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Existence and uniqueness theorem for slant immersions in Kenmotsu space forms

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Abstract

In this paper we have obtained a general existence as well as uniqueness theorem for slant immersions into a Kenmotsu-space form.

Key Words: Kenmotsu manifold, slant immersion, mean curvature, sectional curvature.

1. Introduction

B. Y. Chen has defined and studied slant immersions by generalizing the concept of holomorphic and totally real immersions [5]. Latter, it was A. Lotta [14], who introduced the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold. B. Y. Chen and Y. Tazawa [8] have obtained examples of n-dimensional proper slant submanifolds in the complex Euclidean n-space C^n. On the other hand, Chen and Vrancken [6] have established the existence of n-dimensional proper slant submanifolds into a non-flat complex space form $\bar{M}^n(4c)$ and in contact geometry J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, and M. A. Fernandez [2] have established the existence and uniqueness theorem in Sasakian space form. Later, R. S. Gupta, S. M. K. Haider and A. Sharfuddin [10] have obtained the existence and uniqueness theorem into a non-flat cosymplectic space form.

The purpose of the present paper is to establish a general existence and uniqueness theorem for slant immersions in Kenmotsu-space forms.

In section 2, we review some basic formulae and results for our subsequent use.

2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ a vector field, η a 1-form and g is the Riemannian metric on $\overline{M}$. These tensors satisfy [1]

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$$\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \eta(\xi) = 1, \eta(\phi X) = 0$$

and (2.1)

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi)$$

for any X, $Y \in T\bar{M}$, where $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$.

An almost contact metric manifold $\bar{M}$ is called a Kenmotsu manifold if [12],

$$(\overline{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \text{ and } \overline{\nabla}_X \xi = X - \eta(X)\xi \tag{2.2}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

The curvature tensor $\bar{R}$ of Kenmotsu space form $\bar{M}(c)$ is given by [12],

$$\bar{R}(X,Y)Z = \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\left\{\begin{array}{l} \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi \\ -\eta(X)g(Y,Z)\xi - g(\varphi X, Z)\varphi Y \\ +g(\varphi Y, Z)\varphi X + 2g(X, \varphi Y)\varphi Z \end{array}\right\} \tag{2.3}$$

for all $X, Y, Z \in T\bar{M}$.

Now, let M be an m-dimensional Riemannian manifold isometrically immersed in a Kenmotsu manifold $\overline{M}$. Denoting by TM the tangent bundle of M and by $T^{\perp}M$ the set of all vector fields normal to M, we write,

$$\varphi X = PX + FX \text{ and } \varphi N = tN + fN \tag{2.4}$$

for any $X \in TM$ and $N \in T^{\perp}M$, where PX (resp. FX) denotes the tangential (resp. normal) component of φX, and tN (resp. fN) denotes the tangential (resp. normal) component of φN.

From now on, we assume that the structure vector field ξ is tangent to M. We take the orthogonal direct decomposition $TM = D \oplus \{\xi\}$.

A submanifold M is said to be slant if for any non zero X tangent to M at x such that X is not proportional to ξ_x, the angle $\theta(X)$ between φX and T_xM is constant, i.e. $\theta(X)$ is independent of the choice of $x \in M$ and $X \in T_xM - \{\xi_x\}$. Sometime the angle $\theta(X)$ is termed as Wirtinger angle of the slant immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

Let $\bar{\nabla}$ (resp. ∇) denote the Riemannian connection on $\bar{M}$ (resp. M) and $\nabla^{\perp}$ denote the connection in the normal bundle $T^{\perp}M$ of M. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.5}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.6}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$.

The second fundamental forms h and A_N are related by

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.7}$$

Denote by R the curvature tensor of M and by $R^{\perp}$ the curvature tensor of the normal connection. Then the equations of Gauss, Ricci and Codazzi are given by

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)) \tag{2.8}$$

$$\bar{R}(X,Y,U,V) = R^{\perp}(X,Y,U,V) - g([A_U, A_V]X, Y) \tag{2.9}$$

and

$$(\bar{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \tag{2.10a}$$

for all $X,Y,Z,W \in TM$ and $U,V \in T^{\perp}M$, where $(\bar{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\bar{R}(X,Y)Z$, and $(\bar{\nabla}_X h)(Y,Z)$ is given by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.10b}$$

Now if P is the endomorphism given by (2.4), then we have

$$g(PX,Y) + g(X,PY) = 0. \tag{2.11}$$

Thus, it is obvious that the operator P^2, which is denoted by Q, is self adjoint. Also,

$$(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y) \tag{2.12}$$

$$(\nabla_X F)Y = \nabla^{\perp}_X(FY) - F(\nabla_X Y) \tag{2.13}$$

for any $X,Y \in TM$.

Now, Gauss and Weingarten formulae together with (2.2) and (2.9) imply

$$(\nabla_X P)Y = A_{FY}X + th(X,Y) + g(Y,PX)\xi - \eta(Y)PX \tag{2.14}$$

$$\nabla^{\perp}_X(FY) - F(\nabla_X Y) = fh(X,Y) - h(X,PY) - \eta(Y)FX \tag{2.15}$$

for any $X,Y \in TM$.

For each $X \in TM$, we denote

$$X^* = \frac{FX}{\sin\theta}. \tag{2.16}$$

Now, one can define a symmetric bilinear TM-valued form δ on M, given by

$$\delta(X,Y) = th(X,Y). \tag{2.17}$$

Moreover, using (2.2), we have

$$\delta(X,\xi) = 0. \tag{2.18}$$

Also, from (2.4), (2.16) and (2.17), we get

$$\varphi\delta(X,Y) = P\delta(X,Y) + sin\theta\,\delta^*(X,Y). \tag{2.19}$$

Now using (2.4) and (2.17), we get

$$\varphi h(X,Y) = \delta(X,Y) + \sigma^*(X,Y), \tag{2.20}$$

where σ is a symmetric bilinear D-valued form on M. Applying φ on (2.10) and using (2.19) with (1.4), we find

$$-h(X,Y) = P\delta(X,Y) + \sin\theta\,\delta^*(X,Y) + t\sigma^*(X,Y) + f\sigma^*(X,Y). \tag{2.21}$$

Equating tangential as well as normal parts in the above equation, we have

(a) $P\delta(X,Y) = -t\sigma^*(X,Y)$

and

(b) $-h(X,Y) = \sin\theta\,\delta^*(X,Y) + f\sigma^*(X,Y)$.

Moreover, $\varphi^2\sigma(X,Y) = -\sigma(X,Y) = P^2\sigma(X,Y) + FP\sigma(X,Y) + tF\sigma(X,Y) + fF\sigma(X,Y)$

Comparison of tangential and normal parts yields

(c) $-\sin^2\theta\,\sigma(X,Y) = tF\sigma(X,Y)$

and

(d) $FP\sigma(X,Y) = -fF\sigma(X,Y)$.

