

Cartan Calculus on the Quantum Space \mathcal{R}_q^3

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Abstract

To give a Cartan calculus on the extended quantum 3d space, the noncommutative differential calculus on the extended quantum 3d space is extended by introducing inner derivations and Lie derivatives.

Key Words: Quantum space, Hopf algebra, Lie algebra, inner derivation, Lie derivation.

1. Introduction

The noncommutative differential geometry of quantum groups was introduced by Woronowicz [11, 12]. In this approach the differential calculus on the group is deduced from the properties of the group and it involves functions on the group, differentials, differential forms and derivatives. The other approach, initiated by Wess and Zumino [10], followed Manin's emphasis [5] on the quantum spaces as the primary objects. Differential forms are defined in terms of noncommuting coordinates, and the differential and algebraic properties of quantum groups acting on these spaces are obtained from the properties of the spaces.

The differential calculus on the quantum 3d space similarly involves functions on the 3d space, differentials, differential forms and derivatives. The exterior derivative is a linear operator d acting on k -forms and producing $(k + 1)$ -forms, such that for scalar functions (0-forms) f and g we have

$$d(1) = 0, \quad d(fg) = (df)g + (-1)^{\deg(f)} f(dg),$$

where $\deg(f) = 0$ for even variables and $\deg(f) = 1$ for odd variables, and for a k -form ω_1 and any form ω_2

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (d\omega_2).$$

A fundamental property of the exterior derivative d is

$$d \wedge d =: d^2 = 0.$$

There is a relationship of the exterior derivative with the Lie derivative and to describe this relation, we introduce a new operator: the inner derivation. Hence the differential calculus on the quantum 3d space can

be extended into a large calculus. We call this new calculus the Cartan calculus. The connection of the inner derivation denoted by i_a and the Lie derivative denoted by \mathcal{L}_a is given by the Cartan formula:

$$\mathcal{L}_a = i_a \circ d + d \circ i_a.$$

This and other formulae are explained in Ref. [6–8]. We now shall give a brief overview without much discussion.

Let us begin with some information about the inner derivations. Generally, for a smooth vector field X on a manifold the inner derivation, denoted by i_X , is a linear operator which maps k -forms to $(k - 1)$ -forms. If we define the inner derivation i_X on the set of all differential forms on a manifold, we know that i_X is an antiderivation of degree -1 :

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X\beta),$$

where α and β are both differential forms. The inner derivation i_X acts on 0- and 1-forms as follows:

$$i_X(f) = 0, \quad i_X(df) = X(f).$$

We know, from the classical differential geometry, that the Lie derivative \mathcal{L} can be defined as a linear map from the exterior algebra into itself which takes k -forms to k -forms. For a 0-form, that is, an ordinary function f , the Lie derivative is just the contraction of the exterior derivative with the vector field X :

$$\mathcal{L}_X f = i_X df.$$

For a general differential form, the Lie derivative is likewise a contraction, taking into account the variation in X :

$$\mathcal{L}_X \alpha = i_X d\alpha + d(i_X \alpha).$$

The extended calculus on the quantum plane was introduced in Ref. 1 using the approach of Ref. [7]. In this work we explicitly set up the Cartan calculus on the quantum 3d space using approach of Ref. [2].

2. Review of the Calculus on the Quantum 3d Space

In this section we give some information on the differential calculus on the quantum 3d space [3] which we shall use in order to establish our notions.

2.1. The algebra of polynomials on the quantum 3d space

The quantum three dimensional space is defined as an associative algebra generated by three noncommuting coordinates x , y and z with three quadratic relations:

$$xy = qyx, \quad yz = qzy, \quad xz = qzx, \tag{1}$$

where q is a non-zero complex numbers. This associative algebra over the complex number is known as the algebra of polynomials over the quantum 3d space and we shall denote it by \mathcal{R}_q^3 . We define the extended quantum 3d space to be the algebra that contains \mathcal{R}_q^3 , the unit and x^{-1} , the inverse of x , which obeys

$$xx^{-1} = 1 = x^{-1}x.$$

We denote the extended algebra by \mathcal{A} . We know that the algebra \mathcal{A} is a Hopf algebra [3].

