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Strong Convergence Theorems by an Extragradient Method for Solving Variational Inequalities and Equilibrium Problems in a Hilbert Space*

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Abstract

In this paper, we introduce an iterative process for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for monotone, Lipschitz-continuous mappings. The iterative process is based on the so-called extragradient method. We show that the sequence converges strongly to a common element of the above three sets under some parametric controlling conditions. This main theorem extends a recent result of Yao, Liou and Yao [Y. Yao, Y. C. Liou and J.-C. Yao, "An Extragradient Method for Fixed Point Problems and Variational Inequality Problems," *Journal of Inequalities and Applications* Volume 2007, Article ID 38752, 12 pages doi:10.1155/2007/38752] and many others.

Key Words: Nonexpansive mapping; Equilibrium problem; Fixed point; Lipschitz-continuous mappings; Variational inequality; Extragradient method.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping T of H into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \longrightarrow \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \longrightarrow H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). In

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1997 Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let $A : C \longrightarrow H$ be a mapping. The classical variational inequality, denoted by $VI(A,C)$, is to find $x^* \in C$ such that $\langle Ax^*, v - x^*\rangle \geq 0$ for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [12, 15] and the references therein. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \tag{1.2}$$

for all $u, v \in C$. A is called k*-Lipschitz-continuous* if there exists a positive constant k such that for all $u, v \in C$

$$\|Au - Av\| \leq k\|u - v\|. \tag{1.3}$$

We denote by $F(S)$ the set of fixed points of S. For finding an element of $F(S) \cap VI(A,C)$, Takahashi and Toyoda [9] introduced the iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.4}$$

for every $n = 0, 1, 2, ...,$ where $x_0 = x \in C, \alpha_n$ is a sequence in (0, 1), and λ_n is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [6] and Zeng and Yao [16] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A,C)$.

The algorithm suggested by Takahashi and Toyoda [9] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of “hybrid” or “outer-approximation” types of methods was originally introduced by Haugazeau in 1968; see [3] for more details.

In 1976, Korpelevich [4] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = P_C(x_n - \lambda_n A\bar{x}_n) \end{cases} \tag{1.5}$$

for all $n \geq 0$, where $\lambda_n \in (0, \frac{1}{k}), C$ is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C in to $\mathbb{R}^n$. He proved that if $VI(C,A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.5), converge to the same point $z \in VI(C,A)$.

Motivated by the idea of Korpelevichs extragradient method Zeng and Yao [16] introduced a new extragradient method for finding an element of $F(S) \cap VI(C,A)$ and proved the following strong convergence theorem.

Theorem 1.1 ([16, Theorem 3.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and k-Lipschitz-continous mapping of C into H. Let S be a nonexpansive mappings from C into itself such that* $\mathrm{F}(S) \cap VI(C,A) \neq \emptyset$. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = \alpha_n x_0 + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \ \ \forall n \geq 0, \end{cases} \tag{1.6}$$

where $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *satisfy the conditions*

(i) $\lambda_n k \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$*;*

(ii) $\alpha_n \subset (0,1), \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\longrightarrow\infty} \alpha_n = 0$*,*

Then the sequence $\{x_n\}$ *and* $\{y_n\}$ *converges strongly to the same point* $P_{F(S)\cap VI(C,A)}x_0$ *provied that* $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.

In 2007, Yao, Liou and Yao [14] introduced the following iterative scheme: Let C be a closed convex subset of real Hilbert space H. Let A be a monotone k-Lipschitz-continous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A,C) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$. They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for a monotone k-Lipschitz-continous mapping under some parameters controlling conditions.

