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Dual Quaternions in Spatial Kinematics in an Algebraic Sense

Bedia Akyar

Abstract

This paper presents the finite spatial displacements and spatial screw motions by using dual quaternions and Hamilton operators. The representations are considered as 4×4 matrices and the relative motion for three dual spheres is considered in terms of Hamilton operators for a dual quaternion. The relation between Hamilton operators and the transformation matrix has been given in a different way. By considering operations on screw motions, representation of spatial displacements is also given.

Key Words: Dual quaternions, Hamilton operators, Lie algebras.

1. Introduction

The matrix representation of spatial displacements of rigid bodies has an important role in kinematics and the mathematical description of displacements. Veldkamp and Yang-Freudenstein investigated the use of dual numbers, dual numbers matrix and dual quaternions in instantaneous spatial kinematics in [9] and [10], respectively. In [10], an application of dual quaternion algebra to the analysis of spatial mechanisms was given. A comparison of representations of general spatial motion was given by Rooney in [8]. Hiller-Woernle worked on a unified representation of spatial displacements. In their paper [7], the representation is based on the screw displacement pair, i.e., the dual number extension of the rotational displacement pair, and consists of the dual unit vector of the screw axis

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and the associate dual angle of the amplitude. Chevallier gave a unified algebraic approach to mathematical methods in kinematics in [4]. This approach requires screw theory, dual numbers and Lie groups. Agrawal [1] worked on Hamilton operators and dual quaternions in kinematics. In [1], the algebra of dual quaternions is developed by using two Hamilton operators. Properties of these operators are used to find some mathematical expressions for screw motion. The expressions of finite spatial displacements and finite infinitesimal displacements by using dual orthogonal matrices of the Lie algebra of dual numbers and exponentials were given by Akyar-Köse in [2].

In the present paper, the spatial motion of a point on a dual sphere is given by using dual quaternions and Hamilton operators. The aim is to find an aspect of Hamilton operators for dual quaternions to screw motions. It is well-known that the set of oriented lines corresponds to the set of all unit dual vector quaternions so a dual quaternion can be used to write the spatial motion associated with screws. One can extend the Hamilton operators for any quaternion to the ones for any dual quaternion. The point in the present paper is to obtain the relation between Hamilton matrices and the transformation matrix in a different way. We are considering dual spherical motions and combining the Hamilton matrices with the transformation matrix. That is why we may write the finite screw motion in terms of the transition matrix for Hamilton operators. We take 3 dual unit spheres and give the relations among them in view of spherical motion. Although the necessary 3×3 dual transition matrices are already given in [2], in the present work we express the relative motion in terms of Hamilton operators and give the relation among the Hamilton matrices for the transformation matrices. Furthermore, we give the rates of changes of the points on the dual spheres in terms of Hamilton operators for dual quaternions. By considering the instantaneous screw axes of the motions with respect to each other, we give the derivative of Hamilton matrices for a dual quaternion in terms of the Hamilton matrices for the derivative of the same dual quaternion. At the end of the work, we give an aspect of representation of spatial displacements by using the Lie bracket and write the velocity in terms of the Lie bracket of Hamilton operators for dual quaternions.

2. Dual Quaternions

Definition 2.1 *A dual number $\hat{a}$ has the form $a + \varepsilon a'$, where $a, a' \in \mathbb{R}$ and $\varepsilon^2 = 0$. Dual numbers denoted by Δ form an abelian ring with unity and divisors of zero.*

Definition 2.2 *An ordinary quaternion is defined as*

$$\mathbf{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ *and* $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ *of* $\mathbf{q}$ *may be interpreted as the four basis vectors of a Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules* $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. *An ordinary quaternion may be defined as a pair* $(s, \mathbf{v})$, *where* s *is the scalar part (Sc) and* $\mathbf{v}$ *is the vector part of* $\mathbf{q}$. *The set of quaternions,* $\mathbb{H}$, *is a real space with zero* $0 = (0, \mathbf{0})$ *under the usual addition and a multiplication by a scalar defined by*

$$\mathbf{q}_1 + \mathbf{q}_2 = (s_1 + s_2, \mathbf{v}_1 + \mathbf{v}_2),\ \ \lambda(s, \mathbf{v}) = (\lambda s,\ \lambda \mathbf{v}),\ \ \lambda \in \mathbb{R}, \tag{1}$$

respectively. It is a skew field with unity $e = (1, \mathbf{0})$ *under the usual addition and multiplication. The latter is defined by*

$$\mathbf{q}_1.\mathbf{q}_2 = (s_1.s_2 - \mathbf{v}_1.\mathbf{v}_2, s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \wedge \mathbf{v}_2). \tag{2}$$

Here, "." denotes the scalar multiplication and "$\wedge$" denotes the vectorial multiplication of vectors. Finally, it is an associative algebra over $\mathbb{R}$ *under all the operations (1) and (2).*

Definition 2.3 *A dual quaternion* $\hat{\mathbf{q}}$ *is defined by*

$$\hat{\mathbf{q}} = \hat{a}_0 + \hat{a}_1\mathbf{i} + \hat{a}_2\mathbf{j} + \hat{a}_3\mathbf{k}.$$

In other words, this may also be given as $\hat{\mathbf{q}} = \mathbf{q} + \varepsilon\mathbf{q}'$, *where* $\mathbf{q}$ *and* $\mathbf{q}'$ *are* **real and pure dual quaternion components**, *respectively. The set of dual quaternions is denoted by* $\mathcal{H}$ *and we may write* $[\mathbf{q}, \mathbf{q}'] \in \mathcal{H}$. *Under the usual operations (addition, multiplication by a scalar) defined by*

$$\hat{\mathbf{q}}_1 + \hat{\mathbf{q}}_2 = (\mathbf{q}_1 + \mathbf{q}_2, \mathbf{q}_1{}' + \mathbf{q}_2{}'),\quad \lambda(\mathbf{q}, \mathbf{q}') = (\lambda\mathbf{q}, \lambda\mathbf{q}'),\ \ \lambda \in \mathbb{R},$$

$(\mathcal{H}, +)$ *is an abelian group and* Δ *is an abelian ring with the unity* e. $\mathcal{H}$ *is a* Δ*-module and it is an associative algebra under the* **usual quaternion multiplication** *defined by*

$$\hat{\mathbf{q}}_1.\hat{\mathbf{q}}_2 = (\mathbf{q}_1.\mathbf{q}_2, \mathbf{q}_1\mathbf{q}_2{}' + \mathbf{q}_2\mathbf{q}_1{}').$$

Though we omit ".", we still mean the usual multiplication. There is no difference between writing $\hat{a}_1\mathbf{i}$ or $\mathbf{i}\hat{a}_1$, since $\varepsilon\mathbf{i} = \mathbf{i}\varepsilon$. The properties of the units for real and dual parts of the quaternion are obtained by the rules of ordinary algebra.

The scalar part of a dual quaternion $\hat{\mathbf{q}}$ is $S\hat{\mathbf{q}} = \hat{a}_0$.

The dual vector part of a dual quaternion $\hat{\mathbf{q}}$ is $V\hat{\mathbf{q}} = \hat{a}_1\mathbf{i} + \hat{a}_2\mathbf{j} + \hat{a}_3\mathbf{k}$.

