Turk J Math
32 (2008) , 393 – 408.

Certain Rings Whose Simple Singular Modules Are *nil*-injective*

Junchao Wei

Abstract

In this paper, we first study some characterizations of left min-abel ring, strongly left min-abel ring and left $MC2$ ring. Next, we discuss and generalize some well known results for a ring whose simple singular left modules are *nil*- injective. Finally, as a byproduct of these results we are able to show that if R is a left $GMC2$ left Goldie ring whose every simple singular left $R-$ module is $YJ-$ injective, then R is a finite product of simple left Goldie ring.

Key Words: Left minimal elements, Left min-abel rings, Strongly left min-abel rings, Left $MC2$ rings, Simple singular modules, Left $nil-$ injective modules.

Introduction

Throughout this paper R denotes an associative ring with identity, and $R-$modules are unital. For $a \in R$, $r(a)$ and $l(a)$ denote the right annihilator of a and the left annihilator of a, respectively. We write $J(R)$, $Z_l(R)$, $N(R)$, $N_1(R)$ and $S_l(R)$ for the Jacobson radical, the left singular ideal, the set of nilpotent elements, the set of non-nilpotent elements and the left socle of R, respectively. An element $k \in R$ is called left minimal if Rk is a minimal left ideal of R. An element $e \in R$ is called left minimal

2000 *AMS Mathematics Subject Classification:* 16E50, 16D30
*Project supported by the Foundation of Natural Science of China (10771182 and 10771183), and the Natural Science Foundation of Jiangsu Province

idempotent if e is a left minimal element and $e^2 = e$. An idempotent $e \in R$ is called left (resp, right) semicentral if $ae = eae$ (resp, $ea = eae$) for all $a \in R$.

A ring R is called left min-abel if every left minimal idempotent element of R is left semicentral and R is said to be NI [9, 20] if $N(R)$ is an ideal of R. A ring R is called 2-prime if $N(R)$ coincides with its prime radical. Clearly, a 2-prime ring is NI.

A ring R is called strongly left min-abel if for every left minimal idempotent element $e \in R$, $Re = eR$.

Since an abelian ring (that is, every idempotent of a ring R is central) is strongly left min-abel, a ZI ring (cf. [13, 14]) (that is, $ab = 0$ implies $aRb = 0$ for all $a, b \in R$) and so a ZC ring (cf. [13]) (that is, $ab = 0$ implies $ba = 0$ for all $a, b \in R$) is strongly left min-abel because a ZC ring is ZI and a ZI ring is Abelian.

Recall that a ring R is left $MC2$ [17] if for left minimal element $k \in R$, Rk is a summand in ${}_RR$, whenever Rk is projective as left $R-$ module.

Recall that a ring R is left PS [16] if for every left minimal element $k \in R$, $l(k)$ is a summand of ${}_RR$, in other word, Rk is a projective in ${}_RR$. [16] proved that R is a left PS ring if and only if $S_l(R) \cap Z_l(R) = 0$.

A ring R is said to be left universally mininjective [17] if for every left minimal element $k \in R$, Rk is a summand of ${}_RR$. For convenience, these rings are also called left DS by author in [4]. In [17] it is proved that R is left universally mininjective if and only if $S_l(R) \cap J(R) = 0$. And, in [4], a lot of characterization of left universally mininjective rings are given. For example, R is left universally mininjective if and only if R is left PS and left $MC2$.

Call a ring R strongly left DS if for every left minimal element $k \in R$, $k^2 \neq 0$. Obviously, reduced rings (that is, $a^2 = 0$ implies $a = 0$ for all $a \in R$) are strongly left DS.

Call a ring R left $GMC2$ if for any $a \in R$, any left minimal idempotent $e \in R$, $aRe = 0$ implies $eRa = 0$. Clearly, a left $GMC2$ ring is left $MC2$.

Left $R-$ module M is called $p-$ injective [10, 11] if, for any $0 \neq a \in R$, and any left $R-$ homomorphism of Ra into M extends to one of R into M. And M is said to be $YJ-$ injective [6, 7, 8, 19] if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any left $R-$ homomorphism of Ra^n into M extends to one of R into M.

