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Modified Szász-Mirakjan-Kantorovich Operators Preserving Linear Functions

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Abstract

In this paper, we introduce a modification of the Szász-Mirakjan-Kantorovich operators, which preserve the linear functions. This type of operator modification enables better error estimation on the interval $[1/2, +\infty)$ than the classical Szász-Mirakjan-Kantorovich operators. We also obtain a Voronovskaya-type theorem for these operators.

Key Words: Szász-Mirakjan operators, Szász-Mirakjan-Kantorovich operators, the Korovkin-type approximation theorem, modulus of continuity, Lipschitz class functionals, Voronovskaya type theorem

1. Introduction

Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in approximation theory, which allows us to approximate much faster to the function being approximated. In [1, 6, 11], various approximation properties of the classical Szász-Mirakjan operators and Szász-Mirakjan-Kantorovich operators were investigated. Recently, in [3], by modifying the Szász-Mirakjan operators, we have showed that our modified operators have better error estimation than the classical ones. We should recall that such investigations were accomplished for Bernstein polynomials by King [7], for Meyer-König and Zeller operators by Özarslan and Duman [9] and for Szász-Mirakjan-Beta operators by Duman, Özarslan and Aktuğlu [4]. In this paper, we apply our method to the classical Szász-Mirakjan-Kantorovich operators.

Consider the Banach lattice

$$C_\gamma[0,+\infty) := \{f \in C[0,+\infty) : |f(t)| \leq M(1+t)^\gamma \text{ for some } M > 0,\ \gamma > 0\}.$$

Then, the classical Szász-Mirakjan operators are defined by

$$S_n(f;x) := e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

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where $f \in C_\gamma[0,+\infty)$, $x \geq 0$ and $n \in \mathbb{N}$. Various approximation properties of the Szász-Mirakjan operators and their iterates may be found in [1, 3, 4, 5, 6, 8, 10, 11, 12] and the references cited therein.

The Kantorovich version of the Szász-Mirakjan operators are defined by

$$K_n(f;x) := ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{I_{n,k}} f(t)dt, \tag{1.1}$$

where $I_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right]$ and $f \in C_\gamma[0,+\infty)$.

Now, for the Szász-Mirakjan-Kantorovich operators K_n given by (1.1), the following lemma follows from [6] immediately.

Lemma A [6]**.** *Let* $e_i(x) = x^i$, $i = 0,1,2,3,4$. *Then, for each* $x \geq 0$, and $n > 1$, *we have*

(*a*) $K_n(e_0;x) = 1$,

(*b*) $K_n(e_1;x) = x + \dfrac{1}{2n}$,

(*c*) $K_n(e_2;x) = x^2 + \dfrac{2x}{n} + \dfrac{1}{3n^2}$,

(*d*) $K_n(e_3;x) = x^3 + \dfrac{9x^2}{2n} + \dfrac{7x}{2n^2} + \dfrac{1}{4n^3}$,

(*e*) $K_n(e_4;x) = x^4 + \dfrac{8x^3}{n} + \dfrac{15x^2}{n^2} + \dfrac{6x}{n^3} + \dfrac{1}{5n^4}$.

2. Construction of the Operators

The set $\{e_0, e_1, e_2\}$ is a K_+-subset of $C_\gamma[0,+\infty)$ for $\gamma \geq 2$; also the space $C_\gamma[0,+\infty)$ is isomorphic to $C[0,1]$. Recall that a subset H of $C_\gamma[0,+\infty)$ is called a Korovkin subset with respect to positive linear operators or, briefly, a K_+-subset of $C_\gamma[0,+\infty)$ if it satisfies the following property:

> *if* $\{L_n\}$ *is an arbitrary sequence of positive linear operators from* $C_\gamma[0,+\infty)$ *into itself such that* $\lim_{n\to\infty} L_n(h) = h$ *for all* $h \in H$, *then* $\lim_{n\to\infty} L_n(f) = f$ *for every* $f \in C_\gamma[0,+\infty)$

(see [2] for details).