The Hamilton conjugate of a dual quaternion $\hat{\mathbf{q}}$ is

$$\hat{\mathbf{q}}^* = \hat{a}_0 - (\hat{a}_1\mathbf{i} + \hat{a}_2\mathbf{j} + \hat{a}_3\mathbf{k}).$$

The norm of a dual quaternion $\hat{\mathbf{q}}$ is

$$N(\hat{\mathbf{q}}) = \hat{\mathbf{q}}\hat{\mathbf{q}}^* = \hat{\mathbf{q}}^*\hat{\mathbf{q}} = \hat{a}_0^2 + \hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2 = \mathbf{q}\mathbf{q}^* + \varepsilon(\mathbf{q}{\mathbf{q}'}^* + \mathbf{q}'\mathbf{q}^*).$$

This is, in general, not a real number but a dual number. If $N(\hat{\mathbf{q}}) = 1$ then $\hat{\mathbf{q}}$ is called a dual unit quaternion.

Definition 2.4 *The scalar product of two dual quaternions $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ is defined by*

$$\begin{aligned}(\hat{\mathbf{p}}, \hat{\mathbf{q}}) &:= \frac{1}{2}(\hat{\mathbf{p}}^*\hat{\mathbf{q}} + \hat{\mathbf{q}}^*\hat{\mathbf{p}}) = \frac{1}{2}(\hat{\mathbf{p}}\hat{\mathbf{q}}^* + \hat{\mathbf{q}}\hat{\mathbf{p}}^*)\\ &= \hat{a}_0\hat{b}_0 + \hat{a}_1\hat{b}_1 + \hat{a}_2\hat{b}_2 + \hat{a}_3\hat{b}_3 = (\hat{\mathbf{q}}, \hat{\mathbf{p}}),\end{aligned}$$

where $\hat{\mathbf{p}} = \mathbf{p} + \varepsilon\mathbf{p}'$ and $\hat{\mathbf{q}} = \mathbf{q} + \varepsilon\mathbf{q}'$. If $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are dual quaternions, then the quaternion multiplication is given by

$$\begin{aligned}\hat{\mathbf{p}}\hat{\mathbf{q}} &= (\hat{a}_0\hat{b}_0 - \hat{a}_1\hat{b}_1 - \hat{a}_2\hat{b}_2 - \hat{a}_3\hat{b}_3) + (\hat{a}_1\hat{b}_0 + \hat{a}_0\hat{b}_1 - \hat{a}_3\hat{b}_2 + \hat{a}_2\hat{b}_3)\mathbf{i}\\ &+ (\hat{a}_2\hat{b}_0 + \hat{a}_3\hat{b}_1 + \hat{a}_0\hat{b}_2 - \hat{a}_1\hat{b}_3)\mathbf{j} + (\hat{a}_3\hat{b}_0 - \hat{a}_2\hat{b}_1 + \hat{a}_1\hat{b}_2 + \hat{a}_0\hat{b}_3)\mathbf{k}.\end{aligned}$$

The quaternion multiplication is in general not commutative. We may consider a dual quaternion as a tetrad of dual numbers which can also be written in a column matrix form as

$$\hat{\mathbf{q}} = [\hat{a}_0\ \hat{a}_1\ \hat{a}_2\ \hat{a}_3]^T,$$

where

$$\mathbf{q} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{q}' = \begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \\ a'_3 \end{pmatrix}$$

are 4×1 column vectors.

Definition 2.5 *The vector product of two dual quaternions* $\hat{\mathbf{p}}$ *and* $\hat{\mathbf{q}}$ *is defined by*

$$\hat{\mathbf{p}} \times \hat{\mathbf{q}} = \frac{1}{2}(\hat{\mathbf{p}}\hat{\mathbf{q}} - \hat{\mathbf{q}}\hat{\mathbf{p}}).$$

3. Dual Unit Quaternions and Spherical Displacements

A dual unit vector quaternion corresponds to a directed line. We may consider a dual quaternion as

$$\hat{\mathbf{q}} = \mathbf{q} + \varepsilon\mathbf{q}' = \quad a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + \varepsilon(a_0{}' + a_1{}'\mathbf{i} + a_2{}'\mathbf{j} + a_3{}'\mathbf{k}).$$

There is a 1-1 correspondence between the set of oriented lines and the set of all unit dual vector quaternions, that is, $a_0 = a_0{}' = 0$ (see [2]). Then we get $\mathbf{q} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{q}' = a_1{}'\mathbf{i} + a_2{}'\mathbf{j} + a_3{}'\mathbf{k}$. The vector quaternions $\mathbf{q}$ and $\mathbf{q}'$ may be interpreted as the vectors of an unambiguously determined line L having $\mathbf{q}$ as its direction vector and passing through the point $\mathbf{p} = \mathbf{q}\times\mathbf{q}'$. This line becomes an oriented line. The Plücker coordinates of an arbitrary line $\hat{\mathbf{q}}$ in space are $(a_1, a_2, a_3, a_1{}', a_2{}', a_3{}')$. The dual quaternion can be used to express the spatial transformations associated with screws. A screw motion is described as dual angular displacement about a dual vector axis. Here the dual angular displacement consists of a rotation (the real part of the dual angle) about the screw axis and a translation (the dual part of the dual angle) along the screw axis. That is, the dual vector screw axis consists of a direction (the real part of the dual vector) and its moment (the dual part of the dual vector) about the origin.

4. Hamilton Operators

We can recall some definitions of Hamilton operators as follows.

Definition 4.1 *Let* $\mathbf{q}$ *be a real quaternion then Hamilton operators* $\overset{+}{H}$, $\overset{-}{H}$ *are respectively defined as*

$$\overset{+}{H}(\mathbf{q}) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}$$

and

$$\overset{-}{H}(\mathbf{q}) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix}.$$

The multiplication of the two dual quaternions $\hat{\mathbf{p}}$ *and* $\hat{\mathbf{q}}$

$$\hat{\mathbf{p}}\hat{\mathbf{q}} = \overset{+}{H}(\hat{\mathbf{p}})\hat{\mathbf{q}} = \overset{-}{H}(\hat{\mathbf{q}})\hat{\mathbf{p}} \tag{3}$$

follows from the definition of $\overset{+}{H}$ *and* $\overset{-}{H}$ *and it is very useful to prove several identities ([1]). Here* $\overset{+}{H}$ *and* $\overset{-}{H}$ *are seen as left (or right) translations.*

Since $\overset{+}{H}$, $\overset{-}{H}$ are linear in their elements, it follows that

$$\begin{aligned} \overset{+}{H}(\mathbf{q}) &= a_0 \overset{+}{H}(1) + a_1 \overset{+}{H}(\mathbf{i}) + a_2 \overset{+}{H}(\mathbf{j}) + a_3 \overset{+}{H}(\mathbf{k}) \\ \overset{-}{H}(\mathbf{q}) &= a_0 \overset{-}{H}(1) + a_1 \overset{-}{H}(\mathbf{i}) + a_2 \overset{-}{H}(\mathbf{j}) + a_3 \overset{-}{H}(\mathbf{k}). \end{aligned}$$

We extend the definition of the Hamilton operators to dual quaternions by setting

$$\begin{aligned} \overset{+}{H}(\hat{\mathbf{q}}) &= \overset{+}{H}(\mathbf{q}) + \varepsilon \overset{+}{H}(\mathbf{q}'), \\ \overset{-}{H}(\hat{\mathbf{q}}) &= \overset{-}{H}(\mathbf{q}) + \varepsilon \overset{-}{H}(\mathbf{q}'). \end{aligned}$$