Now from (a) and (c), we get

$$\sigma(X,Y) = \csc\theta\, P\delta(X,Y) \tag{2.22}$$

Also, (b) and (d) after making use of (2.22), give

$$h(X,Y) = -\csc\theta\,\delta^*(X,Y) \tag{2.23}$$

Using (2.19), we get

$$h(X,Y) = \csc^2\theta\,\left(P\delta(X,Y) - \varphi\delta(X,Y)\right). \tag{2.24}$$

Now, from (2.14)

$$g((\nabla_X P)Y,Z) = -g(\delta(X,Z),Y) + g(\delta(X,Y),Z) + \eta(Z)g(PX,Y) + \eta(Y)g(X,PZ). \tag{2.25}$$

From (2.3), we have

$$\bar{R}(X,Y,Z,W) = \tfrac{c-3}{4}\{g(Y,Z)g(X,W) - g(X,Z)\,g(Y,W)\}$$

$$+\tfrac{c+1}{4}\left\{\begin{array}{c} \eta(X)\eta(Z)g(Y,W) - \eta(Y)\,\eta(Z)g(X,W) \\ +\eta(Y)\eta(W)g(X,Z) - \eta(X)\,\eta(W)g(Y,Z) - g(\varphi X,Z)\,g(\varphi Y,W) \\ +g(\varphi Y,Z)g(\varphi X,W) + 2g(X,\varphi Y)\,g(\varphi Z,W) \end{array}\right\} \tag{2.26}$$

for $X, Y, Z, W \in TM$.

Using (2.1), (2.4) and (2.8) in (2.26), we find

$$R(X,Y,Z,W)-g(h(X,W),h(Y,Z))+g(h(X,Z),h(Y,W))=\tfrac{c-3}{4}\{g(Y,Z)g(X,W)-$$

$$g(X,Z)(Y,W)\}+\tfrac{c+1}{4}\left\{\begin{array}{l}\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)\\ +\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z)-g(PX,Z)g(PY,W)\\ +g(PY,Z)g(PX,W)+2g(X,PY)g(PZ,W)\end{array}\right\}, \tag{2.27}$$

which in the view of (2.23) and the relation $g(FX,FY)=\sin^2\theta\,\{g(X,Y)-\eta(X)\eta(Y)\}$, gives

$$\begin{aligned}R(X,Y,Z,W)&=\csc^2\theta\,\{g(\delta(X,W),\delta(Y,Z))-g(\delta(X,Z),\delta(Y,W))\}\\ &+\tfrac{c-3}{4}\{g(Y,Z)g(X,W)-g(X,Z)(Y,W)\}\\ &+\tfrac{c+1}{4}\left\{\begin{array}{l}\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)\\ +\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z)\\ -g(PX,Z)g(PY,W)+g(PY,Z)g(PX,W)+2g(X,PY)g(PZ,W)\end{array}\right\}.\end{aligned} \tag{2.28}$$

Now taking normal part of equation (2.3), we get

$$[\bar{R}(X,Y)Z]^{\perp}=\frac{c+1}{4}\{-g(PX,Z)FY+g(PY,Z)FX+2g(X,PY)FZ\} \tag{2.29}$$

We have,

$$\begin{aligned}\nabla^{\perp}_X(h(Y,Z))&=\nabla^{\perp}_X(-\csc\theta\,\delta^*(Y,Z))\\ &=-\csc^2\theta\,\nabla^{\perp}_X(F\delta(Y,Z))\\ &=-\csc^2\theta\,\{(\nabla_XF)\delta(Y,Z)+F(\nabla_X\delta(Y,Z))\}\end{aligned}$$

Using (2.15), we get

$$\nabla^{\perp}_X(h(Y,Z))=-\csc^2\theta\left\{\begin{array}{l}fh(X,\delta(Y,Z))-h(X,P\delta(Y,Z))+F((\nabla_X\delta)(Y,Z)\\ +\delta(\nabla_XY,Z)+\delta(Y,\nabla_XZ))\end{array}\right\}.$$

From (2.23), we obtain

$$h\ (\nabla_XY,Z)=-\csc\ \theta\,\delta^*(\nabla_XY,Z)=-\csc^2\theta\,F\delta(\nabla_XY,Z)$$

Also, $h\ (Y,\nabla_XZ)=-\csc^2\theta\,F\delta(Y,\nabla_XZ)$.

Hence using (2.10) (b), we get

$$(\bar{\nabla}_Xh)(Y,Z)=-\csc^2\theta\,\{fh\,(X,\delta(Y,Z))-h(X,P\delta(Y,Z)+F((\nabla_X\delta)(Y,Z)\}\,.$$

Since, $fh\,(X,Y) = \csc^2\theta\, FP\delta(X,Y)$, we have

$$(\bar{\nabla}_X h)(Y,Z) = -\csc^2\theta\,[\csc^2\theta\, FP\delta(X,\delta(Y,Z)) + \\ + \csc^2\theta\, F\delta(X, P\delta(Y,Z) + F((\nabla_X\delta)(Y,Z)]. \tag{2.30}$$

Now using (2.29) and (2.30) in Codazzi equation, we obtain

$$\begin{aligned}&(\nabla_X\delta)(Y,Z) + \csc^2\theta\,\{P\delta(X,\delta(Y,Z)) + \delta(X,P\delta(Y,Z))\} + \\ &+ \tfrac{c+1}{4}\sin^2\theta\{g(X,PZ)(Y-\eta(Y)\xi) + g(X,PY)(Z-\eta(Z)\xi)\} \\ &= (\nabla_Y\,\delta)(X,Z) + \csc^2\theta\,\{P\delta(Y,\delta(X,Z)) + \delta(Y,P\delta(X,Z))\} + \\ &+ \tfrac{c+1}{4}\sin^2\theta\ \{g(Y,PZ)(X-\eta(X)\xi) + g(Y,PX)(Z-\eta(Z)\xi)\}\ .\end{aligned} \tag{2.31}$$

3. Existence theorem for slant immersions into Kenmotsu space form

In this section we shall obtain a general existence theorem for slant immersions into Kenmotsu space form. In order to prove the existence theorem, we need the following result.

Theorem A ([9]). *Let us take a manifold S with complete connection $\bar{D}$ having parallel torsion and curvature tensors. Let M be a simply connected manifold and E be a vector bundle with connection $\bar{D}$ over M having the algebraic structure $(\bar{R},\bar{T})$ of S. Let $F: TM \to E$ be a vector bundle homomorphism satisfying the equations*

$$\bar{D}_V F(W) - \bar{D}_W F(V) - F([V,\,W]) = \bar{T}(F(V),\,F(W))$$

$$\bar{D}_V \bar{D}_W U - \bar{D}_W \bar{D}_V U - \bar{D}_{[V,\,W]}U = \bar{R}(F(V),\,F(W))U$$

for any sections V, W of TM and U of E. Then there exists a smooth map $f: M \to S$ and a parallel bundle isomorphism $\bar{\Phi}: E \to f^{*}TS$ preserving $\bar{T}$ and $\bar{R}$ such that $df = \bar{\Phi}\circ F$.