Call a left $R-$ module M $nil-$ (resp, $Gnp-$) injective if for each nilpotent element (resp, non-nilpotent element) $k \in R$, there exists a positive integer n such that $k^n \neq 0$ and any left $R-$ morphism $Rk^n \longrightarrow M$ extends to R.

Call a left $R-$ module M $Jcp-$ (resp, $np-$ [12]) injective if for each $k \notin Z_l(R)$ (resp, $k \in N_1(R)$), any left $R-$ morphism $Rk \longrightarrow M$ extends to R.

Examples of these modules include left $p-$ injective modules and $YJ-$ injective modules. R is called left $nil-$ (resp, $Jcp-$, $np-$ and $Gnp-$) injective ring if ${}_RR$ is $nil-$ (resp, $Jcp-$, $np-$ and $Gnp-$) injective.

A ring R is called left weakly continuous [18] if $J(R) = Z_l(R), R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. Every regular ring is left weakly continuous. Clearly, R is a regular ring if and only if R is a left weakly continuous left nonsingular ring.

In section 1, we introduce some rings characterized by minimal left ideals, give some characterizations of these rings, study the relations among these rings. Such as Theorem 1.2: R is a left quasi-duo ring if and only if R is a left min-abel $MELT$ ring; Theorem 1.8: R is a strongly left min-abel ring if and only if R is a left min-abel left $MC2$ ring; And Theorem 1.11: R is a strongly left DS ring if and only if R is a left PS strongly left min-abel ring.

In section 2, we investigate the rings whose simple singular modules are $nil-$ injective, generalize some known results appeared in [6, 7, 8, 11]. As a byproduct of these results the author shows that if R is a left $GMC2$ left Goldie ring whose every simple singular left $R-$ module is $YJ-$ injective, then R is a finite product of simple left Goldie rings.

1. Characterizations of left min-abel rings

Theorem 1.1 *The following conditions are equivalent for a ring R:*

(1) R is a left min-abel ring.

(2) For every left minimal element $k \in R$, $k^2 = 0$ always implies $kRk = 0$.

(3) For every left minimal element $e^2 = e \in R$, $ae = 0$ implies $aRe = 0$ for all $a \in R$.

(4) For every left minimal element $e^2 = e \in R$, we have $eke = ke$ for all left minimal elements $k \in R$.

(5) For every left minimal element $e^2 = e \in R$, we have $eke = ke$ for all left minimal elements $k \in R$ with $k^2 = 0$.

(6) For every left minimal element $e^2 = e \in R$, $fe = 0$ implies $fRe = 0$ for all $f^2 = f \in R$.

(7) For every left minimal element $e^2 = e \in R$, $Re \subseteq eR$.

(8) For every left minimal element $e^2 = e \in R$, $R(1 - e)$ is an ideal of R.

Proof. (1) $\Longrightarrow$ (2). Suppose that $kRk \neq 0$, then $Rk = Re$ where $e^2 = e \in R$. By (1), e is left semi-central, so $k = ke = eke$. Write $e = ck, c \in R$, then $k = eke = ckke = ck^2e = 0$ by hypothesis, which is a contradiction. Hence $kRk = 0$.

(2) $\Longrightarrow$ (1). Assume that $e^2 = e \in R$ is a left minimal element. Write $h = ae - eae$ for an $a \in R$. If $h \neq 0$, then h is a left minimal element with $he = h, eh = 0, h^2 = 0$. By (2), $hRh = 0$. But $Rh = Rhe \subseteq Re$, so $Rh = Re$ because Re is a minimal left ideal of R. Hence $Re = ReRe = RhRh = 0$, which is a contradiction. Thus $h = 0$, which implies that R is a left min-abel ring.

(1) $\Longrightarrow$ (3). Since R is a left min-abel ring, $Re = eRe$. Hence, by hypothesis $aRe = aeRe = 0$.

(3) $\Longrightarrow$ (1). Since $(1-e)e = 0$, $(1-e)Re = 0$ by (3). Hence $(1-e)ae = 0$ for all $a \in R$, so $ae = eae$ for all $a \in R$, which implies R is a left min-abel ring.