2.2. Differential algebra

A deformed differential calculus on the quantum 3d space is as follows [3]:

- the commutation relations with the coordinates of differentials

$$\begin{aligned}
 xdx &= dxx, & xdy &= qdyx, & xdz &= qdzx, \\
 ydx &= q^{-1}dxy, & ydy &= dydy, & ydz &= qdzzy, \\
 zdx &= q^{-1}dxz, & zdy &= q^{-1}dyz, & zdz &= dzdz;
 \end{aligned} \tag{2}$$

- the commutation relations between the differentials

$$\begin{aligned}
 dx \wedge dy &= -qdy \wedge dx, & dx \wedge dx &= 0, \\
 dx \wedge dz &= -qdz \wedge dx, & dy \wedge dy &= 0, \\
 dy \wedge dz &= -qdz \wedge dy, & dz \wedge dz &= 0,
 \end{aligned} \tag{3}$$

- the relations of the coordinates with their partial derivatives

$$\begin{aligned}
 \partial_x x &= 1 + x\partial_x, & \partial_x y &= q^{-1}y\partial_x, & \partial_x z &= q^{-1}z\partial_x, \\
 \partial_y x &= qx\partial_y, & \partial_y y &= 1 + y\partial_y, & \partial_y z &= q^{-1}z\partial_y, \\
 \partial_z x &= qx\partial_z, & \partial_z y &= qy\partial_z, & \partial_z z &= 1 + z\partial_z;
 \end{aligned} \tag{4}$$

- the relations of partial derivatives

$$\partial_x \partial_y = q\partial_y \partial_x, \quad \partial_x \partial_z = q\partial_z \partial_x, \quad \partial_y \partial_z = q\partial_z \partial_y; \tag{5}$$

- the relations between partial derivatives and differentials are found as

$$\begin{aligned}
 \partial_x dx &= dx\partial_x, & \partial_x dy &= q^{-1}dy\partial_x, & \partial_x dz &= q^{-1}dz\partial_x, \\
 \partial_y dx &= qdx\partial_y, & \partial_y dy &= dy\partial_y, & \partial_y dz &= q^{-1}dz\partial_y, \\
 \partial_z dx &= qdx\partial_z, & \partial_z dy &= qdy\partial_z, & \partial_z dz &= dz\partial_z.
 \end{aligned} \tag{6}$$

These relations will be used in section 5.

3. Hopf Algebra Structure of Quantum Lie Algebra

In this short section we shall give the Hopf algebra structure of the Lie algebra generators. In Ref. [3], the commutation rules of the quantum Lie algebra generators found as

$$[T_x, T_y] = 0, \quad [T_x, T_z] = 0, \quad [T_y, T_z] = 0. \tag{7}$$

The Hopf algebra structure of the Lie algebra generators is given by

$$\begin{aligned}
 \Delta(T_x) &= T_x \otimes \mathbf{1} + \mathbf{1} \otimes T_x, \\
 \Delta(T_y) &= T_y \otimes \mathbf{1} + q^{T_x} \otimes T_y, \\
 \Delta(T_z) &= T_z \otimes \mathbf{1} + q^{T_x} \otimes T_z, \\
 \epsilon(T_x) &= 0, \quad \epsilon(T_y) = 0, \quad \epsilon(T_z) = 0, \\
 S(T_x) &= -T_x, \quad S(T_y) = -q^{-T_x} T_y, \quad S(T_z) = -q^{-T_x} T_z.
 \end{aligned} \tag{8}$$

We show in the next section that this Hopf algebra structure are consistent with the dual Hopf algebra structure.

4. The Dual of the Hopf Algebra \mathcal{A}

In this section, in order to obtain the dual of the Hopf algebra \mathcal{A} defined in section 2, we shall use the method of Refs. [4] and [9].

As a Hopf algebra \mathcal{A} is generated by the elements x , x^{-1} , y and z , and a basis is given by all monomials of the form

$$f = x^k y^l z^m$$

where $k, l, m \in \mathcal{Z}_+$. Let us denote the dual algebra by \mathcal{U}_q and its generating elements by X , Y and Z .

