{(1-b)\gamma(T)}$. Clearly $\gamma(S)\leq\gamma(A)+\gamma(T)$. Therefore $\frac{1}{\gamma(A)+\gamma(T)}\leq K_S$. □

Recall that a closed operator A from $\mathcal{H}$ into $\mathcal{K}$ is called left semi-Fredholm if $\dim\mathcal{N}(A)<\infty$ and $\mathcal{R}(A)$ is closed. It is called right semi-Fredholm if $codim\mathcal{R}(A)<\infty$ and $\mathcal{R}(A)$ is closed. We say a closed operator A is semi-Fredholm if it is left or right semi-Fredholm.

Remark 2.10 *Suppose that A is a T-bounded operator with a T-bound b smaller than 1 and constant $a=0$, and $\mathcal{N}(A)=\mathcal{N}(T)$. If T is a closed operator and has the Hyers-Ulam stability. Then, by Theorem 2.9, the operator $S:=A+T$ is closed and has the Hyers-Ulam stability. So that $\mathcal{R}(S)$ is closed.*
The conclusion that S is closed has already obtained in [4, Theorem V.3.6] under the different assumption that the operator T is semi-Fredholm.

Corollary 2.11 *Suppose that A is a left semi-Fredholm and T-bounded operator with constant $a=0$ and a T-bound b smaller than 1, and T is a closed operator such that $\mathcal{N}(A)=\mathcal{N}(T)$. Then $S:=T+A$ is a left semi-Fredholm operator.*

Theorem 2.12 *Suppose that A is a T-bounded operator with a T-bound b smaller than 1 and constant $a=0$, and $\mathcal{N}(A)=\mathcal{N}(T)$. If $S=T+A$ has the Hyers-Ulam stability then T has the Hyers-Ulam stability. Moreover if S is a closed operator then $\mathcal{R}(S)$ and $\mathcal{R}(T)$ are closed.*

Proof. The operator S has the Hyers-Ulam stability thus there exists a constant $K>0$ with the following property:

For any $x\in\mathcal{D}(S)=\mathcal{D}(T)$ there exists a $x_0\in\mathcal{N}(S)$ such that $\|x-x_0\|\leq K\|Sx\|$.

Since A is a T-bounded operator and, by the proof of Corollary 2.8, $\mathcal{N}(T)=\mathcal{N}(S)$, we have

$$\|x-x_0\|\leq K\|Sx\|\leq K(\|Ax\|+\|Tx\|)\leq K(b+1)\|Tx\|.$$

Therefore T has the Hyers-Ulam stability.

Now assume that S is a closed operator. Then so is T. In view of S and T having the Hyers-Ulam stability, $\mathcal{R}(S)$ and $\mathcal{R}(T)$ are closed. □

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Mohammad Sal MOSLEHIAN[1,2], Ghadir SADEGHI[1,2]
[1]Department of Pure Mathematics,
Ferdowsi University of Mashhad, P. O. Box
1159, Mashhad 91775, IRAN
e-mail: moslehian@ferdowsi.um.ac.ir and moslehian@ams.org
[2]Center of Excellence in Analysis
on Algebraic Structures (CEAAS),
Ferdowsi University of Mashhad-IRAN
e-mail: ghadir54@yahoo.com

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