(1) $\Longrightarrow$ (4) $\Longrightarrow$ (5) are evidently.

(5) $\Longrightarrow$ (1). Assume that $e^2 = e \in R$ is a left minimal element, write $h = ae - eae$ for an $a \in R$. If $h \neq 0$, then h is a left minimal element with $he = h, eh = 0, h^2 = 0$. By (5), $ehe = he$. Hence $h = he = ehe = 0$, which is a contradiction. Thus $h = 0$, which implies that R is left min-abel.

Clearly, an idempotent $e \in R$ is left semicentral if and only if $Re \subseteq eR$. Hence (1) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (8). □

According to [14], a ring R is left quasi-duo if every maximal left ideal of R is an ideal, and R is $MELT$ if every essential maximal left ideal of R is an ideal. In terms of left min-abel rings, we have the following theorem.

Theorem 1.2 *R is a left quasi-duo ring if and only if R is a left min-abel $MELT$ ring.*

Proof. Assume that R is a left min-abel MELT ring and M is any maximal left ideal of R. If M is essential, then, certainly, M is an ideal of R. Otherwise, $M = Re, e^2 = e \in R$, then $1 - e$ is a left minimal idempotent, so $M = Re$ is an ideal of R by Theorem 1.1. This implies that R is left quasi-duo.

Conversely, if R is a left quasi-duo ring, then R is $MELT$ ring. Now let $e^2 = e \in R$ be any left minimal element, then $R(1-e) = l(e)$ is a maximal left ideal of R, so $R(1-e)$ is an ideal. By Theorem 1.1, R is left min-abel. □

Corollary 1.3 *The following conditions are equivalent for a MELT ring R:*

(1) R is a left quasi-duo ring.

(2) For each left minimal element h, $Rh + R(hc - 1) = R$ for all $c \in R$.

(3) For each left minimal nilpotent element h, $Rh + R(hc - 1) = R$ for all $c \in R$.

Proof. (1) $\Longrightarrow$ (2). This is a direct result of [13, Theorem 3.2].

(2) $\Longrightarrow$ (3) is evident.

(3) $\Longrightarrow$ (1). Assume that $e^2 = e \in R$ is a left minimal element. If there exists an $a \in R$ such that $h = ae - eae \neq 0$, then $Rh = Re, he = h, eh = 0, h^2 = 0$. Let $e = ch, c \in R$. By hypothesis, $Rh + R(hc - 1) = R$. Write $1 = dh + u(hc - 1)$, where $d, u \in R$. Clearly, $h = dh^2 + u(hc - 1)h = u(hch - h) = u(he - h) = u(h - h) = 0$, which is a contradiction. This implies that e is left semicentral in R, so R is a left min-abel ring. By Theorem 1.2, R is a left quasi-duo ring. □

Theorem 1.4 *(1) If $N(R) \subseteq J(R)$, then R is a left min-abel ring.*

(2) Let A be an ideal of R such that R/A is a left min-abel ring. If A contains no left mininal idempotent of R, then R is a left min-abel ring.

Proof. (1) Assume that $e^2 = e \in R$ is a left minimal element. Write $h = ae - eae$ for an $a \in R$. If $h \neq 0$, then h is a left minimal element with $he = h, eh = 0, h^2 = 0$ and so $h \in N(R) \subseteq J(R)$. Hence $e \in J(R)$ because $Rh = Re$, which is a contradiction. Thus $h = 0$, which implies that R is a left min-abel ring.

(2) We denote $a + A \in R/A = \bar{R}$ by $\bar{a}$ where $a \in R$. Assume that $e^2 = e \in R$ is a left minimal element, then $e \notin A$. We claim that $\bar{e}$ is a left minimal idempotent of R/A. In fact, if $a \in R$ such that $ae \notin A$, then $Rae = Re$ because $ae \neq 0$ and Re is a minimal left ideal of R. write $e = bae, b \in R$, then $\bar{R}\bar{e} = \bar{R}\bar{b}\bar{a}\bar{e} = \bar{R}\bar{a}\bar{e}$, so $\bar{e}$ is a left minimal element of $\bar{R}$. Let $h = be - ebe$ for $b \in R$. If $h \neq 0$, then h is a left minimal element of R with $he = h, eh = 0, h^2 = 0, Re = Rh$ and so $h \notin A$. Hence $\bar{0} \neq \bar{h} = \bar{b}\bar{e} - \bar{e}\bar{b}\bar{e} = \bar{0}$, which is a contradiction. Thus $h = 0$, which implies that R is a left min-abel ring. □

Remark: From (1) of Theorem 1.4, if N(R) is a one-sided ideal of R, then R is a left min-abel ring. In particular, NI-rings are left min-abel rings.