Theorem 3.1 *(Existence). Let c and θ be two constants with $0 < \theta \le \pi/2$ and M be a simply-connected $(m+1)$-dimensional Riemannian manifold with metric tensor g. Suppose that there exists a unit global vector field ξ on M, an endomorphism P of the tangent bundle TM and a symmetric bilinear TM-valued form δ on M such that*

$$P(\xi) = 0, \quad g(\delta(X,Y),\xi) = 0, \quad \nabla_X\xi = X - \eta(X)\xi \tag{3.1}$$

$$P^2X \;=\; -\cos^2\theta\,(X - \eta(X)\xi) \tag{3.2}$$

$$g(PX,Y) + g(X,PY) = 0 \tag{3.3}$$

$$\delta(X,\xi) = 0 \tag{3.4}$$

$$g((\nabla_X P)Y,Z) \;=\; g(\delta(X,Y),Z) - g(\delta(X,Z),Y) + g(PX,Y)\,\eta(Z) + g(X,PZ)\,\eta(Y) \tag{3.5}$$

$$R(X,Y,Z,W) = \csc^2\theta\,\{g(\delta(X,W),\delta(Y,Z)) - g(\delta(X,Z),\delta(Y,W))\}$$
$$+\tfrac{c-3}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$
$$+\tfrac{c+1}{4}\left\{\begin{array}{l}\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z)\\ -\eta(X)\eta(W)g(Y,Z) - g(PX,Z)g(PY,W) + g(PY,Z)g(PX,W)\\ +2g(X,PY)g(PZ,W)\end{array}\right\} \quad (3.6)$$

and

$$\begin{aligned}&(\nabla_X\delta)(Y,Z) + \csc^2\theta\,\{P\delta(X,\delta(Y,Z)) + \delta(X,P\delta(Y,Z))\}\\ &+\tfrac{c+1}{4}\sin^2\theta\{g(X,PZ)(Y-\eta(Y)\xi) + g(X,PY)(Z-\eta(Z)\xi)\}\\ &= (\nabla_Y\delta)(X,Z) + \csc^2\theta\,\{P\delta(Y,\delta(X,Z)) + \delta(Y,P\delta(X,Z))\}\\ &+\tfrac{c+1}{4}\sin^2\theta\,\{g(Y,PZ)(X-\eta(X)\xi) + g(Y,PX)(Z-\eta(Z)\xi)\}\end{aligned} \quad (3.7)$$

for all X, Y, $Z \in TM$, where η denotes the dual 1-form of ξ. Then there exists a θ-slant immersion from M into Kenmotsu space form $\bar{M}^{2m+1}(c)$ whose second fundamental form $h(X,Y) = csc^2\theta(P\delta(X,Y) - \varphi\delta(X,Y))$ is given by the relation

$$h(X,Y) = \csc^2\theta(P\delta(X,Y) - \varphi\delta(X,Y)). \quad (3.8)$$

Proof. Let c, θ, M, ξ, P and δ satisfy the conditions stated above. Suppose $TM \oplus D$ be the Whitney sum. We identify $(X,0)$ with X for each $X \in TM$, and $(0,Z)$ by Z^* for each $Z \in D$. In particular, we identify $(\xi,0)$ with $\hat{\xi}$ for ξ. We denote the product metric on $TM \oplus D$ by $\hat{g}$. Hence, if we denote the dual 1-form of $\hat{\xi}$ by $\hat{\eta}$ then, $\hat{\eta}(X,Z) = \eta(X)$, for any $X \in TM$ and $Z \in D$.

The endomorphism $\hat{\phi}$ on $TM \oplus D$ can be defined as

$$\hat{\phi}(X,0) = (PX, \sin\theta\,(X - \eta(X)\xi)), \quad \hat{\phi}(0,Z) = (-\sin\theta\, Z, -PZ) \quad (3.9)$$

for any $X \in TM$ and $Z \in D$.

It is easy to see that $\hat{\phi}^2(X,0) = -(X,0) + \hat{\eta}(X,0)\hat{\xi}$, and $\hat{\phi}^2(0,Z) = -(0,Z)$.
Thus, $\hat{\phi}^2(X,Z) = -(X,Z) + \hat{\eta}(X,Z)\,\hat{\xi}$, for any $X \in TM$ and $Z \in D$.

Now, using (3.2), (3.3) and (3.9) it can readily be seen that $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ is an almost contact metric structure on $TM \oplus D$.

Now we can take A, h and $\nabla^\perp$ as following:

$$A_{Z^*}X = \csc\theta\,\{(\nabla_X P)Z - \delta(X,Z) - g(Z,PX)\xi\} \quad (3.10)$$

$$h(X,Y) = -\csc\theta\,\delta^*(X,Y) \quad (3.11)$$

$$\nabla^\perp_X Z^* = (\nabla_X Z - \eta(\nabla_X Z)\xi)^* + \csc^2\theta\,\{(P\delta(X,Z))^* + \delta^*(X,PZ)\} \quad (3.12)$$

for any $X, Y \in TM$ and $Z \in D$. It is easy to check that A is an endomorphism on TM, h is a $(D)^*$-valued symmetric bilinear form on TM and $\nabla^\perp$ is a metric connection of the vector bundle $(D)^*$ over M.

Let $\overset{\wedge}{\nabla}$ denote the connection on $TM \oplus D$ induced from equations (3.10)-(3.12). Then from (3.1), (3.2), (3.4) and (3.9), we have

$$(\overset{\wedge}{\nabla}_{(X,0)} \hat{\varphi})(Y,0) = g(\hat{\varphi}(X,0),(Y,0))\,\hat{\xi} - \hat{\eta}(Y,0)\,\hat{\varphi}(X,0), \qquad (\overset{\wedge}{\nabla}_{(X,0)} \hat{\varphi})(0,Z) = 0$$

for any $X,\ Y \in TM$ and $Z \in D$.