Recently, Takahashi and Takahashi [10] introduced an iterative scheme:

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, & \forall u \in C; \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T y_n, \ \ n \geq 1 \end{cases}$$

for approximatiing a common element of the set of fixed points of a non-self nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

In this paper, motivated and inspired by the above results, we introduce a new iterative scheme by the extragradient method as follows: For $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C; \\ y_n = P_C(u_n - \lambda_n A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \ \ n \geq 1, \end{cases} \tag{1.8}$$

for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for a monotone k-Lipschitz-continous mapping in a real Hilbert space. Moreover, we obtain a strong convergence theorem which is connected with Yao, Liou and Yao's result [14], Takahashi and Tada's result [9] and Zeng and Yao's result [16].

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. Let H be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda A u), \lambda > 0. \tag{2.6}$$

We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I-\lambda A)u - (I-\lambda A)v\|^2 &= \|(u-v) - \lambda(Au - Av)\|^2 \\ &= \|u-v\|^2 - 2\lambda\langle u-v, Au-Av\rangle + \lambda^2\|Au-Av\|^2 \\ &\leq \|u-v\|^2 + \lambda(\lambda - 2\alpha)\|Au-Av\|^2. \end{aligned} \tag{2.7}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1 (See Osilike and Igbokwe [7].) *Let $(E, \langle .,.\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 2.2 (See Suzuki [8]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3 (**Demiclosedness Principle**; cf. Goebel and Kirk [5].) *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \longrightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 2.4 (See Xu [11]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that:

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \longrightarrow \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\longrightarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.5 (See Blum and Oettli [1]) *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0 \textit{ for all } y \in C.$$

The following lemma was also given in [2].

Lemma 2.6 (See Combettes and Hirstoaga [2].) *Assume that $F : C \times C \longrightarrow \mathbf{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. *T_r is single- valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

3. Main Results

In this section, we introduce an iterative process by the extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for a monotone k-Lipschitz-continous mapping in a real Hilbert space. We prove that the iterative sequences converges strongly to a common element of the above three sets.

Theorem 3.1 *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \longrightarrow \mathbf{R}$ satisfying (A1)–(A4) and let A be a monotone k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A,C) \cap EP(F) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are given by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C; \\ y_n = P_C(u_n - \lambda_n A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{3.1}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$, $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$ and $\{r_n\} \subset (0, \infty)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(iv) $\liminf_{n\longrightarrow\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

(v) $\lim_{n\longrightarrow\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(A,C)\cap EP(F)}u$.

Proof. For all $x, y \in C$, we note that

$$\begin{aligned} \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|(x - y) - \lambda_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda_n^2 k^2\|x - y\|^2 = (1 + \lambda_n^2 k^2)\|x - y\|^2, \end{aligned} \tag{3.2}$$

which implies that

$$\|(I - \lambda_n A)x - (I - \lambda_n A)y\| \leq (1 + \lambda_n k)\|x - y\|. \tag{3.3}$$

Let $x^* \in F(S) \cap VI(A,C) \cap EP(F)$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.6 and $u_n = T_{r_n}x_n$. Then $x^* = P_C(x^* - \lambda_n A x^*) = T_{r_n}x^*$. Put $v_n = P_C(x_n - \lambda_n A y_n)$. For any $n \in \mathbb{N}$, we get

$$\|u_n - x^*\| = \|T_{r_n}x_n - T_{r_n}x^*\| \leq \|x_n - x^*\|.$$

From (2.5) and the monotonicity of A, we have

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \|x_n - \lambda_n Ay_n - x^*\|^2 - \|x_n - \lambda_n Ay_n - v_n\|^2 \\ &= \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n\langle Ay_n, u - v_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n(\langle Ay_n - Au, x^* - y_n\rangle + \langle Au, x^* - y_n\rangle) + \langle Ay_n, y_n - v_n\rangle \\ &\leq \|x_n - x^*\|^2 - \|x_n - v_n\|^2 + 2\lambda_n\langle Ay_n, y_n - v_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 + 2\lambda_n\langle Ay_n, y_n - v_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle.\end{aligned}$$