Theorem 4.2 [*1*] *If* $\hat{\mathbf{p}}$ *and* $\hat{\mathbf{q}}$ *are two dual quaternions and* $\overset{+}{H}$, $\overset{-}{H}$ *are the operators as defined before, then we have the following identities:*

$$\begin{aligned} &(i) && \overset{+}{H}(\hat{\mathbf{p}}\hat{\mathbf{q}}) = \overset{+}{H}(\hat{\mathbf{p}})\overset{+}{H}(\hat{\mathbf{q}}) \\ &(ii) && \overset{-}{H}(\hat{\mathbf{p}}\hat{\mathbf{q}}) = \overset{-}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{p}}) \\ &(iii) && \overset{+}{H}(\overset{-}{H}(\hat{\mathbf{p}})\hat{\mathbf{q}}) = \overset{+}{H}(\hat{\mathbf{q}})\overset{+}{H}(\hat{\mathbf{p}}) \\ &(iv) && \overset{-}{H}(\overset{+}{H}(\hat{\mathbf{p}})\hat{\mathbf{q}}) = \overset{-}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{p}}) \\ &(v) && \overset{+}{H}(\hat{\mathbf{p}})\overset{-}{H}(\hat{\mathbf{q}}) = \overset{-}{H}(\hat{\mathbf{q}})\overset{+}{H}(\hat{\mathbf{p}}). \end{aligned}$$

Proof. Since the first four identities are clear, we can only consider the proof of (v) which follows by using (3) two times. That is, $\forall r \in \mathbb{H}$, $\hat{\mathbf{p}}\hat{\mathbf{r}}\hat{\mathbf{q}}$ may be written in two different ways:

$$\begin{aligned} \hat{\mathbf{p}}\hat{\mathbf{r}}\hat{\mathbf{q}} &= \overset{+}{H}(\hat{\mathbf{p}})\hat{\mathbf{r}}\hat{\mathbf{q}} = \overset{+}{H}(\hat{\mathbf{p}})\overset{-}{H}(\hat{\mathbf{q}})\hat{\mathbf{r}} \\ \hat{\mathbf{p}}\hat{\mathbf{r}}\hat{\mathbf{q}} &= \overset{-}{H}(\hat{\mathbf{q}})\hat{\mathbf{p}}\hat{\mathbf{r}} = \overset{-}{H}(\hat{\mathbf{q}})\overset{+}{H}(\hat{\mathbf{p}})\hat{\mathbf{r}}. \end{aligned}$$

So one gets (v). □

5. Dual Matrices

Definition 5.1 *A matrix $\hat{\mathbf{A}}$ is called an orthogonal dual matrix (ogdm) if $\hat{\mathbf{A}}^T\hat{\mathbf{A}} = \hat{\mathbf{A}}\hat{\mathbf{A}}^T = \hat{a}I$, where $\hat{a}$ is a dual number and I is the identity matrix. If $\hat{a} = 1$, then a matrix $\hat{\mathbf{A}}$ is called an orthonormal dual matrix (ondm). The row vectors (column vectors) of a dual matrix are mutually orthogonal dual unit vectors, that is, the scalar product of two different row (column) vectors is zero, otherwise 1. The set of orthonormal dual $n \times n$ matrices is denoted by $\hat{O}(n)$. We denote the set of special orthonormal dual matrices by*

$$\widehat{SO}(n) = \{\hat{\mathbf{A}} \in \hat{O}(n) \mid det\hat{\mathbf{A}} = 1\}.$$

The space $M_n(\Delta)$ of all $n \times n$ matrices over the ring Δ, forms a Lie algebra under the Lie bracket operation $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ for $\mathbf{A}, \mathbf{B} \in M_n(\Delta)$. The Lie algebra of $\widehat{SO}(n)$ is isomorphic to the Lie algebra of vectors with the Lie bracket taken to be the vector product.

Definition 5.2 *The symplectic group is defined by*

$$Sp(n) = \{A \in M_n(\mathbb{H}) \mid AxAy = xy \text{ for all } x, y \in \mathbb{H}^n\}.$$

$Sp(1)$ is the group of all quaternions of unit length. That is, $Sp(1) = \{q \in \mathbb{H} \mid N(q) = 1\}$.

If we define $S^{k-1} = \{x \in \mathbb{R}^k \mid \|x\| = 1\}$ to be the unit $(k-1)$-sphere, one can see that $Sp(1) = S^3$, that is, $Sp(1)$ is the unit 3-sphere in $\mathbb{R}^4$; here $\mathbb{R}^4$ is identified with $\mathbb{H}$. Define a map $\rho : S^3 \to SO(3)$ for $\mathbf{q} \in S^3$ and $\alpha \in \mathbb{H}$ by $\rho(\mathbf{q})(\alpha) = \mathbf{q}\alpha\mathbf{q}^*$. Here, we do a left translation by $\mathbf{q}$ and a right translation by $\mathbf{q}^*$. In fact $\rho(\mathbf{q}) \in O(4)$. Note that, $\rho(\mathbf{q}^*)$ is the inverse of $\rho(\mathbf{q})$ in the group $O(4)$. Moreover real quaternions commute with all other quaternions. That is, if $\mathrm{Sc}\alpha = \alpha$, then $\rho(\mathbf{q})\alpha = \mathbf{q}\alpha\mathbf{q}^* = \mathbf{q}^*\mathbf{q}\alpha = \alpha$. If α is in $\mathrm{Span}\{i, j, k\}$ then one can show that $\rho(\mathbf{q})\alpha$ is also in $\mathrm{Span}\{i, j, k\}$. These two observations imply that $\rho(\mathbf{q})$ maps the 3-space spanned by i, j, k to itself. Thus $\rho(\mathbf{q})$ can be considered as an element of $O(3)$. By $\det\rho(\mathbf{q}) = 1$, we identify $\rho(\mathbf{q})_{|\mathrm{Span}\{i,j,k\}}$ with an element in $SO(3)$.

Proposition 5.3 [1] *$\rho : S^3 \to SO(3)$ is a surjective homomorphism and $Ker\rho = \{1, -1\}$.*

In order to give a relation between an ondm and the Hamilton operators, we will need the following theorems.

Theorem 5.4 [1] *Matrices generated by operators $\overset{+}{H}$ and $\overset{-}{H}$ are orthogonal matrices, i.e.,*

i) $\overset{+}{H}(\hat{\mathbf{q}})^T \overset{+}{H}(\hat{\mathbf{q}}) = \overset{+}{H}(\hat{\mathbf{q}}) \overset{+}{H}(\hat{\mathbf{q}})^T = N(\hat{\mathbf{q}})I$,

ii) $\overset{-}{H}(\hat{\mathbf{q}})^T \overset{-}{H}(\hat{\mathbf{q}}) = \overset{-}{H}(\hat{\mathbf{q}}) \overset{-}{H}(\hat{\mathbf{q}})^T = N(\hat{\mathbf{q}})I$ *and*

iii) $\overset{+}{H}(\hat{\mathbf{q}})$ *and* $\overset{-}{H}(\hat{\mathbf{q}})$ *are orthonormal dual matrices if and only if $\hat{\mathbf{q}}$ is a unit dual quaternion, that is, $N(\hat{\mathbf{q}}) = 1$.*

Theorem 5.5 [1] *Let $\hat{\mathbf{A}}$ be a 3×3 orthonormal dual matrix with $\hat{\mathbf{a}}_{.k} = \mathbf{a}_{.k} + \varepsilon\mathbf{a}'_{.k}$ as its k-th column vector. Then there exists one and only one vector $\mathbf{a}$ such that*

$$\mathbf{a} \times \mathbf{a}_{.k} = \mathbf{a}'_{k}, \ k = 1, 2, 3.$$

Then any 3×3 orthonormal dual matrix can be represented as

$$\hat{\mathbf{A}} = mat(\mathbf{a}_{.1} + \varepsilon\mathbf{a} \times \mathbf{a}_{.1}, \mathbf{a}_{.2} + \varepsilon\mathbf{a} \times \mathbf{a}_{.2}, \mathbf{a}_{.3} + \varepsilon\mathbf{a} \times \mathbf{a}_{.3}),$$

where $\mathbf{A} = (\mathbf{a}_{ik})$ is a real orthogonal matrix.