Let $R^{\perp}$ denote the curvature tensor associated with the connection $\nabla^{\perp}$ on $(D)^{*}$, that is $R^{\perp}(X,Y)Z^{*} = \nabla^{\perp}_{X}\nabla^{\perp}_{Y}Z^{*} - \nabla^{\perp}_{Y}\nabla^{\perp}_{X}Z^{*} - \nabla^{\perp}_{[X,\,Y]}Z^{*}$, for any $X,\ Y \in TM$ and $Z \in D$. Then by virtue of (3.1), (3.2), (3.3), (3.4), (3.7) and (3.12), a straightforward computation yields

$$\begin{aligned} R^{\perp}(X,Y)Z^{*} = \;& \{R(X,Y)Z - \eta(R(X,Y)Z)\xi\}^{*} \\ & + \tfrac{c+1}{4}\left[\begin{array}{l} P\{g(Y,PZ)X + 2g(Y,PX)Z - g(X,PZ)Y\} \\ + \{g(Y,P^{2}Z)(X-\eta(X)\xi) + 2g(Y,PX)PZ - g(X,P^{2}Z)(Y-\eta(Y)\xi)\} \end{array}\right]^{*} \\ & + \csc^{2}\theta \left\{\begin{array}{l} (\nabla_{X}P)\delta(Y,Z) - (\nabla_{Y}P)\delta(X,Z) - \eta(\nabla_{X}(P\delta(Y,Z)))\xi + \eta(\nabla_{Y}(P\delta(X,Z)))\xi \\ +\delta(Y,(\nabla_{X}P)Z) - \delta(X,(\nabla_{Y}P)Z) - \eta(\nabla_{X}(\delta(Y,PZ)))\xi + \eta(\nabla_{Y}(\delta(X,PZ)))\xi \end{array}\right\}^{*} \\ & + \{\eta(X)\eta(\nabla_{Y}Z)\xi - \eta(Y)\eta(\nabla_{X}Z)\xi - \eta(\nabla_{Y}Z)X + \eta(\nabla_{X}Z)Y\}^{*}. \end{aligned} \tag{3.13}$$

Also, from (3.1), (3.5), (3.10), and (3.11), we have

$$\begin{aligned} \sin^{2}\theta\, g([A_{Z^{*}},\, A_{W^{*}}]X,\, Y) = \;& g((\nabla_{Y}P)Z,(\nabla_{X}P)W) - g((\nabla_{Y}P)Z,\delta(X,W)) \\ & -g(W,PX)\,\eta((\nabla_{Y}P)Z) - g(\delta(Y,Z),(\nabla_{X}P)W) + g(\delta(Y,Z),\delta(X,W)) \\ & -g(Z,PY)\,\eta((\nabla_{X}P)W) + g(Z,PY)\,g(W,PX) - g((\nabla_{Y}P)W,(\nabla_{X}P)Z) \\ & +g((\nabla_{Y}P)W,\delta(X,Z)) + g(Z,PX)\,\eta((\nabla_{Y}P)W) + g(\delta(Y,W),(\nabla_{X}P)Z) \\ & -g(\delta(Y,W),\delta(X,Z)) + \eta((\nabla_{X}P)Z)\,g(W,PY) - g(W,PY)\,g(Z,PX). \end{aligned} \tag{3.14}$$

From (3.3), we have

$$g(\delta(Y,Z),PW) + g(P\delta(Y,Z),W) = 0. \tag{3.15}$$

Taking covariant derivative of (3.15) with respect to X and using (3.3), we get

$$g(\delta(Y,Z),(\nabla_{X}P)W) + g((\nabla_{X}P)\delta(Y,Z),W) = 0. \tag{3.16}$$

Moreover, by virtue of (3.5), we get

$$\begin{aligned} g((\nabla_{Y}P)W,(\nabla_{X}P)Z) = \;& g(\delta(Y,W),(\nabla_{X}P)Z) - g(\delta(Y,(\nabla_{X}P)Z),\,W) + \\ & + g(PY,W)\,\eta((\nabla_{X}P)Z) + g(Y,P((\nabla_{X}P)Z))\eta(W). \end{aligned} \tag{3.17}$$

Also,

$$g((\nabla_Y P)Z, (\nabla_X P)W) = g((\nabla_X P)W, (\nabla_Y P)Z)$$
$$= g(\delta(X,W), (\nabla_Y P)Z) - g(\delta(X, (\nabla_Y P)Z), W) + g(PX, W)\eta((\nabla_Y P)Z).$$

Using this in equations (2.3), (3.2), (3.3) and (3.18), we get

$$\begin{aligned} & g(R^{\perp}(X,Y)Z^*, W^*) - g([A_{Z*}, A_{W*}]X, Y) \\ &= \left(\tfrac{c+1}{4}\right)\left[\sin^2\theta\,\{\, g(Y,Z)\, g(X,W) - g(X,Z)\, g(Y,W)\} + 2g(Y,PX)\, g(PZ,W)\right]. \end{aligned} \tag{3.18}$$

Equations (2.3), (3.2), (3.3) and (3.18) imply that $(M,\, A,\, \nabla^{\perp})$ satisfies the Ricci equation for an (m+1)-dimensional θ-slant submanifold in $\bar{M}^{2m+1}(c)$. Moreover, (2.28) and (2.31) imply that $(M,\, h)$ satisfies the equations of Gauss and Codazzi for a θ-slant submanifold. Thus, we have a vector bundle $(TM \oplus D)$ over M equipped with product metric $\hat{g}$, the shape operator A, the second fundamental form h and the connections $\nabla^{\perp}$ and $\overset{\wedge}{\nabla}$ satisfying the structure equations of (m+1)-dimensional θ-slant submanifold in $\bar{M}^{2m+1}(c)$. Therefore, from theorem A, there exists a θ-slant isometric immersion of M in $\bar{M}^{2m+1}(c)$ with h as its second fundamental form, A as its shape operator and $\nabla^{\perp}$ as its normal connection. □

4. Uniqueness theorem for slant immersions into Kenmotsu space form

Theorem 4.1 *(Uniqueness). Let x^1, $x^2 : M \to \bar{M}(c)$ be two slant immersions with slant angle θ $(0 < \theta \leq \pi/2)$, of a connected Riemannian manifold M^{m+1} into a Kenmotsu space-form $\bar{M}^{2m+1}(c)$. Let h^1, h^2 denote the second fundamental forms of x^1 and x^2, respectively. Suppose that there is a vector field $\hat{\xi}$ on M such that $x^i_{*p}(\hat{\xi}_p) = \xi_{x^i(p)}$ for any $i = 1, 2$ and $p \in M$; and*

$$g(h^1(X,Y), \varphi x^1_* Z) = g(h^2(X,Y), \varphi x^2_* Z) \tag{4.1}$$

for all vector fields X, Y, Z tangent to M. Moreover, we also assume that one of the following conditions hold:

(i) $\theta = \pi/2$

(ii) There exists a point p of M such that $P_1 = P_2$.

(iii) $c \neq -1$.

Then, there exists an isometry ϕ of $\bar{M}^{2m+1}(c)$ such that $x^1 = \phi \circ x^2$.

Proof. Let us take any point p of M. We may assume that $x^1(p) = x^2(p)$ and $x^1_*(p) = x^2_*(p)$. Take a geodesic γ through $p = \gamma(0)$. Now, we define $\gamma_1 = x^1(\gamma)$ and $\gamma_2 = x^2(\gamma)$. To prove the theorem it is sufficient to show that γ_1 and γ_2 coincide. It is known that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1'(0) = \gamma_2'(0)$. Let $E_1, E_2, ..., E_m, \hat{\xi}$ be any orthonormal frame along γ. We can define a frame along γ_1 and γ_2 as follows:

$a_i = x^1_*(E_i)$, $B_i = x^2_*(E_i)$, $A_{n+i} = (x^1_*(E_i))^*$, $B_{n+i} = (x^2_*(E_i))^*$, where, $X^* = \frac{FX}{\sin\theta}$

for any $X \in D$.