Since A is k-Lipschitz-continuous, it follows that

$$\begin{aligned}\langle x_n - \lambda_n Ay_n - y_n, v_n - y_n\rangle &= \langle x_n - \lambda_n Ax_n - y_n, v_n - y_n\rangle + \langle \lambda_n Ax_n - \lambda_n Ay_n, v_n - y_n\rangle \\ &\leq \langle \lambda_n Ax_n - \lambda_n Ay_n, v_n - y_n\rangle \\ &\leq \lambda_n k\|x_n - y_n\|\|v_n - y_n\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n k\|x_n - y_n\|\|v_n - y_n\| \\ &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + \lambda_n^2 k^2(\|x_n - y_n\|^2 + \|v_n - y_n\|^2) \\ &= \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \qquad (3.4) \\ &\leq \|x_n - x^*\|^2.\end{aligned}$$

Then, we have also

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n Sv_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|v_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_0 - x^*\|\}\end{aligned}$$

Therefore $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}$ and $\{v_n\}$ are also bounded. Moreover, we observe that

$$\begin{aligned}\|v_{n+1} - v_n\| &= \|P_C(x_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(x_n - \lambda_n Ay_n)\| \\ &\leq \|(x_{n+1} - \lambda_{n+1}Ay_{n+1}) - (x_n - \lambda_n Ay_n)\| \\ &= \|(x_{n+1} - x_n) - \lambda_{n+1}(Ay_{n+1} - Ay_n) - (\lambda_{n+1} - \lambda_n)Ay_n\| \\ &\leq \|x_{n+1} - x_n\| + \lambda_{n+1}k\|y_{n+1} - y_n\| + |\lambda_{n+1} - \lambda_n|\|Ay_n\| \\ &\leq \|x_{n+1} - x_n\| + \lambda_{n+1}k\|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|Ay_n\|. \qquad (3.5)\end{aligned}$$

On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C \qquad (3.6)$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \text{ for all } y \in C. \tag{3.7}$$

Putting $y = u_{n+1}$ in (3.6) and $y = u_n$ in (3.7), we obtain

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

It follows from (A2) that

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Since $\liminf_{n\longrightarrow\infty} r_n > 0$, without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbf{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{L}{c}|r_{n+1} - r_n|, \end{aligned} \tag{3.8}$$

where $L = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Substituting (3.8) into (3.5), we have

$$\begin{aligned} \|v_{n+1} - v_n\| &\leq \|x_{n+1} - x_n\| + k\lambda_{n+1}\{\|x_{n+1} - x_n\| + \frac{L}{c}|r_{n+1} - r_n|\} + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\ &\leq (1 + k\lambda_{n+1})\|x_{n+1} - x_n\| + k\lambda_{n+1}\frac{L}{c}|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}|\|Ay_n\|. \end{aligned} \tag{3.9}$$

Let $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$. Thus, we get

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n SP_C(x_n - \lambda_n Ay_n)}{1 - \beta_n} = \frac{\alpha_n u + \gamma_n Sv_n}{1 - \beta_n}$$

and hence we have

$$
\begin{aligned}
z_{n+1}-z_n &= \frac{\alpha_{n+1}u+\gamma_{n+1}Sv_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n u+\gamma_n Sv_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}u+\gamma_{n+1}Sv_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_{n+1}u+\gamma_{n+1}Sv_n}{1-\beta_{n+1}}+\frac{\alpha_{n+1}u+\gamma_{n+1}Sv_n}{1-\beta_{n+1}}-\frac{\alpha_n u+\gamma_n Sv_n}{1-\beta_n}\\
&= (\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n})u+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(Sv_{n+1}-Sv_n)+(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n})Sv_n. \qquad (3.10)
\end{aligned}
$$