The pairing is defined through the tangent vectors as

$$\begin{aligned}
 \langle X, f \rangle &= k \delta_{l,0} \delta_{m,0}, \\
 \langle Y, f \rangle &= \delta_{l,1} \delta_{m,0}, \\
 \langle Z, f \rangle &= \delta_{l,0} \delta_{m,1}.
 \end{aligned} \tag{9}$$

We also have

$$\langle 1_{\mathcal{U}}, f \rangle = \epsilon_{\mathcal{A}}(f) = \delta_{k,0}.$$

Using relations (9), one gets

$$\langle XY, f \rangle = \delta_{l,1} \delta_{m,0} \quad \text{and} \quad \langle YX, f \rangle = \delta_{l,1} \delta_{m,0}$$

where differentiation is from the right as this is most suitable for differentiation in this basis. Thus we obtain one of the commutation relations in the algebra \mathcal{U}_q dual to \mathcal{A} as:

$$XY = YX.$$

Similarly, one has

$$XZ = ZX, \quad YZ = ZY.$$

The Hopf algebra structure of this algebra can be deduced by using the duality. The coproduct of the elements of the dual algebra is given by

$$\begin{aligned}
 \Delta_{\mathcal{U}}(X) &= X \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes X, \\
 \Delta_{\mathcal{U}}(Y) &= Y \otimes q^{-X} + 1_{\mathcal{U}} \otimes Y, \\
 \Delta_{\mathcal{U}}(Z) &= Z \otimes q^{-X} + 1_{\mathcal{U}} \otimes Z.
 \end{aligned}$$

The counit is given by

$$\epsilon_U(X) = 0, \quad \epsilon_U(Y) = 0, \quad \epsilon_U(Z) = 0.$$

The coinverse is given as

$$S_U(X) = -X, \quad S_U(Y) = -Yq^X, \quad S_U(Z) = -Zq^X.$$

We can now transform this algebra to the form obtained in section 3 (eq. 8) by making the identities

$$T_x \equiv X, \quad T_y \equiv q^{X/2}Yq^{X/2}, \quad T_z \equiv q^{X/2}Zq^{X/2},$$

which are consistent with the commutation relation and the Hopf structures.

5. Extended Calculus On The Quantum 3d Space

The Lie derivative is closely related to the exterior derivative. The exterior derivative and the Lie derivative are set to cover the idea of a derivative in different ways. These differences can be hasped together by introducing the idea of an antiderivation which is called an inner derivation.

5.1. Inner derivations

In order to obtain the commutation rules of the coordinates with inner derivations, we shall use the approach of Ref. [2]. Similarly other relations can also obtain.

We now wish to find the commutation relations between the coordinates x, y, z and the inner derivations associated with them. In order to obtain the commutation rules of the coordinates with inner derivations, we shall assume that they are of the form

$$\begin{aligned} i_x x &= A_1 x i_x + A_2 y i_y + A_3 z i_z, \\ i_x y &= A_4 y i_x + A_5 x i_y, \\ i_x z &= A_6 z i_x + A_7 x i_z, \\ i_y x &= A_8 x i_y + A_9 y i_x, \\ i_y y &= A_{10} y i_y + A_{11} x i_x + A_{12} z i_z, \\ i_y z &= A_{13} z i_y + A_{14} y i_z, \\ i_z x &= A_{15} x i_z + A_{16} z i_x, \\ i_z y &= A_{17} y i_z + A_{18} z i_y, \\ i_z z &= A_{19} z i_z + A_{20} x i_x + A_{21} y i_y. \end{aligned} \tag{10}$$

The coefficients A_k ($1 \leq k \leq 21$) will be determined in terms of the deformation parameter q . But the use of the relations (1) does not give rise any solution in terms of the parameter q . However, we have, at least, the system

$$\begin{aligned} A_5(A_1 - qA_8) &= 0, \quad A_2A_{11} - q^2A_5A_9 = 0, \quad A_2A_{14} = 0, \\ A_2(A_{10} - qA_4) &= 0, \quad A_3A_{20} - q^2A_7A_{16} = 0, \quad A_3A_{18} = 0, \quad etc. \end{aligned}$$