From (3.11),$h(X,Y) = -\csc\theta\,\delta^*(X,Y)$, and therefore $h^i = -\csc\,\theta\,(\delta^i)^*$ for any $i = 1\,,\,2$.

From (4.1), we have

$$g\,(\csc\theta\,(\delta^1)^*(X,Y),\,\varphi\,x^1_*Z) = g\,(\csc\theta\,(\delta^2)^*(X,Y),\,\varphi\,x^2_*Z)$$

$$g\,((\delta^1)^*(X,Y),\,F\,x^1_*Z) = g\,((\delta^2)^*(X,Y),\,F\,x^2_*Z)$$

$$g\,(\delta^1(X,Y),\,x^1_*Z) = g\,(\delta^2(X,Y),\,x^2_*Z)$$

Since, $x^1_*(p) = x^2_*(p)$ and Z is arbitrary, we conclude that $\delta^1 = \delta^2$.

Now, we have to show that $P_1 = P_2$.

If (i) is satisfied then we see that $P_1 = P_2 = 0$.

And if (ii) is satisfied, it follows from (3.5) that,

$$g(\nabla_X(P_1 - P_2)Y,\,Z) = g((P_1 - P_2)X,\,Y)\,\eta(Z) + g(X,\,(P_1 - P_2)Z)\,\eta(Y).$$

Since it is true for any X, Y, Z and we have $P_1 = P_2$ at any point $p \in M$, therefore we have $P_1 = P_2$ everywhere.

Now suppose that (iii) is satisfied and assume that $P_1 \neq P_2$ and (i) and (ii) are not satisfied. First we want to show that $P_1 = -P_2$.

From (3.6), we find that

$$\begin{aligned} g(P_1X,W)\,g(P_1Y,Z)\; &- g(P_1X,Z)\,g(P_1Y,W) + 2g(P_1Z,W)\,g(P_1Y,X) \\ =\; g(P_2X,W)\,g(P_2Y,Z) &- g(P_2X,Z)\,g(P_2Y,W) + 2g(P_2Z,W)\,g(P_2Y,X). \end{aligned} \tag{4.2}$$

Putting $X = W,\ Y = Z$, and using skew symmetric property of P_1 and P_2, equation (4.2) reduces to

$$g(P_1Y,X)^2 = g(P_2Y,X)^2. \tag{4.3}$$

Now putting $e_1 = X$ and $e_2 = P_1X$, and letting that P_2e_1 has a component in the direction of vector e_3 which is orthogonal to both e_1 and e_2, a contradiction follows from (4.3) which states that

$$g(P_2e_1,e_3)^2 = g(P_1e_1,e_3)^2 = g(e_2,e_3)^2 = 0.$$

Now using (3.2) and (3.3), we have $P_1\nu = \pm P_2\nu$ for any tangent vector ν.

We choose a basis $\{e_1,...,e_m,\,e_{m+1}\}$ of the tangent space at a point $p \in M$. Then there exists a number $\varepsilon_i \in \{-1,\,1\}$ such that $P_1e_i = \varepsilon_iP_2e_i$. Hence we have

$$\pm P_1(e_i + e_j) = P_2(e_i + e_j) = \varepsilon_iP_1e_i + \varepsilon_jP_1e_j.$$

Thus the above formula shows that all ε_i have to be same, and so either $P_1\nu = P_2\nu$ or $P_1\nu = -P_2\nu$ for all $\nu \in T_pM$.

Since M is connected, we have either $P_1 = P_2$ or $P_1 = -P_2$ in case (iii).

Now, assume that we have two immersions with $P_1 = -P_2$. From (3.5) it follows that

$$g((\nabla_X P_1)Y, Z) = g(\delta^1(X, Y), Z) - g(\delta^1(X, Z), Y) + g(P_1 X, Y)\,\eta(Z) + g(X, P_1 Z)\,\eta(Y)$$

and

$$\begin{aligned} g((\nabla_X P_2)Y, Z) &= -g((\nabla_X P_1)Y, Z) \\ &= g(\delta^2(X, Y), Z) - g(\delta^2(X, Z), Y) + g(P_2 X, Y)\,\eta(Z) + g(X, P_2 Z)\,\eta(Y). \end{aligned}$$

Since $\delta^1 = \delta^2 = \delta$, we get

$$g(\delta(X, Y), Z) = g(\delta(X, Z), Y). \tag{4.4}$$

Writing equation (3.7) for both the immersions, we get

$$\begin{aligned} \big\{(\nabla_X \delta^1)(Y, Z) - (\nabla_Y \delta^1)(X, Z)\big\} &= \csc^2\theta\,\{P_1\delta^1(Y,\,\delta^1(X, Z)) + \delta^1(Y,\,P_1\delta^1(X, Z)) \\ &-P_1\delta^1(X,\,\delta^1(Y, Z)) - \delta^1(X, P_1\,\delta^1(Y, Z))\} + \tfrac{c+1}{4}\sin^2\theta\,\{g(Y, P_1 Z)(X - \eta(X)\xi) \\ &-g(X, P_1 Z)(Y - \eta(Y)\xi) - 2g(X, P_1 Y)(Z - \eta(Z)\xi)\} \end{aligned}$$

and

$$\begin{aligned} \{\nabla_X \delta^2)(Y, Z) - (\nabla_Y \delta^2)(X, Z)\} &= \csc^2\theta\,\{P_2\delta^2(Y, \delta^2(X, Z)) + \delta^2(Y, P_2\delta^2(X, Z)) \\ &-P_2\delta^2(X, \delta^2(Y, Z)) - \delta^2(X, P_2\delta^2(Y, Z))\} + \tfrac{c+1}{4}\sin^2\theta\{g(Y, P_2 Z)(X - \eta(X)\xi) \\ &-g(X, P_2 Z)(Y - \eta(Y)\xi) - 2g(X, P_2 Y)(Z - \eta(Z)\xi)\}. \end{aligned}$$