Let $\{r_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0,+\infty)$ with $0 \leq r_n(x) < +\infty$. Then we have

$$K_n(f;r_n(x)) := ne^{-nr_n(x)} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{k!} \int_{I_{n,k}} f(t)dt.$$

Now, if we replace $r_n(x)$ by $r_n^*(x)$ defined as

$$r_n^*(x) := x - \frac{1}{2n}, \quad x \geq \frac{1}{2} \text{ and } n \in \mathbb{N}, \tag{2.2}$$

then we get the following positive linear operators:

$$K_n^*(f;x) := ne^{\frac{1-2nx}{2}} \sum_{k=0}^{\infty} \frac{(2nx-1)^k}{2^k k!} \int_{I_{n,k}} f(t)dt, \tag{2.3}$$

where $f \in C_\gamma[0,+\infty)$, $\gamma > 0$ and $x \geq 1/2$. Observe that if $x \in [1/2,+\infty)$, then $r_n^*(x)$ given by (2.2) belongs to the interval $[0,+\infty)$.

On the other hand, from Lemma A we obtain the following result at once.

Lemma 2.1 *For each $x \geq 1/2$, we have*

(a) $K_n^*(e_0;x) = 1$,

(b) $K_n^*(e_1;x) = x$,

(c) $K_n^*(e_2;x) = x^2 + \dfrac{x}{n} - \dfrac{5}{12n^2}$,

(d) $K_n^*(e_3;x) = x^3 + \dfrac{3x^2}{n} - \dfrac{x}{4n^2} - \dfrac{1}{2n^3}$,

(e) $K_n^*(e_4;x) = x^4 + \dfrac{6x^3}{n} + \dfrac{9x^2}{2n^2} - \dfrac{7x}{2n^3} - \dfrac{1}{80n^4}$.

By Lemma 2.1, it is clear that the positive linear operators K_n^* given by (2.3) preserve the linear functions, that is, for $h(t) = ct + b$ (c and d are any real numbers), $K_n^*(h;x) = h(x)$ for all $x \geq 1/2$ and $n \in \mathbb{N}$.

Now, fix $b > 1/2$ and consider the lattice homomorphism $T_b : C[0,+\infty) \to C[0,b]$ defined by $T_b(f) := f|_{[0,b]}$ for every $f \in C[0,+\infty)$, where $f|_{[0,b]}$ denotes the restriction of the domain of f to the interval $[0,b]$. In this case, we see that, for each $i = 0,1,2$,

$$\lim_{n\to\infty} T_b\left(K_n^*(e_i)\right) = T_b(e_i) \quad \text{uniformly on } [1/2,b]. \tag{2.4}$$

Thus, by using (2.4) and with the universal Korovkin-type property with respect to monotone operators (see Theorem 4.1.4 (vi) of [2, p. 199]) we have the following Korovkin-type approximation result.

Theorem 2.2 $\lim_{n\to\infty} K_n^*(f;x) = f(x)$ *uniformly with respect to $x \in [1/2,b]$ provided $f \in C_\gamma[0,+\infty)$, $\gamma \geq 2$ and $b > 1/2$.*

In order to get uniform convergence on $[1/2,+\infty)$ of the sequence $\{K_n^*(f)\}$ we consider the following subspace E of $C_\gamma[0,+\infty)$:

$$E := \left\{ f \in C[0,+\infty) : \lim_{t\to+\infty} f(t) \text{ is finite} \right\}$$

endowed with the sup-norm.

For a given $\lambda > 0$, consider the function $f_\lambda(t) := e^{-\lambda t}$, $(t \geq 0)$. Then, for every $x \geq 1/2$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} K_n^*(f_\lambda; x) &= ne^{\frac{1-2nx}{2}} \sum_{k=0}^{\infty} \frac{(2nx-1)^k}{2^k k!} \int_{I_{n,k}} e^{-\lambda t} dt \\ &= \frac{n\left(1-\exp(-\lambda/n)\right)}{\lambda} \times \exp\left(-n(x-1/2n)\right) \sum_{k=0}^{\infty} \frac{\left(n(x-1/2n)e^{-\lambda/n}\right)^k}{k!} \\ &= \frac{n\left(1-\exp(-\lambda/n)\right)}{\lambda} \times \exp\left\{-n\left(x-\frac{1}{2n}\right)\left(1-\exp\left(-\lambda/n\right)\right)\right\}. \end{aligned}$$

Since $\lim_{n\to\infty} n\left(1-\exp(-\lambda/n)\right) = \lambda$, we conclude that

$$\lim_{n\to\infty} K_n^*(f_\lambda) = f_\lambda \quad \text{uniformly on } [1/2, +\infty).$$

Hence using this limit and applying Proposition 4.2.5-(7) of [2, p. 215] one can obtain the next result at once.