Combining (3.9) and (3.10), we obtain

$$
\begin{aligned}
\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\le |\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|u\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|v_{n+1}-v_n\|\\
&\quad +|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}|\|Sv_n\|-\|x_{n+1}-x_n\|\\
&\le |\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|u\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(1+\lambda_{n+1}k)\|x_{n+1}-x_n\|\\
&\quad +\frac{\gamma_{n+1}}{(1-\beta_{n+1})}\frac{L}{c}\lambda_{n+1}k|r_{n+1}-r_n|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_n-\lambda_{n+1}|\|Ay_n\|\\
&\quad +|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}|\|Sv_n\|-\|x_{n+1}-x_n\|\\
&\le |\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|u\|+\|Sv_n\|)+\frac{\gamma_{n+1}\lambda_{n+1}k-\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|\\
&\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}\{\lambda_{n+1}k\frac{L}{c}|r_{n+1}-r_n|+|\lambda_n-\lambda_{n+1}|\|Ay_n\|\}.
\end{aligned}
$$

This together with (ii), (iv) and (v) imply that

$$\limsup_{n\longrightarrow\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, by Lemma 2.2, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \qquad (3.11)$$

Consequently,

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=\lim_{n\longrightarrow\infty}(1-\beta_n)\|z_n-x_n\|=0. \qquad (3.12)$$

From (iv), (v), (3.5) and (3.8), we also have $\|v_{n+1}-v_n\|\longrightarrow 0, \|u_{n+1}-u_n\|\longrightarrow 0$ and $\|y_{n+1}-y_n\|\longrightarrow 0$ as $n\longrightarrow\infty$. Since

$$x_{n+1}-x_n=\alpha_n u+\beta_n x_n+\gamma_n Sv_n-x_n=\alpha_n(u-x_n)+\gamma_n(Sv_n-x_n),$$

it follows by (ii) and (3.12) that

$$\lim_{n\to\infty}\|x_n-Sv_n\|=0. \qquad (3.13)$$

We note that

$$\begin{aligned}\|y_n - v_n\| &\leq \|P_C(u_n - \lambda_n A u_n) - P_C(x_n - \lambda_n A y_n)\| \\ &\leq \|(u_n - \lambda_n A u_n) - (x_n - \lambda_n A y_n)\| \\ &\leq \|u_n - x_n\| + \lambda_n \|A u_n - A y_n\| \\ &\leq \|u_n - x_n\| + \lambda_n k \|u_n - y_n\| \\ &\leq \|u_n - x_n\|,\end{aligned}$$

since $\lambda_n \leq 1$, hence we also have

$$\|y_n - v_n\|^2 \leq \|u_n - x_n\|^2. \tag{3.14}$$

From this and by (3.4) and (3.14) we obtain when $n \geq N$ that

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \\ &\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \\ &\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2.\end{aligned}$$

So, from this, we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + \beta_n x_n + \gamma_n S v_n - x^*\|^2 \leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|S v_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|v_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2\} \\ &= \alpha_n \|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 + \gamma_n (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - x_n\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}(1 - \lambda_n^2 k^2)\|x_n - u_n\|^2 &\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - x^*\| - \|x_{n+1} - x^*\|).\end{aligned}$$

Since $\alpha_n \longrightarrow 0$, $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{k})$ and $\|x_{n+1} - x_n\| \longrightarrow 0$, imply that

$$\lim_{n \longrightarrow \infty} \|x_n - u_n\| = 0. \tag{3.15}$$

Since $\liminf_{n \longrightarrow \infty} r_n > 0$, we get

$$\lim_{n \longrightarrow \infty} \|\frac{x_n - u_n}{r_n}\| = \lim_{n \longrightarrow \infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.16}$$

By (3.4), we note that

$$\|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2. \tag{3.17}$$

Thus, from Lemma 2.1 and (3.17), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|Sv_n-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\{\|x_n-x^*\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|^2\}\\ &\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|^2.\end{aligned} \tag{3.18}$$

Therefore, we have

$$\begin{aligned}(1-\lambda_n^2k^2)\|x_n-y_n\|^2 &\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &= \alpha_n\|u-x^*\|^2+\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|).\end{aligned} \tag{3.19}$$