Now using $P_1 = -P_2 = P$ in the above equations, and subtracting the two, we get

$$\begin{aligned} 0 \;=\; & 2\csc^2\theta\{P\delta(Y,\,\delta(X, Z)) + \delta(Y,\,P\delta(X, Z)) - P\delta(X,\,\delta(Y, Z)) - \delta(X, P\delta(Y, Z))\} \\ & +2\,\tfrac{c+1}{4}\sin^2\theta\,\{g(Y, PZ)(X - \eta(X)\xi) - g(X, PZ)(Y - \eta(Y)\xi) - 2g(X, PY)(Z - \eta(Z)\xi)\}, \end{aligned}$$

or

$$\begin{aligned} &\{P\delta(Y,\,\delta(X, Z)) + \delta(Y,\,P\delta(X, Z)) - P\delta(X,\,\delta(Y, Z)) - \delta(X, P\delta(Y, Z))\} \\ &+ \tfrac{c+1}{4}\sin^4\theta\,\{g(Y, PZ)(X - \eta(X)\xi) - g(X, PZ)(Y - \eta(Y)\xi) - 2g(X, PY)(Z - \eta(Z)\xi)\} = 0, \end{aligned}$$

or

$$\begin{aligned} &P\delta(X,\,\delta(Y, Z)) + \delta(X, P\delta(Y, Z)) - P\delta(Y,\,\delta(X, Z)) - \delta(Y,\,P\delta(X, Z)) \\ &+ \tfrac{c+1}{4}\sin^4\theta\,\{g(X, PZ)(Y - \eta(Y)\xi) - g(Y, PZ)(X - \eta(X)\xi) + \\ &+2g(X, PY)(Z - \eta(Z)\xi)\} = 0. \end{aligned} \tag{4.5}$$

Taking inner product of equation (4.5) with a vector W, we get

$$\begin{aligned} &g(P\delta(X,\,\delta(Y, Z)), W) + g(\delta(X, P\delta(Y, Z)), W) - g(P\delta(Y,\,\delta(X, Z)), W) - g(\delta(Y,\,P\delta(X, Z)), W) \\ &+ \tfrac{c+1}{4}\sin^4\theta\,\{g(X, PZ)\,g(Y, W) - g(X, PZ)\eta(Y)\eta(W) - g(Y, PZ)g(X, W) + g(Y, PZ)\eta(X)\eta(W) \\ &+2g(X, PY)\,g(Z, W) - 2g(X, PY)\eta(Z)\eta(W)\} = 0, \end{aligned}$$

or

$$\begin{aligned}
&-g(\delta(X,\ PW),\delta(Y,Z))\ +g(\delta(Y,\ PW),\delta(X,Z))\ +g(\delta(X,W),P\delta(Y,Z))-\\
&-g(\delta(Y,\ W),P\delta(X,Z))+\ \tfrac{c+1}{4}\sin^4\theta\,\{g(X,PZ)\,g(Y,W)-g(X,PZ)\eta(Y)\eta(W)-\\
&-g(Y,PZ)g(X,W)+g(Y,PZ)\eta(X)\eta(W)+\\
&+2g(X,PY)\,g(Z,W)-2g(X,PY)\eta(Z)\eta(W)\}=0.
\end{aligned}\tag{4.6}$$

If δ vanishes identically at a point, then a contradiction follows from (4.6) since $c \neq -1$.

Now we take a fixed point p of M and consider a function f defined on the set of all unit tangent vectors UM_p by

$f(\nu) = g(\delta(\nu,\nu),\nu)$, for all $\nu \in UM_p$.

Since UM_p is compact there exists a vector u such that f attains an absolute maximum at u. Let w be a unit vector orthogonal to u. Then the function $f(t) = f(g(t))$, where the relation $g(t) = (\cos\, t)\, u + (\sin\, t)\, w$ satisfies the conditions $f'(0) = 0$ and $f''(0) \leq 0$. The first condition implies that $g(\delta(u,u),w) = 0$, whereas the second condition implies $g(\delta(u,w),w) \leq \frac{1}{2}\, g(\delta(u,u),u)$.

Using the total symmetry of δ, it follows that we can choose an orthonormal basis $e_1 = u,\ e_2,\ e_3 ...,e_m, e_{m+1}$ such that

$$\delta(e_1,\, e_1) = \lambda_1 e_1, \quad \delta(e_1,\, e_i) = \lambda_i\, e_i \tag{4.7}$$

with $i > 1$ and $\lambda_i \leq \frac{1}{2}\lambda_1$. Since δ is not identically 0, it follows from total symmetry of (4.4) that $\lambda_1 > 0$.

Using (4.4) and (4.7) in (4.6), with $X = Z = W = e_1$ and $Y = e_i$, we find

$$\begin{aligned}
&-g(\delta(e_1, Pe_1),\ \delta(e_i,\, e_1)) + g(\delta(e_i,\ Pe_1),\ \delta(e_1,\, e_1)) + g(\delta(e_1,\, e_1),\ P\delta(e_i,\, e_1))\\
&-g(\delta(e_i,\, e_1),\ P\delta(e_1,\, e_1)) + \tfrac{c+1}{4}\sin^4\theta\,\{g(e_1, Pe_1)\,g(e_i,\, e_1) - g(e_1, Pe_1)\,\eta(e_i)\,\eta(e_1)\\
&-g(e_i, Pe_1)\,g(e_1,\, e_1) + g(e_i,\ Pe_1)\,\eta(e_1)\,\eta(e_1)\ + 2g(e_1, Pe_i)\,g(e_1, e_1) - 2g(e_1, Pe_i)\,\eta(e_1)\,\eta(e_1)\} = 0,
\end{aligned}$$

or

$$\begin{aligned}
&-g(\delta(e_1, Pe_1),\ \lambda_i\, e_i) + g(\delta(e_i, Pe_1),\ \lambda_1 e_1) + g(\lambda_1 e_1,\ P\lambda_i e_i) - g(\lambda_i e_i,\ P\delta(e_1,\, e_1)\\
&\qquad \tfrac{c+1}{4}\sin^4\theta\,\{-g(e_i,\ pe_1)\,g(e_1,\, e_1) + 2g(e_1,\ Pe_i)\,g(e_1, e_1)\} = 0,
\end{aligned}$$

or

$$(\lambda_i^2\ + \lambda_i\,\lambda_1\ + 3\frac{c+1}{4}\sin^4\theta)\,g(Pe_1,\ e_i) = 0. \tag{4.8}$$

Now, we show that Pe_1 is an eigen vector of $\delta(e_1,\, .)$. In order to do so, we put $X = Z = W = e_1$, $W = e_j$ and $Y = e_i$ for $i,\, j > 1$. Then, we get

$$\begin{aligned}
&-g(\delta(e_1, Pe_j),\ \delta(e_i, e_1)) + g(\delta(e_i,\ Pe_j), \delta(e_1, e_1)) + g(\delta(e_1, e_j), P\delta(e_i, e_1))\\
&-g(\delta(e_i, e_j), P\delta(e_1, e_1)) + \tfrac{c+1}{4}\sin^4\theta\,\{g(e_1, Pe_1)g(e_i, e_j) - g(e_1, Pe_1)\,\eta(e_i)\,\eta(e_j)\\
&-g(e_i,\ Pe_1)\,g(e_1, e_j) + g(e_i, Pe_1)\,\eta(e_1)\,\eta(e_j) + 2\,g(e_1, Pe_i)\,g(e_1, e_j) - 2\,g(\,e_1, Pe_i)\,\eta(e_1)\,\eta(e_j)\} = 0,
\end{aligned}$$

or

$$(\lambda_i^2 - \lambda_i \lambda_1 + \lambda_i \lambda_j)\, g(Pe_j, e_i) + \lambda_1\, g(\delta(e_i, e_j), Pe_1) = 0. \tag{4.9}$$