Theorem 2.3 $\lim_{n\to\infty} K_n^*(f) = f$ *uniformly on* $[1/2, +\infty)$ *provided* $f \in E$.

We can also give an L_p-approximation for the operators $K_n^*(f;x)$ by using Proposition 4.2.5-(2) of [2, p. 215] as follows.

Corollary 2.4 *Let* $1 \leq p < +\infty$. *Then, for all* $f \in L_p[0, +\infty)$, $\lim_{n\to\infty} K_n^*(f;x) = f(x)$ *uniformly with respect to* $x \in [1/2, +\infty)$.

3. Better Error Estimation

In this section we compute the rate of convergence of the operators K_n^* defined by (2.3). Then, we will show that our operators have a better error estimation on the interval $[1/2, +\infty)$ than the Szász-Mirakjan-Kantorovich operators K_n given by (1.1). To achieve this we use the modulus of continuity and the elements of Lipschitz class functionals.

If we define the function ψ_x, $(x \geq 0)$, by $\psi_x(t) = t - x$, then by Lemma 2.1 one can get the following result, immediately.

Lemma 3.1 *For every* $x \geq 1/2$, *we have*

(a) $K_n^*(\psi_x; x) = 0$,

(b) $K_n^*(\psi_x^2; x) = \dfrac{x}{n} - \dfrac{5}{12n^2}$,

(c) $K_n^*(\psi_x^3; x) = \dfrac{x}{n^2} - \dfrac{1}{2n^3}$,

$(d)\quad K_n^*(\psi_x^4;x)=\dfrac{3x^2}{n^2}-\dfrac{3x}{2n^3}-\dfrac{1}{80n^4}.$

Let $f\in C_B[0,+\infty)$, the space of all bounded functions on $[0,+\infty)$, and $x\geq 1/2$. Then, for $\delta_x>0$, the modulus of continuity of f denoted by $\omega(f,\delta_x)$, is defined to be

$$\omega(f,\delta_x)=\sup_{x-\delta_x\leq t\leq x+\delta_x;\ t\in[0,+\infty)}\left|f(t)-f(x)\right|.$$

Then we have the following theorem.

Theorem 3.2 *For every* $f\in C_B[0,+\infty)$, $x\geq 1/2$ *and* $n\in\mathbb{N}$, *we have*

$$\left|K_n^*(f;x)-f(x)\right|\leq 2\omega(f,\delta_{n,x}),$$

where $\delta_{n,x}:=\sqrt{\dfrac{x}{n}-\dfrac{5}{12n^2}}$.

Proof. Now, let $f\in C_B[0,+\infty)$ and $x\geq 0$. Using linearity and monotonicity of K_n^* we easily get, for $\delta_x>0$ and $n\in\mathbb{N}$, that

$$\left|K_n^*(f;x)-f(x)\right|\leq\omega(f,\delta)\left\{1+\frac{1}{\delta}\sqrt{K_n^*\left(\psi_x^2;x\right)}\right\}.$$

Now applying Lemma 3.1 (b) and choosing $\delta=\delta_{n,x}$, the proof is complete. □

Remark. For the Szász-Mirakjan-Kantorovich operators given by (1.1) we may write that, for every $f\in C_B[0,+\infty)$, $x\geq 0$ and $n\in\mathbb{N}$,

$$\left|K_n(f;x)-f(x)\right|\leq 2\omega(f,\alpha_{n,x}), \tag{3.5}$$

where $\alpha_{n,x}:=\sqrt{\dfrac{x}{n}+\dfrac{1}{3n^2}}$ (see [5, 6]).

Now we claim that the error estimation in Theorem 3.2 is better than that of (3.5) provided $f\in C_B[0,+\infty)$ and $x\geq 1/2$. Indeed, for $x\geq 1/2$ and $n\in\mathbb{N}$, it is clear that

$$\frac{x}{n}-\frac{5}{12n^2}\leq\frac{x}{n}+\frac{1}{3n^2}. \tag{3.6}$$

This guarantees that $\delta_{n,x}\leq\alpha_{n,x}$ for $x\geq 1/2$ and $n\in\mathbb{N}$.