Theorem 5.6 *[1] Let $\hat{\mathbf{A}}$ be a 4×4 orthonormal dual matrix with $\hat{\mathbf{a}}_{.k} = \mathbf{a}_{.k} + \varepsilon\mathbf{a}'_{.k}$ as its k-th column vector. Then there exist two vectors $\mathbf{u}$ and $\mathbf{v}$ (i.e. $\mathbf{u}_0 = \mathbf{v}_0 = 0$) such that $\mathbf{a}'_{.k} = \mathbf{u}\mathbf{a}_{.k} + \mathbf{a}_{.k}\mathbf{v}$, $k = 0, 1, 2, 3$ and the vectors $\mathbf{u}$ and $\mathbf{v}$ are unique.*

Let $\hat{\mathbf{A}}$ be written as $\hat{\mathbf{A}} = \mathbf{A} + \varepsilon\mathbf{A}'$. By using the orthonormality condition, we get $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = I$, $\mathbf{A}'\mathbf{A}^T + \mathbf{A}\mathbf{A}'^T = 0$. Define $\mathbf{A}'\mathbf{A}^T = \mathbf{B}$. It is clear that $\mathbf{B}$ is a skew-symmetric matrix. The matrix $\mathbf{B}$ can be written uniquely as

$$\mathbf{B} = \overset{+}{\mathbf{H}}(\mathbf{u}) + \overset{-}{\mathbf{H}}(\mathbf{v}),$$

where $\mathbf{u}$ and $\mathbf{v}$ are two vectors, i.e., $u_0 = v_0 = 0$. If we consider a dual quaternion vector $\hat{\mathbf{u}}$, we can define Hamilton matrices $\overset{+}{H}(\hat{\mathbf{u}})$ and $\overset{-}{H}(\hat{\mathbf{u}})$ which are skew-symmetric matrices. Here these two matrices $\overset{+}{H}(\hat{\mathbf{u}})$, $\overset{-}{H}(\hat{\mathbf{u}})$ are generated by using the vector component of a quaternion only and they are written as

$$\overset{+}{H}(V\hat{\mathbf{u}}) = \begin{pmatrix} 0 & -\hat{u}_1 & -\hat{u}_2 & -\hat{u}_3 \\ \hat{u}_1 & 0 & -\hat{u}_3 & \hat{u}_2 \\ \hat{u}_2 & \hat{u}_3 & 0 & -\hat{u}_1 \\ \hat{u}_3 & -\hat{u}_2 & \hat{u}_1 & 0 \end{pmatrix} \tag{4}$$

and

$$\overset{-}{H}(V\hat{\mathbf{u}}) = \begin{pmatrix} 0 & -\hat{u}_1 & -\hat{u}_2 & -\hat{u}_3 \\ \hat{u}_1 & 0 & \hat{u}_3 & -\hat{u}_2 \\ \hat{u}_2 & -\hat{u}_3 & 0 & \hat{u}_1 \\ \hat{u}_3 & \hat{u}_2 & -\hat{u}_1 & 0 \end{pmatrix} \tag{5}$$

respectively.

6. The Transformation Matrix

Let us consider a dual-quaternion vector $\hat{\mathbf{x}}$, that is, $D\hat{\mathbf{x}} = 0$. We assume that we are given two triples of points on the dual unit sphere by means of two orthonormal trihedra $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. So we may write any point on the dual sphere as a linear combination of the orthonormal trihedra

$$\hat{X}_1\hat{\mathbf{i}}_1 + \hat{X}_2\hat{\mathbf{i}}_2 + \hat{X}_3\hat{\mathbf{i}}_3 = \hat{x}_1\hat{\mathbf{e}}_1 + \hat{x}_2\hat{\mathbf{e}}_2 + \hat{x}_3\hat{\mathbf{e}}_3. \tag{6}$$

The column vectors $\hat{\mathbf{X}}_i$ and $\hat{\mathbf{x}}_j$ are the position vectors of $\hat{\mathbf{x}}$ with respect to the row vectors $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, respectively, where $i = 1, 2, 3$ and $j = 1, 2, 3$. From

(6), we get

$$\hat{X}_i = (\hat{\mathbf{i}}_i\hat{\mathbf{e}}_1)\hat{x}_1 + (\hat{\mathbf{i}}_i\hat{\mathbf{e}}_2)\hat{x}_2 + (\hat{\mathbf{i}}_i\hat{\mathbf{e}}_3)\hat{x}_3. \tag{7}$$

By putting $(\hat{\mathbf{i}}_i\hat{\mathbf{e}}_k) = \hat{\mathbf{a}}_{ik}$, and introducing the dual matrix $\hat{\mathbf{A}} = (\hat{\mathbf{a}}_{ik})$, where $k = 1, 2, 3$, we see that (7) expresses that $\hat{\mathbf{X}} = \hat{\mathbf{A}}\hat{\mathbf{x}}$. Here $\hat{\mathbf{A}}$ is an orthogonal dual matrix, called the transformation matrix, from the trihedron $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ onto $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$. For further information see [2]. The aim is to combine this matrix with Hamilton operators and see the role of the Hamilton operators in dual motions. Before giving the relation between Hamilton operators and dual motion, let us consider dual spherical motions.

7. Dual Spherical Motions

By using the notations given in the previous section, we suppose that $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ is fixed and the vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ of $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ are functions of a real parameter t. Then $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ moves with respect to $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ for a dual unit sphere K_2 rigidly connected with $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ and a dual unit sphere K_1 rigidly connected with $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$. Here K_1 is called the fixed and K_2 the moving sphere. This motion is called a dual spherical motion and denoted by K_2/K_1. If $\hat{\mathbf{x}} \in K_2$ coincides with $\hat{\mathbf{X}} \in K_1$ at t, we have $\hat{\mathbf{X}} = \hat{\mathbf{A}}\hat{\mathbf{x}}$, where $\hat{\mathbf{A}} = (\hat{\mathbf{a}}_{ik}) = \mathbf{A} + \varepsilon\mathbf{A}'$ is an ondm.

8. The Relation Between $\hat{A}$ and the Hamilton Operators

We have already written a unique skew-symmetric matrix $\mathbf{B} = \overset{+}{\mathbf{H}}(\mathbf{u}) + \overset{-}{\mathbf{H}}(\mathbf{v})$ for two vectors $\mathbf{u}$ and $\mathbf{v}$ in Theorem 5.6. We would like to get a relation between $\hat{A}$, given in Theorem 5.6, and the Hamilton operators $\overset{+}{H}$ and $\overset{-}{H}$ for a dual quaternion. In other words, we want to reduce $\overset{+}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{q}}^*)$ from $\hat{A}, \mathbf{u}, \mathbf{v}$ mentioned in Theorem 5.6.