Interchanging the indices i and j in (4.9), we get

$$(\lambda_j^2 - \lambda_j \lambda_1 + \lambda_i \lambda_j)\, g(Pe_i, e_j) + \lambda_1\, g(\delta(e_i, e_j), Pe_1) = 0. \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$(\lambda_i^2 - \lambda_i \lambda_1 + \lambda_j^2 - \lambda_j \lambda_1 + 2\lambda_i \lambda_j)\, g(Pe_j, e_i) = 0,$$

or

$$(\lambda_i + \lambda_j)(\lambda_1 - \lambda_i - \lambda_j)\, g(Pe_j, e_i) = 0. \tag{4.11}$$

Since, $\lambda_1 \geq 2\lambda_i$, we get that $\lambda_1 - \lambda_i - \lambda_j = 0$ only if $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$.

Now if we put $X = W = e_1$, $Z = e_j$ and $Y = e_i$ for $i, j > 1$ in (4.6), we find that

$$-g(\delta(e_1, Pe_1), \delta(e_i, e_j)) + \lambda_j\, g(\delta(e_i, e_j), Pe_1) - \lambda_i \lambda_j\, g(e_i, Pe_j) -$$
$$-\lambda_1 g(\delta(e_i, e_j), Pe_1) - \tfrac{c+1}{4} \sin^4\theta \{g(e_i, Pe_j)\} = 0,$$

or

$$\begin{aligned} &g(\delta(e_1, Pe_1), \delta(e_i, e_j)) - \lambda_j\, g(\delta(e_i, e_j), Pe_1) + \lambda_i \lambda_j\, g(e_i, Pe_j) \\ &+\lambda_1 g(\delta(e_i, e_j), Pe_1) + \tfrac{c+1}{4} \sin^4\theta \{g(e_i, Pe_j)\} = 0. \end{aligned} \tag{4.12}$$

Interchanging the indices i and j in (4.12), we get

$$\begin{aligned} &g(\delta(e_1, Pe_1), \delta(e_i, e_j)) - \lambda_i\, g(\delta(e_i, e_j), Pe_1) + \lambda_i \lambda_j\, g(e_j, Pe_i) + \\ &+\lambda_1 g(\delta(e_i, e_j), Pe_1) + \tfrac{c+1}{4} \sin^4\theta \{g(e_j, Pe_i)\} = 0. \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we get

$$(\lambda_i - \lambda_j)\, g(\delta(e_i, e_j), Pe_1) + 2\lambda_i \lambda_j\, g(e_i, Pe_j) + \frac{c+1}{2} \sin^4\theta\, g(e_i, Pe_j) = 0. \tag{4.14}$$

Now, we summarize the previous equations in the following manner. First, by taking $i = j$ in (4.9), we get

$$g(\delta(e_i, e_i), Pe_1) = 0. \tag{4.15}$$

Hence, we have $g(\delta(\nu, \nu), Pe_1) = 0$ if ν is an eigenvector of $\delta(e_1, .)$. Moreover, the symmetry of δ then implies that $g(\delta(e_i, e_j), Pe_1) = 0$, whenever $\lambda_i = \lambda_j$.

We now consider the following four different cases.

(1) $\lambda_i + \lambda_j \neq 0$, but not $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$. In this case (4.11) implies $g(Pe_i, e_j) = 0$.

(2) $\lambda_i + \lambda_j = 0$, and $\lambda_i \neq 0$. In this case, (4.9) implies $g(\delta(e_i, e_j), Pe_1) = \lambda_i g(Pe_j, e_i)$.

Using the consequence of case (2) in (4.14), we get $g(Pe_j, e_i) = 0$.

(3) $\lambda_i = \lambda_j = 0$. In this case it follows from (4.14) that $g(e_i, Pe_j) = 0$;

(4) $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$.

If $e_{i_1}, e_{i_2}, ..., e_{i_k}$ are eigenvectors belonging to an eigenvalue different from $\frac{1}{2}\lambda_1$, then each Pe_{i_l}, $l = 1, ..., k$, can only have a component in the direction of e_1, say $Pe_{i_l} = \mu_l e_1$. Thus$\mu_l Pe_1 = -\cos^2\theta\, e_{i_l}$. Consequently, either $k = 1$ or there does not exist an eigenvector with eigenvalue different from$\frac{1}{2}\lambda_1$. If $k = 1$, then obviously Pe_1 is an eigenvector. In the latter case $\delta(e_1, .)$, restricted to the space $e_1^{\perp}$, only has one eigenvalue, namely $\frac{1}{2}\lambda_1$. Since, Pe_1 is always orthogonal to e_1, Pe_1 is also an eigenvector in this case. Hence Pe_1 is always an eigenvector of $\delta(e_1, .)$.

Without any loss of generality, we may assume that e_2 is in the direction of Pe_1. Then it follows immediately that $\delta(e_1, Pe_1) = \lambda_2 Pe_1$, where λ_2 satisfies the equation

$$\lambda_2^2 + \lambda_2\lambda_1 + \frac{3(c+1)}{4}\sin^4\theta = 0 \tag{4.16}$$

by virtue of (4.8).

If we choose $X = Z = e_1$, $W = Pe_1$ and $Y = e_i$ for $i > 2$ in (4.6), then

$$g(\delta(e_1, Pe_1), \lambda_i Pe_i) - \lambda_1 g(\delta(e_i, Pe_1), Pe_1) = 0$$

or, $g(\lambda_2 Pe_1, \lambda_i Pe_i) - \lambda_1 g(\delta(e_i, Pe_1), Pe_1) = 0$
or, $\lambda_i \lambda_2 \cos^2\theta\, g(e_1, e_i) - \lambda_1 g(\delta(e_i, Pe_1), Pe_1) = 0$
or, $\lambda_1 g(\delta(e_i, Pe_1), Pe_1) = \lambda_1\, g(\delta(Pe_1, Pe_1), e_i) = 0$.

Thus $\delta(Pe_1, Pe_1) = \lambda_2 \cos^2\theta\, e_1$.