To find the coefficients, we need the commutation relations of the inner derivations with the differentials of x , y and z . Since

$$i_{X_i}(\mathbf{d}X_j) = \delta_{ij},$$

we can assume that the relations between the differentials and the inner derivations are of the form

$$\begin{aligned} i_x \wedge \mathbf{d}x &= 1 + a_1 \mathbf{d}x \wedge i_x + a_2 \mathbf{d}y \wedge i_y + a_3 \mathbf{d}z \wedge i_z, \\ i_x \wedge \mathbf{d}y &= a_4 \mathbf{d}y \wedge i_x + a_5 \mathbf{d}x \wedge i_y, \\ i_x \wedge \mathbf{d}z &= a_6 \mathbf{d}z \wedge i_x + a_7 \mathbf{d}x \wedge i_z, \\ i_y \wedge \mathbf{d}x &= a_8 \mathbf{d}x \wedge i_y + a_9 \mathbf{d}y \wedge i_x, \\ i_y \wedge \mathbf{d}y &= 1 + a_{10} \mathbf{d}y \wedge i_y + a_{11} \mathbf{d}x \wedge i_x + a_{12} \mathbf{d}z \wedge i_z, \\ i_y \wedge \mathbf{d}z &= a_{13} \mathbf{d}z \wedge i_y + a_{14} \mathbf{d}y \wedge i_z, \\ i_z \wedge \mathbf{d}x &= a_{15} \mathbf{d}x \wedge i_z + a_{16} \mathbf{d}z \wedge i_x, \\ i_z \wedge \mathbf{d}y &= a_{17} \mathbf{d}y \wedge i_z + a_{18} \mathbf{d}z \wedge i_y, \\ i_z \wedge \mathbf{d}z &= 1 + a_{19} \mathbf{d}z \wedge i_z + a_{20} \mathbf{d}x \wedge i_x + a_{21} \mathbf{d}y \wedge i_y. \end{aligned} \quad (11)$$

Applying i_x , i_y and i_z to the relations (2), one gets

$$\begin{aligned} A_1 &= 1, & A_2 &= 0, & A_3 &= 0, & A_4 &= q^{-1}, \\ A_5 &= 0, & A_6 &= q^{-1}, & A_7 &= 0, & A_8 &= q, \\ A_9 &= 0, & A_{10} &= 1, & A_{11} &= 0, & A_{12} &= 0, \\ A_{13} &= q^{-1}, & A_{14} &= 0, & A_{15} &= q, & A_{16} &= 0, \\ A_{17} &= q, & A_{18} &= 0, & A_{19} &= 1, & A_{20} &= 0, & A_{21} &= 0, \end{aligned}$$

and

$$\begin{aligned} a_3(qA_1 - A_{15}) &= 0, & A_2 a_9 - a_2 A_9 &= 0, & A_2 a_{12} &= 0, \\ A_3(a_{15} - qa_1) &= 0, & A_3 a_{16} - a_3 A_{16} &= 0, & A_3 a_{18} &= 0, \quad \text{etc.} \end{aligned}$$

To find the coefficients a_k ($1 \leq k \leq 21$), we use the expression

$$i_a \circ \mathbf{d} - F_1 \mathbf{d} \circ i_a = \partial_a, \quad \text{for } a \in \{x, y, z\}.$$

For example, using the first relation in (10) with the relations (4), we obtain

$$F_1 = -1, \quad a_1 = -1, \quad a_2 = 0 = a_3.$$

Other coefficients can be similarly obtained. Consequently, we have the following commutation relations.

- The commutation relations of the inner derivations with x , y and z :

$$\begin{aligned} i_x x &= x i_x, & i_x y &= q^{-1} y i_x, & i_x z &= q^{-1} z i_x, \\ i_y x &= q x i_y, & i_y y &= y i_y, & i_y z &= q^{-1} z i_y, \\ i_z x &= q x i_z, & i_z y &= q y i_z, & i_z z &= z i_z. \end{aligned} \quad (12)$$