By (2) of Theorem 1.4, we have the following corollary.

Corollary 1.5 *(1) If $R/J(R)$ is a left min-abel ring, so is R.*

(2) If $R/Z_l(R)$ is a left min-abel ring, so is R.

(3) Let B be a nil ideal of R such that R/B be a left min-abel ring, so is R.

Theorem 1.6 *The following conditions are equivalent for a ring R:*

(1) R is a left $MC2$ ring.

(2) For any left minimal elements $k, g^2 = g \in R$, $Rk \cong Rg$ as left $R-$ module always implies $Rk = Re, e^2 = e \in R$.

(3) For any left minimal elements $k, g \in R$ with $k^2 = 0, g^2 = g$, $Rk \cong Rg$ as left $R-$ module always implies $Rk = Re, e^2 = e \in R$.

(4) For any left minimal elements $k, e^2 = e \in R$, $kRe = 0$ implies $eRk = 0$.

Proof. (1) $\Longrightarrow$ (2) Assume that R is a left $MC2$ ring and $Rk \cong Rg$ for left minimal elements $k, g^2 = g \in R$. Evidently, there exists an idempotent $h \in R$ such that $hk = k$ and $l(k) = l(h)$. If $(Rk)^2 = 0$, then $kR \subseteq l(k) = l(h)$. Hence $kRh = 0$, so $hRk = 0$ because h is a left minimal idempotent. Consequently, $hRh = 0$ because $hR \subseteq l(k) = l(h)$, which is a contradiction. Hence $(Rk)^2 \neq 0$, which implies $Rk = Re, e^2 = e \in R$.

(2) $\Longrightarrow$ (3) is evident.

(3) $\Longrightarrow$ (4) Let $k, e^2 = e \in R$ be left minimal elements with $kRe = 0$. If $eRk \neq 0$, then $eak \neq 0$ for some $a \in R$. Clearly, the map $Re \longrightarrow Reak$ by $re \longmapsto reak$ implies that it is an isomorphism. Since $(eak)^2 = eakeak = 0$, by hypothesis, $Reak = Rg, g^2 = g \in R$. Hence $Rg = RgRg = ReakReak = Rea(kRe)ak = 0$, which is a contradiction. Therefore $eRk = 0$.

(4) $\Longrightarrow$ (1) Let $k \in R$ be a left minimal element and ${}_RRk$ be projective. Then $l(k) = R(1-e)$ where $e^2 = e \in R$ is a left minimal element. Hence $k = ek$. By (4), $kRe \neq 0$, so $RkRe = Re$. Consequently, $Re = RkRe = RkRkRe$ and so $(Rk)^2 \neq 0$. Since Rk is a minimal left ideal of R, $Rk = Rg, g^2 = g \in R$. This shows that Rk is a direct summand of ${}_RR$. So R is a left $MC2$ ring. □

Theorem 1.7 *The following conditions are equivalent for a ring R:*

(1) R is a left $MC2$ ring.

(2) Every left minimal idempotent element is right minimal.

(3) For each left minimal idempotent $e \in R$, $r(Re \cap l(a)) = (1-e)R + aR$ always holds for all $a \in R$.

Proof. (1) $\Longrightarrow$ (2) Assume that $e^2 = e \in R$ is a left minimal element. Let $a \in R$ with $ea \neq 0$, then, clearly, $Re \cong Rea$. Hence, by Theorem 1.6, $Rea = Rg, g^2 = g \in R$. Write $g = cea, c \in R$, then $ea = eag = eacea$. Let $h = eac$, then $h^2 = h$ and $eaR = hR$. So $l(e) = l(ea) = l(h)$, and so $eR = rl(e) = rl(h) = hR = eacR \subseteq eaR \subseteq eR$. Hence $eR = eaR$, which implies eR is a minimal right ideal and so e is a right minimal element.