Putting $X = Z = W = Pe_1$ and $Y = e_1$ in (4.6), we get

$$-\lambda_2^2 - \lambda_2\lambda_1 + \frac{3(c+1)}{4}\sin^4\theta = 0. \tag{4.17}$$

Now from (4.16) and (4.17) we get $\frac{3(c+1)}{4}\sin^4\theta = 0$, which is a contradiction since $c \neq -1$. Therefore $P_1 = P_2$. It can be easily seen from relations (3.10)-(3.12) that

$g(\gamma_1', A_k) = g(\gamma_2', B_k)$ and $g(\overset{\wedge}{\nabla}_{\gamma} A_k, A_l) = g(\overset{\wedge}{\nabla}_{\gamma} B_k, B_l)$ for $k,\ l = 1, ..., 2m$

such that, by [16, Proposition 3],$\gamma_1 = \gamma_2$. □

5. Applications and examples

Let$\psi = \psi(x), \psi_i = \psi_i(x), i = 1, 2, 3$ be four functions defined on an open interval containing 0. Let c and θ be two constants with $0 < \theta \leq {}^{\pi}/_{2}$ and M be a simply connected open neighbourhood of the origin $(0, 0, 0) \in R^3$. Now we suppose that

$$f(x) = \exp\int \phi_3(x)\, dx \tag{5.1}$$

$$\eta = dz. \tag{5.2}$$

Let the warped metric on M be defined by

$$g = \eta \otimes \eta + e^{2z}\,(dx \otimes dx + f^2(x)\,dy \otimes dy). \tag{5.3}$$

Now, we consider the vectors

$$e_1 = \frac{1}{e^z}\frac{\partial}{\partial x}\ , \qquad e_2 = \frac{1}{f\,e^z}\frac{\partial}{\partial y}\ , \qquad e_3 = \xi = \frac{\partial}{\partial z}\ .$$

Then it can be readily seen that $\{e_1, e_2, e_3\}$ is a local orthonormal frame of TM and that η is a dual 1-form of ξ. Moreover, we have

$$\nabla_{e_1} e_1 = -e_3 \quad \nabla_{e_1} e_2 = 0 \quad \nabla_{e_1} e_3 = e_1$$

$$\nabla_{e_2} e_1 = \frac{\psi_3}{e^z} e_2 \quad \nabla_{e_2} e_2 = -\frac{\psi_3}{e^z} e_1 - e_3 \quad \nabla_{e_2} e_3 = e_2$$

$$\nabla_{e_3} e_1 = 0 \quad \nabla_{e_3} e_2 = 0 \quad \nabla_{e_3} e_3 = 0.$$

Let us define a tensor field φ such that

$$\varphi\, e_1 = e_2\,, \quad \varphi\, e_2 = -e_1\,, \qquad \varphi\, \xi = 0,$$

and a symmetric bilinear TM-valued form δ on M given by

$$\delta(e_1, e_1) = \psi\, e_1 + \psi_1\, e_2, \quad \delta(e_1, e_2) = \psi_1\, e_1 + \psi_2\, e_2, \quad \delta(e_2, e_2) = \psi_2\, e_1 - \psi_1\, e_2 \tag{5.4}$$

$$\delta(e_1, \xi) = 0, \qquad \delta(e_2, \xi) = 0, \qquad \delta(\xi, \xi) = 0. \tag{5.5}$$

It is easy to show that $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold with $(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\,\varphi X$ for any X, $Y \in TM$. By putting $P = \cos\theta\ \varphi$, we can see that (M, g, ξ, P, δ) satisfy equations (3.1), (3.2), (3.3), (3.4) and (3.5). In addition, we can prove that M satisfy conditions (3.6) and (3.7) if we have the following:

$$\psi_3' = -\psi_3^2 - e^{2z}\csc^2\theta\,\{\psi\psi_2 - 2\psi_1^2 - \psi_2^2\} - e^{2z}\frac{(c+1)}{4}(1 + 3\cos^2\theta); \tag{5.6}$$

$$\psi_2' = (-2\psi_2 + \psi)\psi_3 - e^z\ \csc\theta\ \cot\theta\,(\psi_2 + \psi)\psi_1; \tag{5.7}$$

$$\psi_1' = -3\psi_1\psi_3 + e^z\ \csc\theta\ \cot\theta\,(\psi_2 + \psi)\psi_2 + 3\,e^z\frac{(c+1)}{4}\sin^2\theta\ \cos\theta; \tag{5.8}$$

$$\psi_1' = -3\psi_1\psi_3 + e^z\ \csc\theta\ \cot\theta\,(\psi_2 + \psi)\psi_2 - 3\,e^z\frac{(c+1)}{4}\sin^2\theta\ \cos\theta. \tag{5.9}$$

But we see that (5.8) and (5.9) hold simultaneously if and only if

$e^z\,\frac{(c+1)}{4}\sin^2\theta\ \cos\theta = 0$. Since, $0 < \theta \le \frac{\pi}{2}$, we know that $\sin^2\theta \ne 0$, and $e^z \ne 0$ for any $z \in R$. Hence, it must be either $c = -1$ or $\theta = \frac{\pi}{2}$.

By applying theorem (4.1), we have the following result.

Theorem 5.1 *Let $\psi = \psi(x)$ be a function defined on an open interval containing 0 and c_1, c_2, c_3, c and θ be the five constants with $0 < \theta \leq \frac{\pi}{2}$. Consider the following set of first order differential equations in $y_i = y_i(x)$, for $i = 1, 2, 3$*

$$y_1' = -3y_1y_3 + e^z \csc\theta \cot\theta\,(y_2 + \psi)y_2$$

$$y_2' = (-2y_2 + \psi)y_3 - e^z \csc\theta \cot\theta\,(y_2 + \psi)y_1$$

$$y_3' = -y_3^2 - e^{2z}\csc^2\theta\,(\psi\, y_2 - 2y_1^2 - y_2^2) - e^{2z}\frac{(c+1)}{4}(1 + 3\cos^2\theta)$$

with the initial conditions: $y_1(0) = c_1$, $y_2(0) = c_2$, $y_3(0) = c_3$. Let ψ_1, ψ_2 and ψ_3 be the components of the unique solution of this differentiable system on some open interval containing 0. Let M be a simply connected open neighborhood of the origin $(0,0,0) \in R^3$, endowed with the metric given by (51)-(5.3). Let δ be the TM-valued form defined by (5.4) and (5.5). Then, we have

1. *If $c = -1$, there exists a θ-slant isometric immersion from M into $\bar{M}^5(-1)$, whose second fundamental form is given by*

$$h(X,Y) = \csc^2\theta\,(P\delta(X,Y) - \varphi\,\delta(X,Y)).$$

1. *If $\theta = \frac{\pi}{2}$, then there exists an anti-invariant immersion from M into $\bar{M}^5(c)$, whose second fundamental form is given by*

$$h(X,Y) = -\varphi\,\delta(X,Y)).$$

From theorem 5.1, we have the following existence result for three dimensional submanifolds with prescribed scalar curvature parantez or mean curvature.

Corollary 5.2 *For a given constant θ with $0 < \theta < \pi/2$ and a given function $F_1 = F_1(x)$ (resp. $F_2 = F_2(x)$), there exist infinitely many three dimensional θ slant submanifolds in Kenmotsu space form $\bar{M}^5(c)$ with F_1 (resp. F_2) as the prescribed scalar curvature (resp., mean curvature) function for $c = -1$.*

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