- The commutation relations between the differentials and the inner derivations:

$$\begin{aligned}
 i_x \wedge dx &= 1 - dx \wedge i_x, & i_x \wedge dy &= -q^{-1}dy \wedge i_x, \\
 i_y \wedge dx &= -qdx \wedge i_y, & i_y \wedge dy &= 1 - dy \wedge i_y, \\
 i_z \wedge dx &= -qdx \wedge i_z, & i_z \wedge dy &= -qdy \wedge i_z, \\
 i_x \wedge dz &= -q^{-1}dz \wedge i_x, & i_y \wedge dz &= -q^{-1}dz \wedge i_y, \\
 i_z \wedge dz &= 1 - dz \wedge i_z.
 \end{aligned} \tag{13}$$

- The relations of the inner derivations with the partial derivatives $\partial_x, \partial_y, \partial_z$:

$$\begin{aligned}
 i_x \partial_x &= \partial_x i_x, & i_x \partial_y &= q \partial_y i_x, & i_x \partial_z &= q \partial_z i_x, \\
 i_y \partial_x &= q^{-1} \partial_x i_y, & i_y \partial_y &= \partial_y i_y, & i_y \partial_z &= q \partial_z i_y, \\
 i_z \partial_x &= q^{-1} \partial_x i_z, & i_z \partial_y &= q^{-1} \partial_y i_z, & i_z \partial_z &= \partial_z i_z.
 \end{aligned} \tag{14}$$

5.2. Lie derivations

In this section we find the commutation rules of the Lie derivatives with functions, *i.e.* the elements of the algebra \mathcal{A} , their differentials, etc., using the approach of Ref. [2].

For a general differential form, the Lie derivative is given by the formula

$$\mathcal{L}_a \alpha = i_a \mathbf{d} \alpha + \mathbf{d}(i_a \alpha).$$

For example, if we apply this formula to the first relation in (12), using the relations (13) we get

$$\begin{aligned}
 \mathcal{L}_x x &= (i_x \mathbf{d} + \mathbf{d} i_x)x \\
 &= 1 + x i_x \mathbf{d} + x \mathbf{d} i_x \\
 &= 1 + x \mathcal{L}_x.
 \end{aligned}$$

Other relations can be similarly obtained. Consequently, we have the following commutation relations:

- The relations between the Lie derivatives and the elements of \mathcal{A} :

$$\begin{aligned}
 \mathcal{L}_x x &= 1 + x \mathcal{L}_x, & \mathcal{L}_x y &= q^{-1} y \mathcal{L}_x, & \mathcal{L}_x z &= q^{-1} z \mathcal{L}_x, \\
 \mathcal{L}_y x &= q x \mathcal{L}_y, & \mathcal{L}_y y &= 1 + y \mathcal{L}_y, & \mathcal{L}_y z &= q^{-1} z \mathcal{L}_y, \\
 \mathcal{L}_z x &= q x \mathcal{L}_z, & \mathcal{L}_z y &= q y \mathcal{L}_z, & \mathcal{L}_z z &= 1 + z \mathcal{L}_z.
 \end{aligned} \tag{15}$$

- The relations of the Lie derivatives with the differentials:

$$\begin{aligned}
 \mathcal{L}_x dx &= dx \mathcal{L}_x, & \mathcal{L}_x dy &= q^{-1} dy \mathcal{L}_x, & \mathcal{L}_x dz &= q^{-1} dz \mathcal{L}_x, \\
 \mathcal{L}_y dx &= q dx \mathcal{L}_y, & \mathcal{L}_y dy &= dy \mathcal{L}_y, & \mathcal{L}_y dz &= q^{-1} dz \mathcal{L}_y, \\
 \mathcal{L}_z dx &= q dx \mathcal{L}_z, & \mathcal{L}_z dy &= q dy \mathcal{L}_z, & \mathcal{L}_z dz &= dz \mathcal{L}_z.
 \end{aligned} \tag{16}$$

Other commutation relations can be similarly obtained. To complete the description of the above scheme, we get below the remaining commutation relations as follows:

- The Lie derivatives and partial derivatives:

$$\begin{aligned}
 \mathcal{L}_x \partial_x &= \partial_x \mathcal{L}_x, & \mathcal{L}_x \partial_y &= q \partial_y \mathcal{L}_x, & \mathcal{L}_x \partial_z &= q \partial_z \mathcal{L}_x, \\
 \mathcal{L}_y \partial_x &= q^{-1} \partial_x \mathcal{L}_y, & \mathcal{L}_y \partial_y &= \partial_y \mathcal{L}_y, & \mathcal{L}_y \partial_z &= q \partial_z \mathcal{L}_y, \\
 \mathcal{L}_z \partial_x &= q^{-1} \partial_x \mathcal{L}_z, & \mathcal{L}_z \partial_y &= q^{-1} \partial_y \mathcal{L}_z, & \mathcal{L}_z \partial_z &= \partial_z \mathcal{L}_z.
 \end{aligned} \tag{17}$$

- The inner derivations:

$$\begin{aligned}
 i_x \wedge i_y &= -q i_y \wedge i_x, & i_x \wedge i_x &= 0, \\
 i_x \wedge i_z &= -q i_z \wedge i_x, & i_y \wedge i_y &= 0, \\
 i_y \wedge i_z &= -q i_z \wedge i_y, & i_z \wedge i_z &= 0.
 \end{aligned} \tag{18}$$

- The Lie derivatives and the inner derivations:

$$\begin{aligned}
 \mathcal{L}_x i_x &= i_x \mathcal{L}_x, & \mathcal{L}_x i_y &= q i_y \mathcal{L}_x, & \mathcal{L}_x i_z &= q i_z \mathcal{L}_x, \\
 \mathcal{L}_y i_x &= q^{-1} i_x \mathcal{L}_y, & \mathcal{L}_y i_y &= i_y \mathcal{L}_y, & \mathcal{L}_y i_z &= q i_z \mathcal{L}_y, \\
 \mathcal{L}_z i_x &= q^{-1} i_x \mathcal{L}_z, & \mathcal{L}_z i_y &= q^{-1} i_y \mathcal{L}_z, & \mathcal{L}_z i_z &= i_z \mathcal{L}_z.
 \end{aligned} \tag{19}$$

- The Lie derivatives:

$$\mathcal{L}_x \mathcal{L}_y = q \mathcal{L}_y \mathcal{L}_x, \quad \mathcal{L}_x \mathcal{L}_z = q \mathcal{L}_z \mathcal{L}_x, \quad \mathcal{L}_y \mathcal{L}_z = q \mathcal{L}_z \mathcal{L}_y. \tag{20}$$

These commutation relations are the same with the relations (5).

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References

- [1] Chryssomalakos, C., Schupp, P. and Zumino, B.: *Induced extended calculus on the quantum plane*, hep-th /9401141.
- [2] Çelik, S.: *Cartan Calculi on The Quantum Superplane*, J. Math. Phys. **47** (8), Art. No: 083501 (2006).
- [3] Çelik, S. A. and Yasar, E.: *Differential Geometry of the Quantum 3-Dimensional Space*, Czech. J. Phys. **56**, 229-236 (2006).
- [4] Dobrev, V. K.: *Duality for the Matrix Quantum Group*, J. Math. Phys. **33**, 3419-3430 (1992).
- [5] Manin, Yu I.: *Quantum groups and noncommutative geometry*, Montreal Univ. Preprint, (1988).

- [6] Schupp, P.: *Cartan calculus: Differential geometry for quantum groups*, hep-th/9408170.
- [7] Schupp, P., Watts, P. and Zumino, B.: *Differential Geometry on Linear Quantum Groups*, Lett. Math. Phys. **25**, 139-147 (1992).
- [8] Schupp, P., Watts, P. and Zumino, B.: *Cartan calculus on quantum Lie algebras*, hep-th/9312073.
- [9] Sudbery, A.: *Non-Commuting Coordinates and Differential Operators*, Proc. Workshop on Quantum Groups, Argonne, eds. Curtright T, Fairlie D and Zachos C, 33-51 (1990).
- [10] Wess, J. and Zumino, B.: *Covariant Differential Calculus on the Quantum Hyperplane*, Nucl. Phys. B **18**, 302-312 (1990).
- [11] Woronowicz, S. L.: *Compact Matrix Pseudogroups*, Commun. Math. Phys. **111**, 613-665 (1987).
- [12] Woronowicz, S. L.: *Differential Calculus on Compact Matrix Pseudogroups*, Commun. Math. Phys. **122**, 125-170 (1989).

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