Now we can also compute the rate of convergence of the operators K_n^* by means of the elements of the Lipschitz class $Lip_M(\alpha)$, $(\alpha\in(0,1])$. As usual, we say that a function $f\in C_B[0,+\infty)$ belongs to $Lip_M(\alpha)$ if the inequality

$$\left|f(t)-f(x)\right|\leq M\left|t-x\right|^{\alpha} \tag{3.7}$$

holds for all $t\in[0,+\infty)$ and $x\in[1/2,+\infty)$.

Theorem 3.3 *For every* $f\in Lip_M(\alpha)$, $x\geq 1/2$ *and* $n\in\mathbb{N}$, *we have*

$$\left|K_n^*(f;x)-f(x)\right|\leq M\left\{\frac{x}{n}-\frac{5}{12n^2}\right\}^{\frac{\alpha}{2}}.$$

Proof. Since $f \in Lip_M(\alpha)$ and $x \geq 0,$ using inequality (3.7) and then applying the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ we get

$$|K_n^*(f;x) - f(x)| \leq K_n^*\left(|f(t)-f(x)|\,;x\right) \leq M\,K_n^*\left(|t-x|^{\alpha}\,;x\right) \leq M\,\left\{K_n^*\left(\psi_x^2;x\right)\right\}^{\frac{\alpha}{2}} \leq M\left\{\frac{x}{n} - \frac{5}{12n^2}\right\}^{\frac{\alpha}{2}},$$

whence the result. □

Notice that as in the proof of Theorem 3.2, since $K_n(\psi_x^2;x) = \dfrac{x}{n} + \dfrac{1}{3n^2}$, the Szász-Mirakjan-Kantorovich operators defined by (1.1) satisfy

$$|K_n(f;x) - f(x)| \leq M\left\{\frac{x}{n} + \frac{1}{3n^2}\right\}^{\frac{\alpha}{2}} \tag{3.8}$$

for every $f \in Lip_M(\alpha)$, $x \geq 1/2$ and $n \in \mathbb{N}$. So, it follows from (3.6) that the above claim also holds for Theorem 3.2, i.e., the rate of convergence of the operators K_n^* by means of the elements of the Lipschitz class functionals is better than the ordinary error estimation given by (3.8) whenever $x \geq 1/2$ and $n \in \mathbb{N}$.

4. A Voronovskaya-Type Theorem

In this section, we prove a Voronovskaya-type theorem for the operators K_n^* given by (2.3).

We first need the following lemma.

Lemma 4.1 $\lim_{n\to\infty} n^2 K_n^*\left(\psi_x^4;x\right) = 3x^2$ *uniformly with respect to* $x \in [1/2,b]\,(b > 1/2)$.

Proof. Then, by Lemma 3.1 (d), we may write that

$$n^2 K_n^*\left(\psi_x^4;x\right) = 3x^2 - \frac{3x}{2n} - \frac{1}{80n^2}.$$

Now taking limit as $n \to \infty$ on the both sides of the above equality the proof is complete. □

Theorem 4.2 *For every* $f \in C_\gamma[0,+\infty)$ *such that* f', $f'' \in C_\gamma[0,+\infty)$, $\gamma \geq 4$, *we have*

$$\lim_{n\to\infty} n\left\{K_n^*(f;x) - f(x)\right\} = \frac{1}{2}\,x\,f''(x)$$

uniformly with respect to $x \in [1/2,b]$ $(b > 1/2)$.

Proof. Let f, f', $f'' \in C_\gamma[0,+\infty)$ and $x \geq 1/2$. Define

$$\Psi(t,x) = \begin{cases} \dfrac{f(t) - f(x) - (t-x)f'(x) - \frac{1}{2}(t-x)^2 f''(x)}{(t-x)^2}, & \text{if } t \neq x \\ 0, & \text{if } t = x. \end{cases}$$

Then by assumption we have $\Psi(x,x)=0$ and the function $\Psi(\cdot,x)$ belongs to $C_\gamma[0,+\infty)$. Hence, by Taylor's theorem we get