(2) $\Longrightarrow$ (3). Assume that $a \in R$. If $ea = 0$, then $Re \cap l(a) = Re$, $aR \subseteq r(e) = (1-e)R$. Hence $r(Re \cap l(a)) = r(Re) = (1-e)R = (1-e)R + aR$. If $ea \neq 0$, then $r(e) + aR = R$ because e is a right minimal element and $aR \nsubseteq r(e)$. Since Re is a minimal left ideal and $ea \neq 0$, $Re \cap l(a) = 0$. Hence $r(Re \cap l(a)) = r(0) = R = r(e) + aR = (1-e)R + aR$.

(3) $\Longrightarrow$ (1). Assume $k, e^2 = e \in R$ are left minimal elements with $kRe = 0$. If $eRk \neq 0$, then $eak \neq 0$ for some $a \in R$. Hence $Re \cap l(ak) = 0$ because Re is a minimal left ideal, so, by hypothesis, $R = r(0) = r(Re \cap l(ak)) = (1-e)R + akR$. Write $1 = (1-e)u + akv, u, v \in R$, then $e = (1-e)ue + akve = (1-e)ue$. Consequently, $e = ee = e(1-e)ue = 0$, which is a contradiction. Hence $eRk = 0$, which implies that R is a left $MC2$ ring. □

Theorem 1.8 *The following conditions are equivalent for a ring R:*

(1) R is a strongly left min-abel ring.

(2) R is a left min-abel left $MC2$ ring.

(3) $Rkl = Rlk$ for any left minimal elements k, l of R.

Proof. (1) $\Longrightarrow$ (2) Clearly, R is a left min-abel ring R by Theorem 1.1. Now let $k, e^2 = e \in R$ be left minimal elements with $kRe = 0$, but $eRk \neq 0$. Then $ke = 0$ and $eak \neq 0$ for some $a \in R$. Since $eak \in eR = Re$, $eak = eake = 0$, which is a contradiction. So $eRk = 0$ and then R is a left $MC2$ ring.

(2) $\Longrightarrow$ (3) Assume that $e^2 = e \in R$ is any left minimal element. Since R is a left min-abel ring, e is left semicentral. We claim that e is right semicentral. Otherwise there exists a $b \in R$ such that $h = eb - ebe \neq 0$. Then $eh = h, he = 0, h^2 = hh = heh = 0$ and ${}_RRh \cong_R Re$. Since R is a left $MC2$ ring, $Rh = Rg, g^2 = g \in R$. Since R is a left min-abel ring, g is left semicentral. Hence $h = hg = ghg$. Write $g = ch, c \in R$. Then $h = ghg = chhg = ch^2g = 0$, which is a contradiction. Hence e is right semicentral and so e is central. Now let $k, l \in R$ be left minimal elements. If $kl = 0$, then $lk = 0$. Otherwise $Rk = Rlk$. Write $k = clk, c \in R$. Then $k = clk = clclk$, which implies $(Rl)^2 \neq 0$. Hence $Rl = Re, e^2 = e \in R$. Therefore $Rk = Rlk = Rek = Rke = Re$

because e is central. Hence $0 = Rkl = Rel = Rle = Re$, which is a contradiction. Hence $lk = 0$ and so $Rkl = 0 = Rlk$. If $kl \neq 0$, then, by a similarly proof of above, we have $lk \neq 0$. Hence $Rl = Rkl$ and $Rk = Rlk$. Therefore $Rk = Rlk = Rklk \subseteq RkRk$ and so $Rk = Rg, g^2 = g \in R$. Thus $Rl = Rkl = Rgl = Rlg = Rg$ because g is central. Hence $Rkl = Rg = Rk = Rlk$.