The following theorem is motivated by Blaschkes's book [3].

Theorem 8.1 *There is a surjective map $\hat{\rho} : \widehat{Sp}(1) \to \widehat{SO}(3)$ with $Ker\hat{\rho} = \{-1, 1\}$. More precisely, there is a 1-1 correspondence between $\hat{A}$ and $\overset{+}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{q}}^*)$ for $\hat{\mathbf{q}} \in \widehat{Sp}(1)$, up to sign.*

Proof. Let $\hat{\mathbf{q}} \in \widehat{Sp}(1)$. We wish to argue that $\hat{A}$ defined by $\hat{A}\hat{\mathbf{x}} = \hat{\mathbf{q}}\hat{\mathbf{x}}\hat{\mathbf{q}}^*$, $\hat{\mathbf{x}} \in \text{Span}\{i, j, k\}$, is in $\widehat{SO}(3)$. We can easily show that $\hat{A}$ is in $\widehat{SO}(3)$. The following calculation shows that $\hat{A}$ is in $\widehat{SO}(4)$. Indeed,

$$\hat{A}\hat{A}^T = \overset{+}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{q}}^*)[\overset{+}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{q}}^*)]^T = I.$$

If $\hat{\mathbf{x}}$ is real then $\hat{A}\hat{\mathbf{x}} = \hat{\mathbf{x}}$ and if $\hat{\mathbf{x}} \in \text{Span}\{i, j, k\}$ then $\hat{A}\hat{\mathbf{x}} \in \text{Span}\{i, j, k\}$. We identify $\mathbb{R}^3$ with $\text{Span}\{i, j, k\}$ and since $\det\hat{A} = 1$, we get $\hat{A} \in \widehat{SO}(3)$. Now, let $\hat{A} \in \widehat{SO}(3)$. We want to show that there exists $\hat{\mathbf{q}} \in \widehat{Sp}(1)$ such that $\hat{A}\hat{\mathbf{x}} = \hat{\mathbf{q}}\hat{\mathbf{x}}\hat{\mathbf{q}}^*$, where $\hat{\mathbf{x}} \in \text{Span}\{i, j, k\}$. We need to find $\hat{\mathbf{q}} \in \widehat{Sp}(1)$ such that $\hat{A} = \overset{+}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{q}}^*)$. Let $\hat{\mathbf{q}} = \mathbf{q} + \varepsilon\mathbf{q}'$, $\mathbf{q}, \mathbf{q}' \in \mathbb{H}$.

$$\hat{A} = \overset{+}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{q}}^*) = \overset{+}{H}(\mathbf{q})\overset{-}{H}(\mathbf{q}^*) + \varepsilon[\overset{+}{H}(\mathbf{q})\overset{-}{H}(\mathbf{q}'^*) + \overset{+}{H}(\mathbf{q}')\overset{-}{H}(\mathbf{q}^*)].$$

Here the real and the dual parts of $\hat{A}$ are

$$\begin{aligned} A &= \overset{+}{H}(\mathbf{q})\overset{-}{H}(\mathbf{q}^*) \\ A' &= \overset{+}{H}(\mathbf{q})\overset{-}{H}(\mathbf{q}'^*) + \overset{+}{H}(\mathbf{q}')\overset{-}{H}(\mathbf{q}^*), \end{aligned} \tag{8}$$

respectively. The first equation in (8) can be solved by using proposition 5.3, that is, for every A, there exist two and only two normalized dual quaternions $\mathbf{q}$ and $-\mathbf{q}$ which are mapped onto. On the other hand, by Theorem 5.6, there is a 1-1 correspondence between $\hat{A}$ and triples $A, \mathbf{u}, \mathbf{v}$. If A' and A are orthonormal matrices, they are related to vectors $\mathbf{u}, \mathbf{v}$ by $A'A^T = \overset{+}{H}(\mathbf{u}) + \overset{-}{H}(\mathbf{v})$. The second equation in (8) can be solved by using $A'A^T = \overset{+}{H}(\mathbf{u}) + \overset{-}{H}(\mathbf{v})$, where $\mathbf{u}$ and $\mathbf{v}$ are vectors. So we get

$$\begin{aligned} A'A^T &= [\overset{+}{H}(\mathbf{q})\overset{-}{H}(\mathbf{q}'^*) + \overset{+}{H}(\mathbf{q}')\overset{-}{H}(\mathbf{q}^*)][\overset{+}{H}(\mathbf{q})\overset{-}{H}(\mathbf{q}^*)]^T \\ &= \overset{+}{H}(\mathbf{q}'\mathbf{q}^*) + \overset{-}{H}(\mathbf{q}\mathbf{q}'^*). \end{aligned}$$

Thus $\mathbf{u} = \mathbf{q}'\mathbf{q}^*$ and $\mathbf{v} = \mathbf{q}\mathbf{q}'^*$. We can solve for $\mathbf{q}'$, given $\mathbf{u}, \mathbf{v}, \mathbf{q}$ as

$$\mathbf{q}' = \mathbf{u}\mathbf{q} \text{ or } \mathbf{q}' = \mathbf{v}^*\mathbf{q}.$$

This concludes the proof. $\square$

9. The Transformation Matrix for Hamilton Operators

The transformation law of real vectors, which is written in terms of quaternion form $\mathbf{X} = \mathbf{q}\mathbf{x}\mathbf{q}^*$, can be replaced by the dual quaternions $\hat{\mathbf{X}} = \hat{\mathbf{q}}\hat{\mathbf{x}}\hat{\mathbf{q}}^*$, where $\hat{\mathbf{X}}$ and $\hat{\mathbf{x}}$ are dual vectors in dual unit moving and fixed spheres, respectively, and $\hat{\mathbf{q}}$ is a dual-unit-quaternion, as in see Theorem 8.1. The advantage of dual-quaternions is obvious when finite screw motion in terms of a screw-displacement pair is presented. We can write $\hat{\mathbf{X}}$ by using the Hamilton operators $\overset{+}{H}$ and $\overset{-}{H}$ as:

$$\hat{\mathbf{X}} = \overset{+}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{q}}^*)\hat{\mathbf{x}} = \overset{-}{H}(\hat{\mathbf{q}}^*)\overset{+}{H}(\hat{\mathbf{q}})\hat{\mathbf{x}}.$$

The time derivative of $\hat{\mathbf{X}}$, $\frac{d\hat{\mathbf{X}}}{dt}$, is computed by using $\hat{\mathbf{x}} = \hat{\mathbf{q}}^{-1}\hat{\mathbf{X}}\hat{\mathbf{q}}$, which is equal to $\hat{\mathbf{x}} = \overset{+}{H}(\hat{\mathbf{q}})^T\overset{-}{H}(\hat{\mathbf{q}})\hat{\mathbf{X}} = \overset{-}{H}(\hat{\mathbf{q}})\overset{+}{H}(\hat{\mathbf{q}}^*)\hat{\mathbf{X}}$. Then