Since $\alpha_n \longrightarrow 0$ and $\|x_n-x_{n+1}\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain

$$\lim_{n\longrightarrow\infty}\|x_n-y_n\|=0. \tag{3.20}$$

We note that

$$\begin{aligned}\|v_n-y_n\| &= \|P_C(x_n-\lambda_nAy_n)-P_C(u_n-\lambda_nAu_n)\|\\ &\le \|(x_n-\lambda_nAy_n)-(u_n-\lambda_nAu_n)\|\\ &\le \|x_n-u_n\|+\lambda_n\|Au_n)-Ay_n)\|\\ &\le \|x_n-u_n\|+\lambda_nk\|u_n-y_n\|\\ &\le \|x_n-u_n\|+\lambda_nk\{\|u_n-x_n\|+\|x_n-y_n\|\}\\ &\le (1+\lambda_nk)\|u_n-x_n\|+\lambda_nk\|x_n-y_n\|\end{aligned}$$

since (3.15) and (3.20), we have

$$\lim_{n\longrightarrow\infty}\|v_n-y_n\|=0. \tag{3.21}$$

Since

$$\|Sv_n-v_n\|\le\|Sv_n-x_n\|+\|x_n-y_n\|+\|y_n-v_n\|,$$

and hence

$$\lim_{n\longrightarrow\infty}\|Sv_n-v_n\|=0. \tag{3.22}$$

Next, we show that

$$\limsup_{n\longrightarrow\infty}\langle u-z_0,x_n-z_0\rangle\le 0,$$

where $z_0=P_{F(S)\cap VI(A,C)\cap EP(F)}(u)$. To show this inequality, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle u-z_0,Sv_n-z_0\rangle=\lim_{i\longrightarrow\infty}\langle u-z_0,Sv_{n_i}-z_0\rangle.$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup z$. From $\|Sv_n - v_n\| \longrightarrow 0$, we obtain $Sv_{n_i} \rightharpoonup z$. Let us show $z \in EP(F)$. Since $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

From $\|u_n - x_n\| \longrightarrow 0, \|x_n - Sv_n\| \longrightarrow 0$, and $\|Sv_n - v_n\| \longrightarrow 0$, we get $u_{n_i} \rightharpoonup z$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$, it follows by (A4) that $0 \geq F(y, z)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, z) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From (A3), we have $0 \leq F(z, y)$ for all $y \in C$ and hence $z \in EP(F)$. By the opial's condition, we obtain $z \in F(S)$. Finally, by the same argument as that in the proof of [9, Theorem 3.1, p. 197-198] , we can show that $z \in VI(A, C)$. Hence $z \in F(S) \cap VI(A, C) \cap EP(F)$.

Now from (2.4), we have

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle u - z_0, x_n - z_0\rangle &= \limsup_{n\longrightarrow\infty}\langle u - z_0, Sv_n - z_0\rangle = \lim_{i\longrightarrow\infty}\langle u - z_0, Sv_{n_i} - z_0\rangle \\ &= \langle u - z_0, z - z_0\rangle \leq 0. \end{aligned} \tag{3.23}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \langle \alpha_n u + \beta_n x_n + \gamma_n Sv_n - z_0, x_{n+1} - z_0\rangle \\ &= \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle + \beta_n\langle x_n - z_0, x_{n+1} - z_0\rangle + \gamma_n\langle Sv_n - z_0, x_{n+1} - z_0\rangle \\ &\leq \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle + \frac{1}{2}\gamma_n(\|v_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ &\leq \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle + \frac{1}{2}\gamma_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ &= \frac{1}{2}(1 - \alpha_n)(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\leq \frac{1}{2}\{(1 - \alpha_n)\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\} + \alpha_n\langle u - z_0, x_{n+1} - z_0\rangle\end{aligned}$$

which implies that

$$\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.$$

Finally by (3.23) and Lemma 2.4, we get that $\{x_n\}$ converges to z_0, where $z_0 = P_{F(S)\cap VI(A,C)\cap EP(F)}(u)$. This completes the proof. □

Using Theorem 3.1, we can prove the following result.