(3) $\Longrightarrow$ (1). Let e be a left minimal idempotent of R. Then e is right right semicentral. For if there exists an $a \in R$ such that $h = ea - eae \neq 0$, then $h = eh, he = 0$. By (3), $Rh = Reh = Rhe = 0$ which is a contradiction. Hence e is right semicentral. Furthermore, e is left semicentral. In fact, if there exists a $b \in R$ such that $t = be - ebe \neq 0$, then $te = t, et = 0$. Hence $Rt = Rte = Ret = 0$, which is a contradiction. Hence e is left semicentral. Therefore $Re = eR$ which implies that R is a strongly left min-abel ring. □

From the proof of (2) $\Longrightarrow$ (3) of Theorem 1.8, we can see that R is strongly left min-abel if and only if every left minimal idempotent is central in R. In fact, we can show that a left minimal idempotent of a ring R is right semicentral if and only if it is central. Hence we have the following theorem.

Theorem 1.9 *The following conditions are equivalent for a ring R:*

(1) R is a strongly left min-abel ring.

(2) For every left minimal element $e^2 = e \in R$, $ea = 0$ implies $eRa = 0$ for all $a \in R$.

(3) For every left minimal element $e^2 = e \in R$, $ef = 0$ implies $eRf = 0$ for all $f^2 = f \in R$.

(4) For every left minimal element $e^2 = e \in R$, we have $eke = ek$ for all left minimal element $k \in R$.

(5) For every left minimal element $e^2 = e \in R$, we have $eke = ek$ for all left minimal element $k \in R$ with $k^2 = 0$.

(6) R is a left min-abel left $GMC2$ ring.

Theorem 1.10 *Let R be a left min-abel ring, then the following are equivalent:*

(1) R is a left $MC2$ ring.

(2) Every nonsingular simple left $R-$ module is injective.

(3) Every simple projective left $R-$ module is injective.

(4) Every simple projective left $R-$ module is $p-$ injective.

(5) Every simple projective left $R-$ module is $nil-$ injective.

(6) Every simple projective left $R-$ module is mininjective.

Proof. (1) $\Longrightarrow$ (2). Assume that R is left $MC2$. Now let W be a nonsingular simple left $R-$ module. Then ${}_RW$ is projective and $W \cong R/K$, where K is a maximal left ideal of R and since ${}_RR/K$ is projective, then $R = K \oplus U$, where $U = Re, e^2 = e \in R$, is a minimal left ideal of R. If L is a proper essential left ideal of R, $f : L \longrightarrow U$ a non-zero left $R-$ homomorphism, then $L/N \cong U$, where $N = kerf$ is a maximal left subideal of L. Now $L = N \oplus V$, where $V(\cong U)$ is a minimal left ideal of R. Since R is a left $MC2$ ring, $V = Rg$, where $g^2 = g \in R$. Since R is left min-abel left $MC2$ ring, g is central by the remark above Theorem 1.9. Now for any $y \in L$, let $y = d + ag$, where $d \in N, a \in R$. Then $dg = gd \in N \cap V = 0$, so $f(y) = f(d + ag) = f(ag) = f(dg) + f(ag) = f((d + ag)g) = (d + ag)f(g) = yf(g)$. Hence ${}_RU$ is injective, and so is ${}_RW$.

(2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6) are obvious.

(6) $\Longrightarrow$ (1) Assume that $k, e^2 = e$ are left minimal elements of R with $kRe = 0$. If $eRk \neq 0$, then there exists an $a \in R$ such that $eak \neq 0$. Since $Reak \cong Re$ as left $R-$ module, ${}_RReak$ is mininjective. It is easy to show that there exists a $c \in R$ such that $eak = eakceak$. Since $kce \in kRe = 0$, $eak = 0$, which is a contradiction. Hence $eRk = 0$. By Theorem 1.6, R is a left $MC2$ ring. □

Call an ideal I of ring R children [1], if for any $a, b \in R$, and for every $x \in I$, there exists elements $s, t \in I$ such that $x = at + sb + st$. Obviously 0 and ideals which are direct summand of R as left $R-$ modules, are children ideals [1].

Theorem 1.11 *The following conditions are equivalent for a ring R:*

(1) R is a strongly left DS ring.