$$\begin{aligned}
\frac{d\hat{\mathbf{X}}}{dt} &= \overset{+}{H}\left(\frac{d\hat{\mathbf{q}}}{dt}\right)\overset{-}{H}(\hat{\mathbf{q}}^*)\hat{\mathbf{x}} + \overset{-}{H}\left(\frac{d\hat{\mathbf{q}}^*}{dt}\right)\overset{+}{H}(\hat{\mathbf{q}})\hat{\mathbf{x}} \\
&= \overset{+}{H}\left(\frac{d\hat{\mathbf{q}}}{dt}\right)\overset{-}{H}(\hat{\mathbf{q}}^*)\overset{-}{H}(\hat{\mathbf{q}})\overset{+}{H}(\hat{\mathbf{q}}^*)\hat{\mathbf{X}} + \overset{-}{H}\left(\frac{d\hat{\mathbf{q}}^*}{dt}\right)\overset{+}{H}(\hat{\mathbf{q}})\overset{-}{H}(\hat{\mathbf{q}})\overset{+}{H}(\hat{\mathbf{q}}^*)\hat{\mathbf{X}} \\
&= \overset{+}{H}\left(\frac{d\hat{\mathbf{q}}}{dt}\hat{\mathbf{q}}^*\right)\hat{\mathbf{X}} + \overset{-}{H}\left(\hat{\mathbf{q}}\frac{d\hat{\mathbf{q}}^*}{dt}\right)\hat{\mathbf{X}},
\end{aligned}$$

and $\hat{\mathbf{q}}\hat{\mathbf{q}}^* = 1$ implies $\hat{\mathbf{q}}\frac{d\hat{\mathbf{q}}^*}{dt} = -\frac{d\hat{\mathbf{q}}}{dt}\hat{\mathbf{q}}^*$. Therefore one concludes

$$\frac{d\hat{\mathbf{X}}}{dt} = \overset{+}{H}\left(\frac{d\hat{\mathbf{q}}}{dt}\hat{\mathbf{q}}^*\right)\hat{\mathbf{X}} - \overset{-}{H}\left(\frac{d\hat{\mathbf{q}}}{dt}\hat{\mathbf{q}}^*\right)\hat{\mathbf{X}}.$$

This can also be written in terms of dual-quaternions as

$$\begin{aligned}
\frac{d\hat{\mathbf{X}}}{dt} &= \left(\frac{d\hat{\mathbf{q}}}{dt}\hat{\mathbf{q}}^*\right)\hat{\mathbf{X}} - \hat{\mathbf{X}}\left(\frac{d\hat{\mathbf{q}}}{dt}\hat{\mathbf{q}}^*\right) \\
&= 2\,\dot{\hat{\mathbf{q}}}\,\hat{\mathbf{q}}^* \times \hat{\mathbf{X}} \\
&= \hat{\mathbf{w}} \times \hat{\mathbf{X}},
\end{aligned}$$

where $\hat{\mathbf{w}}$ is the dual vector of velocity screw of a line (or the dual angular velocity of instantaneous screw axis of the spherical motion) and it is given by $\hat{\mathbf{w}} = 2\,\dot{\hat{\mathbf{q}}}\,\hat{\mathbf{q}}^*$ or equivalently $\hat{\mathbf{w}} = 2\overset{+}{H}(\dot{\hat{\mathbf{q}}})\hat{\mathbf{q}}^*$.

10. Relative Motion

Let K_1, K_2 and K_3 be dual unit spheres moving with respect to each other, rigidly connected with the bases $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$, $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, $\{\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3\}$, respectively. Let $\hat{\mathbf{x}} \in K_3$ coincide at t with $\hat{\mathbf{X}} \in K_1$ and $\hat{\xi} \in K_2$, respectively; so we write

$$\hat{\mathbf{X}} = \hat{\mathbf{X}}^T \hat{\mathbf{i}} = \hat{\xi}^T \hat{\mathbf{e}} = \hat{\mathbf{x}}^T \hat{\mathbf{r}},$$

where

$$\hat{\mathbf{X}} = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \end{pmatrix}, \hat{\xi} = \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \end{pmatrix}, \hat{\mathbf{x}} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix},$$

$$\hat{\mathbf{i}} = \begin{pmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{pmatrix}, \hat{\mathbf{e}} = \begin{pmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{pmatrix} \text{ and } \hat{\mathbf{r}} = \begin{pmatrix} \hat{\mathbf{r}}_1 \\ \hat{\mathbf{r}}_2 \\ \hat{\mathbf{r}}_3 \end{pmatrix}.$$

By observing $\hat{\mathbf{e}} = \hat{\mathbf{A}}^T \hat{\mathbf{i}}$, $\hat{\mathbf{r}} = \hat{\mathbf{B}}^T \hat{\mathbf{i}}$, $\hat{\mathbf{r}} = \hat{\mathbf{C}}^T \hat{\mathbf{e}}$, that is, $\hat{\mathbf{A}}^T \hat{\mathbf{i}} := \{\hat{\mathbf{A}}^T \hat{\mathbf{i}}_1, \hat{\mathbf{A}}^T \hat{\mathbf{i}}_2, \hat{\mathbf{A}}^T \hat{\mathbf{i}}_3\}$, $\hat{\mathbf{B}}^T \hat{\mathbf{i}} := \{\hat{\mathbf{B}}^T \hat{\mathbf{i}}_1, \hat{\mathbf{B}}^T \hat{\mathbf{i}}_2, \hat{\mathbf{B}}^T \hat{\mathbf{i}}_3\}$, $\hat{\mathbf{C}}^T \hat{\mathbf{e}} := \{\hat{\mathbf{C}}^T \hat{\mathbf{e}}_1, \hat{\mathbf{C}}^T \hat{\mathbf{e}}_2, \hat{\mathbf{C}}^T \hat{\mathbf{e}}_3\}$, and introducing the dual matrices $\hat{\mathbf{A}} = (\hat{a}_{ik})$, $\hat{\mathbf{B}} = (\hat{b}_{ik})$ and $\hat{\mathbf{C}} = (\hat{c}_{ik})$, we may express that $\hat{\mathbf{X}} = \hat{\mathbf{A}}\hat{\xi}$, $\hat{\mathbf{X}} = \hat{\mathbf{B}}\hat{\mathbf{x}}$ and $\hat{\xi} = \hat{\mathbf{C}}\hat{\mathbf{x}}$. The necessary transformation matrices are already given in [2] with aid of 3×3 dual matrices. Now we want to give the relative motion in terms of Hamilton operators. We have already written the transition matrix with Hamilton operators in the previous section. We may replace the 3×3 matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ with the corresponding Hamilton operators for dual quaternions by Theorem 8.1. Let us write the transition matrices for $\hat{\mathbf{X}}$, $\hat{\mathbf{x}}$ and $\hat{\xi}$ as

$$\hat{\mathbf{X}} = \hat{\mathbf{q}}_{21}\hat{\xi} = \hat{\mathbf{q}}_{21}\hat{\mathbf{q}}_{32}\hat{\mathbf{x}} = \hat{\mathbf{q}}_{31}\hat{\mathbf{x}}, \tag{9}$$

where $\hat{\mathbf{q}}_{21}, \hat{\mathbf{q}}_{31}$ and $\hat{\mathbf{q}}_{32}$ are the corresponding transition matrices in the dual spherical motions K_2/K_1, K_3/K_1 and K_3/K_2, respectively. In other words,
$\hat{\mathbf{q}}_{21} = \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})\ \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})$, $\hat{\mathbf{q}}_{31} = \overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\ \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}})$ and $\hat{\mathbf{q}}_{32} = \overset{+}{H}(\hat{\mathbf{q}}_{\hat{C}})\ \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}})$. Then we substitute the value of $\hat{\xi}$ in $\hat{\mathbf{X}}$, we get

$$\hat{\mathbf{X}} \;=\; \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})\ \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})[\overset{+}{H}(\hat{\mathbf{q}}_{\hat{C}})\ \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}})\hat{\mathbf{x}}] = \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})\ \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\hat{\mathbf{q}}^*_{\hat{A}})\hat{\mathbf{x}}.$$