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+(t-x)^2\Psi(t,x).$$

Now from Lemma 3.1 $(a)-(b)$

$$n\left\{K_n^*(f;x)-f(x)\right\}=\frac{n}{2}\left(\frac{x}{n}-\frac{5}{12n^2}\right)f''(x)+n\,K_n^*\left(\psi_x^2(t)\Psi(t,x);x\right). \tag{4.9}$$

If we apply the Cauchy-Schwarz inequality for the second term on the right-hand side of (4.9), then we conclude that

$$n\left|K_n^*\left(\psi_x^2(t)\Psi(t,x);x\right)\right|\le\left(n^2K_n^*(\psi_x^4(t);x)\right)^{\frac{1}{2}}\left(K_n^*(\Psi^2(t,x);x)\right)^{\frac{1}{2}}. \tag{4.10}$$

Let $\eta(t,x):=\Psi^2(t,x)$. In this case, observe that $\eta(x,x)=0$ and $\eta(\cdot,x)\in C_\gamma[0,+\infty)$. Then it follows from Theorem 2.2 that

$$\lim_{n\to\infty}K_n^*\left(\Psi^2(t,x);x\right)=\lim_{n\to\infty}K_n^*\left(\eta(t,x);x\right)=\eta(x,x)=0 \tag{4.11}$$

uniformly with respect to $x\in[1/2,b]$ $(b>1/2)$. Now considering (4.10) and (4.11), and also using Lemma 4.1, we immediately see that

$$\lim_{n\to\infty}n\,K_n^*\left(\psi_x^2(t)\Psi(t,x);x\right)=0 \tag{4.12}$$

uniformly with respect to $x\in[1/2,b]$. On the other hand, observe now that, by (3.6),

$$\lim_{n\to\infty}\frac{n}{2}\left(\frac{x}{n}-\frac{5}{12n^2}\right)=\frac{1}{2}x. \tag{4.13}$$

Then, taking limit as $n\to\infty$ in (4.9) and using (4.12) and (4.13) we have

$$\lim_{n\to\infty}n\,\left\{K_n^*(f;x)-f(x)\right\}=\frac{1}{2}xf''(x)$$

uniformly with respect to $x\in[1/2,b]$ with $b>1/2$. So, the proof is completed. □

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References

[1] Agrawal, P.N., Kasana, H.S.: On simultaneous approximation by Szász-Mirakjan operators. Bull. Inst. Math. Acad. Sinica 22, 181-188 (1994).

[2] Altomare, F., Campiti, M.: Korovkin-type Approximation Theory and its Application. Walter de Gruyter Studies in Math., vol. 17, de Gruyter & Co., Berlin, 1994.

[3] Duman, O., Özarslan, M.A.: Szász-Mirakjan type operators providing a better error estimation. Appl. Math. Lett. 20, 1184-1188 (2007).

[4] Duman, O., Özarslan, M.A., Aktuğlu, H.: Better error estimation for Szász-Mirakjan-Beta operators. J. Comput. Anal. Appl. 10, 53-59 (2008).

[5] Gupta, V., Noor, M.A.: Convergence of derivatives for certain mixed Szász-Beta operators. J. Math. Anal. Appl. 321, 1-9 (2006).

[6] Gupta, V., Vasishtha, V., Gupta, M.K.: Rate of convergence of the Szász-Kantorovitch-Bezier operators for bounded variation functions. Publ. Inst. Math. (Beograd) (N.S.) 72, 137-143 (2002).

[7] King, J.P.: Positive linear operators which preserve x^2, Acta. Math. Hungar. 99, 203-208 (2003).

[8] Özarslan, M.A., Duman, O.: Local approximation results for Szász-Mirakjan type operators. Arch. Math. (Basel) 90, 144-149 (2008).

[9] Özarslan, M.A., Duman, O.: MKZ type operators providing a better estimation on $[1/2, 1)$. Canad. Math. Bull. 50, 434-439 (2007).

[10] Srivastava, H.M., Gupta, V.: A certain family of summation integral type operators. Math. Comput. Modelling 37, 1307-1315 (2003).

[11] Totik, V: Uniform approximation by Szász-Mirakjan type operators. Acta Math. Hungarica 41, 291-307 (1983).

[12] Della Vecchia, B., Mastroianni, G., Szabados, J.: Weighted approximation of functions by Szász-Mirakyan-type operators. 111, 325-345 (2006).

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