Theorem 3.2 (Yao Liou and Yao [14, Theorem 3.1]) *Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone k-Lipschitz-continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S)\cap VI(A,C)\neq\emptyset$. For fixed $u\in H$ and give $x_0\in H$ arbitrary, let the sequence $\{x_n\},\{y_n\}$ be generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(x_n - \lambda_n Ay_n), \end{cases} \tag{3.24}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0,\frac{1}{k}]$. If $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n\in[a,b]$ for some a,b with $0<a<b<\frac{1}{k}$ and

(i) $\alpha_n+\beta_n+\gamma_n=1$,

(ii) $\lim_{n\longrightarrow\infty}\alpha_n=0, \sum_{n=1}^{\infty}\alpha_n=\infty$,

(iii) $0<\liminf_{n\longrightarrow\infty}\beta_n\leq\limsup_{n\longrightarrow\infty}\beta_n<1$,

(iv) $\lim_{n\longrightarrow\infty}(\lambda_{n+1}-\lambda_n)=0$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(A,C)}x_0$.

Proof. Put $F(x,y)=0$ for all $x,y\in C$ and $r_n=1$ for all $n\in\mathbb{N}$ in Theorem 3.1 .
Then, we have $u_n=P_Cx_n=x_n$. So, from Theorem 3.1 the sequence $\{x_n\}$ generated in Theorem 3.2 converges strongly to $P_{F(S)\cap VI(A,C)}u$. □

Remark 3.3 *In Theorem 3.2, we also obtain Yao et al.'s theorem [14].*

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References

[1] Blum, E. and Oettli, W.: From optimization and variational inequalities to equilibrium problems, *Math. Student* 63, 123-145 (1994).

[2] Combettes, P.L. and Hirstoaga, S.A.: Equilibrium programming using proximal-like algorithms, *Math. Program* 78, 29-41 (1997).

[3] Haugazeau, Y.: Surles inéquations variationnelles et la minimisation de fonctionnelles convexes, Thèse, Université de Paris, Paris, France, 1968.

[4] Korpelevich, G.M.: The extragradient method for finding saddle points and other problems, Matecon 12, 747–756 (1976).

[5] Goebel, K. and Kirk, W. A.: Topics on Metric Fixed-Point Theory, Cambridge University Press, Cambridge. England, 1990.

[6] Nadezhkina, N. and Takahashi, W.: Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* 128, 191-201 (2006).

[7] Osilike, M.O. and Igbokwe, D.I.: Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, *Comp. Math. Appl.* 40, 559-567 (2000).

[8] Suzuki, T.: Strong convergence of Krasnoselskii and Manns type sequences for one-parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. and Appl. 305, 227–239 (2005).

[9] Takahashi, W. and Toyoda, M.: Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* 118, 417-428 (2003).

[10] Takahashi, S. and Takahashi, W.: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, 393 J. Math. Anal. Appl. 311, 506–515 (2007).

[11] Xu, H. K.: Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* 298, 279-291 (2004).

[12] Yao, J.-C. and Chadli, O.: Pseudomonotone complementarity problems and variational inequalities, In: Handbook of Generalized Convexity and Monotonicity (Eds.: J.P. Crouzeix, N. Haddjissas and S. Schaible) 501-558, Springer Netherlands (2005).

[13] Yao, Y. and Yao, J.-C.: On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.* 186, 1551-1558 (2007).

[14] Yao, Y., Liou Y. C. and Yao, J.-C.: An Extragradient Method for Fixed Point Problems and Variational Inequality Problems, *Journal of Inequalities and Applications* Volume 2007, Article ID 38752, 12 pages doi:10.1155/2007/38752 (2007).

[15] Zeng, L.C., Schaible, S. and Yao, J.-C.: Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, *J. Optim. Theory Appl.* 124, 725-738 (2005).

[16] Zeng, L.C. and Yao, J.-C.: Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, *Taiwanese J. Math.* 10, 1293-1303 (2006).

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