Thus from (9), we get

$$\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}}) = \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})\,\overset{-}{H}((\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})^*). \tag{10}$$

We can write the sets of bases $\hat{\mathbf{e}}$, $\hat{\mathbf{i}}$ and $\hat{\mathbf{r}}$ in terms of Hamilton operators

$$\hat{\mathbf{r}} = \hat{\mathbf{q}}_{32}^T\hat{\mathbf{e}} = \hat{\mathbf{q}}_{32}^T\hat{\mathbf{q}}_{21}^T\hat{\mathbf{i}}. \tag{11}$$

The derivative of $\hat{\mathbf{e}}$ given in (11) becomes

$$\frac{d\hat{\mathbf{e}}}{dt} = [\overset{-}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt})^T\,\overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})^T + \overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt})^T\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})^T]\hat{\mathbf{i}}. \tag{12}$$

By substituting $\hat{\mathbf{i}} = \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})\hat{\mathbf{e}}$ in (12), we get

$$\frac{d\hat{\mathbf{e}}}{dt} = [\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}) + \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{A}})]\hat{\mathbf{e}},$$

which follows from $\overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})^T = \overset{+}{H}(\hat{\mathbf{q}}^*_{\hat{A}})$, $\overset{+}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\hat{\mathbf{q}}_{\hat{A}}) = I$, $\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\hat{\mathbf{q}}_{\hat{A}}) = I$ $\overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt})^T = \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt})$ and $\overset{-}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt})^T = \overset{-}{H}(\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt})$.

Let $\hat{\mathbf{\Omega}}_{\hat{A}} = \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}) + \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{A}})$. Thus $\frac{d\hat{\mathbf{e}}}{dt} = \hat{\mathbf{\Omega}}_{\hat{A}}\hat{\mathbf{e}}$, here $\hat{\mathbf{\Omega}}_{\hat{A}}$, is a 4×4 dual skew-symmetric matrix defined in terms of Hamilton operators $\overset{+}{H}(V)$ and $\overset{-}{H}(V)$ given in (4) and (5). From now on, we will use the notation $\hat{\mathbf{\Omega}}_{\hat{-}}$ for the Hamilton operators $\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{-}}\frac{d\hat{\mathbf{q}}_{\hat{-}}}{dt}) + \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{-}}}{dt}\hat{\mathbf{q}}_{\hat{-}})$. Let us compute the rate of change of $\hat{\mathbf{r}}$ with respect to K_1:

$$\frac{d\hat{\mathbf{r}}}{dt} = [\overset{-}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{B}}}{dt})^T\,\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})^T + \overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt})^T\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}})^T]\hat{\mathbf{i}}. \tag{13}$$

By substituting $\hat{\mathbf{i}} = \overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}})\hat{\mathbf{r}}$ in (13), we get

$$\frac{d\hat{\mathbf{r}}}{dt} = [\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}}\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt}) + \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{B}}}{dt}\hat{\mathbf{q}}_{\hat{B}})]\hat{\mathbf{r}}.$$

Similarly, the rate of change of $\hat{\mathbf{r}}$ with respect to K_2 is

$$\frac{\delta\hat{\mathbf{r}}}{dt} = [\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt}) + \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{C}}}{dt}\hat{\mathbf{q}}_{\hat{C}})]\hat{\mathbf{r}}.$$

The rate of change of $\hat{\mathbf{x}}$ with respect to K_1 is

$$\frac{d\hat{\mathbf{x}}}{dt} = (\frac{d\hat{\mathbf{x}}^T}{dt} + \hat{\mathbf{x}}^T[\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}}\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt}) + \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{B}}}{dt}\hat{\mathbf{q}}_{\hat{B}})])\hat{\mathbf{r}}, \tag{14}$$

and the rate of change of $\hat{\mathbf{x}}$ with respect to K_2 is given by

$$\frac{\delta\hat{\mathbf{x}}}{dt} = (\frac{d\hat{\mathbf{x}}^T}{dt} + \hat{\mathbf{x}}^T[\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt}) + \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{C}}}{dt}\hat{\mathbf{q}}_{\hat{C}})])\hat{r}.$$

If $\hat{\mathbf{x}}$ is fixed on K_2 then $\frac{\delta\hat{\mathbf{x}}}{dt} = 0$ so $\frac{d\hat{\mathbf{x}}^T}{dt} = \hat{\mathbf{x}}^T[\overset{+}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt}) - \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})]$. Let us substitute this in (14), we obtain the velocity of $\hat{\mathbf{x}}$ fixed on K_2 as

$$\frac{d_f\hat{\mathbf{x}}}{dt} = \hat{\mathbf{x}}^T([\overset{+}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt}) - \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})] - [\overset{+}{H}(\hat{\mathbf{q}}^*_{\hat{B}}\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt}) - \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}}\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt})])\hat{\mathbf{r}}. \tag{15}$$

We may express the difference in (15) only in terms of Hamilton operators for the matrices $\hat{A}$ and $\hat{C}$. By deriving (10), we get

$$\begin{aligned}
d(\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}})) &= \overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}}) + \overset{-}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{B}}}{dt})\,\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}}) \\
&= \overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt})\,\overset{+}{H}(\hat{\mathbf{q}}_{\hat{C}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}) \\
&+ \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})\,\overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}) \\
&+ \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})\,\overset{+}{H}(\hat{\mathbf{q}}_{\hat{C}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})\,\overset{-}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{C}}}{dt}) \\
&+ \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})\,\overset{+}{H}(\hat{\mathbf{q}}_{\hat{C}})\,\overset{-}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}).
\end{aligned} \tag{16}$$

We know that $\hat{\mathbf{\Omega}}_{\hat{B}} = \overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{B}}}{dt}\hat{\mathbf{q}}_{\hat{B}}) + \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}}\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt})$. So if we multiply $d(\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}}))$ given in (16) by $\overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{B}}}{dt})\,\overset{-}{H}(\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt})$ on the left, we get $\hat{\mathbf{\Omega}}_{\hat{B}}$. That is,

$$\overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{B}}}{dt})\,\overset{-}{H}(\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt})d(\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}})) = \hat{\mathbf{\Omega}}_{\hat{B}}.$$

Thus we get

$$d(\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}})) = \overset{-}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{B}}}{dt})\,\overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{B}}}{dt})\hat{\mathbf{\Omega}}_{\hat{B}}. \tag{17}$$

By doing some calculations and using the fact that $\hat{B} = \hat{A}\hat{C}$, which implies $\hat{\mathbf{q}}_{\hat{B}} = \hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}}$, one can find:

$$\begin{aligned}
\hat{\Omega}_{\hat{B}} - \hat{\Omega}_{\hat{C}} &= 2\overset{-}{H}((\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})^*\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})[1+\overset{+}{H}((\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})^*\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})] \\
&+ 2\overset{+}{H}((\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})^*\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})[1+\overset{-}{H}((\hat{\mathbf{q}}_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})^*\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})] \\
&+ 2\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})[1+\overset{+}{H}((\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})^*\hat{\mathbf{q}}_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})] \\
&+ 2\overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt}^{*}\hat{\mathbf{q}}_{\hat{C}})[1+\overset{-}{H}((\hat{\mathbf{q}}_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})^*\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})] \\
&- \overset{+}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})+\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt}),
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{d_f\hat{\mathbf{x}}}{dt} &= \hat{\mathbf{x}}^T[2\overset{-}{H}((\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})^*\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})[1+\overset{+}{H}((\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})^*\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})] \\
&+ 2\overset{+}{H}((\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})^*\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{q}}_{\hat{C}})[1+\overset{-}{H}((\hat{\mathbf{q}}_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})^*\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})] \\
&+ 2\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{C}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})[\frac{3}{2}+\overset{+}{H}((\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})^*\hat{\mathbf{q}}_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})] \\
&+ 2\overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt}^{*}\hat{\mathbf{q}}_{\hat{C}})[\frac{1}{2}+\overset{-}{H}((\hat{\mathbf{q}}_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{C}}}{dt})^*\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{C}})]]\hat{\mathbf{r}}.
\end{aligned}$$

Now, we define a matrix $\hat{\mathbf{W}}$ as $\hat{\mathbf{W}} = e^{\overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{A}})+\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt})}$ which is an orthogonal dual matrix, since $\overset{+}{H}(\hat{\mathbf{q}})^T = \overset{+}{H}(\hat{\mathbf{q}}^*)$ and $\overset{-}{H}(\hat{\mathbf{q}})^T = \overset{-}{H}(\hat{\mathbf{q}}^*)$. We will denote this by $\hat{\mathbf{W}} = e^{\hat{\mathbf{\Omega}}_{\hat{A}}}$. The matrix $\hat{W}$ defined before turns out to be the matrix $\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}})$ follows from (9) , that is, $\overset{+}{H}(\hat{\mathbf{q}}_{\hat{B}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{B}}) = e^{\hat{\mathbf{\Omega}}_{\hat{A}}}$ for a dual quaternion; then we may write the motion K_3/K_1 as $\hat{\mathbf{X}} = e^{\hat{\mathbf{\Omega}}_{\hat{A}}}\hat{\mathbf{x}}$. Note that $\overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{A}}) - \overset{+}{H}(\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}\hat{\mathbf{q}}^*_{\hat{A}}) = 0$ and $\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}) - \overset{-}{H}(\hat{\mathbf{q}}_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}^{*}) = 0$ for a dual vector quaternion. So one can get the formula for the displacement in terms of dual quaternions as

$$\hat{\mathbf{X}} = \sum_{k=0}^{\infty}\frac{[\overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{A}})+\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt})]^k}{k!}\hat{\mathbf{x}}.$$

11. Representation of Spatial Displacements in $\mathbb{H}$

One can see [4],[5] and [10] to read more about operations on screws and further information.

Lemma 11.1 *The velocity for a rigid body motion can be given by using Lie bracket as*

$$\frac{d\hat{\mathbf{X}}}{dt} = [\hat{\mathbf{w}}, \hat{\mathbf{X}}].$$

Proof. We want to prove

$$\frac{d}{dt}\exp(\overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{A}}) + \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}))\hat{\mathbf{x}} = \left[\hat{\mathbf{w}}, \exp(\overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{A}}) + \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt}))\hat{\mathbf{x}}\right] \quad (18)$$

for a fixed $\hat{\mathbf{X}}$, here $\hat{\mathbf{w}} = 2\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}}$. This is the velocity for a rigid body motion. We have

$$\begin{aligned} \hat{\mathbf{X}} &= \overset{+}{H}(\hat{\mathbf{q}}_{\hat{A}})\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{X}}} &= \overset{+}{H}(\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}})\hat{\mathbf{X}} - \overset{-}{H}(\dot{\hat{\mathbf{q}}}^*_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}})\hat{\mathbf{X}}. \end{aligned}$$

We can write $\hat{\mathbf{X}}$ in terms of dual quaternions as $\dot{\hat{\mathbf{X}}} = (\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}})\hat{\mathbf{X}} - \hat{\mathbf{X}}(\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}})$, which can also be written as

$$\dot{\hat{\mathbf{X}}} = 2(\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}}) \times \hat{\mathbf{X}},$$

since $\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}}$ is a dual vector. We have seen that $\dot{\hat{\mathbf{X}}} = \hat{\mathbf{w}} \times \hat{\mathbf{X}}$. Since $\hat{\mathbf{x}}$ is an arbitrary vector, we get

$$\hat{\mathbf{w}} = 2\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}} \quad (19)$$

or

$$\hat{\mathbf{w}} = 2\,\overset{+}{H}(\dot{\hat{\mathbf{q}}}_{\hat{A}})\hat{\mathbf{q}}^*_{\hat{A}} = 2\,\overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}})\dot{\hat{\mathbf{q}}}_{\hat{A}}$$

and

$$\dot{\hat{\mathbf{q}}}_{\hat{A}} = \frac{1}{2}\hat{\mathbf{w}}\hat{\mathbf{q}} \text{ and } \dot{\hat{\mathbf{q}}}_{\hat{A}} = \frac{1}{2}\overset{+}{H}(\hat{\mathbf{w}})\hat{\mathbf{q}} = \frac{1}{2}\overset{-}{H}(\hat{\mathbf{q}})\hat{\mathbf{w}}.$$

In order to prove (18), we consider the derivation $\frac{d}{dt}(\hat{\mathbf{q}}\hat{\mathbf{x}}\hat{\mathbf{q}^*})$ then by using (19) we get (18) as

$$\begin{aligned}
\frac{d\hat{\mathbf{X}}}{dt} &= \frac{d}{dt}(\hat{\mathbf{q}}_{\hat{A}}\hat{\mathbf{x}}\hat{\mathbf{q}}^*_{\hat{A}}) = \overset{+}{H}(\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}})\hat{\mathbf{X}} - \overset{-}{H}(\dot{\hat{\mathbf{q}}}^*_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}})\hat{\mathbf{X}} \\
&= 2\overset{+}{H}(\dot{\hat{\mathbf{q}}}_{\hat{A}})\hat{\mathbf{q}}^*_{\hat{A}} \times \hat{\mathbf{X}} = \dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}}\hat{\mathbf{X}} - \hat{\mathbf{X}}\dot{\hat{\mathbf{q}}}_{\hat{A}}\hat{\mathbf{q}}^*_{\hat{A}} \\
&= \frac{1}{2}(\hat{\mathbf{w}}\hat{\mathbf{X}} - \hat{\mathbf{X}}\hat{\mathbf{w}}) = [\hat{\mathbf{w}}, \hat{\mathbf{X}}] = [\hat{\mathbf{w}}, e^{\overset{+}{H}(\frac{d\hat{\mathbf{q}}^*_{\hat{A}}}{dt}\hat{\mathbf{q}}_{\hat{A}}) + \overset{-}{H}(\hat{\mathbf{q}}^*_{\hat{A}}\frac{d\hat{\mathbf{q}}_{\hat{A}}}{dt})}\hat{\mathbf{x}}].
\end{aligned}$$

$\square$

12. Conclusion

Hamilton operators for dual quaternions to screw motions have been given in a different aspect. It leads us to give the relation between Hamilton matrices and the transition matrices. The relative motion has been expressed in terms of Hamilton operators and the relation among the Hamilton matrices for the transition matrices has been given.

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Bedia AKYAR
Dokuz Eylül University
Faculty of Arts and Sciences
Department of Mathematics
Kaynaklar Yerleşkesi
35020 İzmir-TURKEY
e-mail: bedia.akyar@deu.edu.tr

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