(2) R is a left universally mininjective left min-abel ring.

(3) R is a left PS strongly left min-abel ring.

(4) Every minimal left ideal of R is a children ideal.

Proof. (1) $\Longrightarrow$ (2) Evidently, R is a left universally mininjective ring. Now assume that $e^2 = e \in R$ is a left minimal element: write $h = ae - eae$ for $a \in R$. If $h \neq 0$, then h is a left minimal element with $he = h, eh = 0$. Hence $h^2 = hh = heh = 0$. This is impossible because R is a strongly left DS ring. Hence $h = 0$ and so e is left semicentral. This implies that R is a left min-abel ring.

(2) $\Longrightarrow$ (3) By [17], R is a left PS ring. By [4], R is a left $MC2$ ring. By Theorem 1.8, R is a strongly left min-abel ring.

(3) $\Longrightarrow$ (4) Assume that k is a left minimal element of R. Since R is a left PS ring, ${}_RRk$ is projective. Hence $l(k) = R(1-e)$ where $e^2 = e \in R$ is a left minimal element. Since R is a strongly left min-abel ring, e is central. Hence $k = ek = ke$, $Rk = Re$, so Rk is an ideal of R and is a direct summand as a left $R-$ module. By [1], Rk is a children ideal.

(4) $\Longrightarrow$ (1) Let Rk be a minimal left ideal of R, then Rk is a children ideal by hypothesis. Hence there exist $t, s \in Rk$ such that $k = kt + sk + st$. If $(Rk)^2 = 0$, then $kRk = 0$ and so $k^2 = 0$. Since $kt \in kRk, sk \in Rk^2, st \in RkRk$, $kt = sk = st = 0$. Hence $k = 0$, which is a contradiction. Consequently, $(Rk)^2 \neq 0$, then $Rk = Re, e^2 = e \in R$. Since $Rk = Re$ is an ideal, e is right semi-central. Hence e is central, so $k = ke = ek \in Rk^2$, which implies that $k^2 \neq 0$. Hence R is a strongly left DS ring. $\square$

The following example implies that there exists a left min-abel ring which is not strongly left min-abel.

Let F be a division ring and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $S_l(R) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is projective, so R is a left PS ring. Since $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $J(R) \cap S_l(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \neq 0$. By [17], R is not left universally mininjective ring. By [4], R is not left $MC2$ ring. On the other hand $R/J(R) \cong \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ is reduced, so $R/J(R)$ and then R is a left min-abel ring by Theorem 1.4. By Theorem 1.8, R is not a strongly left min-abel ring.

2. Certain rings whose simple singular modules are $nil-$ injective

The following theorem is a corollary of [5, Corollary 3.2(1)].

Theorem 2.1 *The following conditions are equivalent for a ring R:*

(1) R is a regular ring.

(2) R is a left weakly continuous ring whose simple singular left $R-$ modules are $nil-$ injective.

Note that $YJ-$ injective modules defined in this paper also called $GP-$ injective in [6]. In fact, $GP-$ injective modules were first given by Roger Yue Chi Ming in his paper "On regular rings and Artinian rings (II)" published in Riv. Math. Univ. Parma 11, 101-109 (1985). Hence the following corollary generalizes [6, Theorem 2].

Corollary 2.2 *(1) [6, Theorem 2] R is regular if and only if R is left weakly continuous whose simple singular left $R-$ modules are $YJ-$ injective.*

(2) Let R be a ring whose simple singular left $R-$ module is nil$-$ injective, then $Z_l(R) = 0$ if and only if $Z_l(R) \subseteq J(R)$.

In [18, Lemma 2.3] it is shown that if R is a left $C2$ ring, then $Z_l(R) \subseteq J(R)$. Hence if R is a right Kasch ring [18] or R is a left $Jcp-$ injective ring, then $Z_l(R) \subseteq J(R)$ because right Kasch rings and left $Jcp-$ injective rings are all left $C2$ ring.

In [12, Proposition 5] it is shown that if R is left $np-$ injective, then $Z_l(R) \subseteq J(R)$.

Certainly, if R is a semiperfect ring, then $Z_l(R) \subseteq J(R)$.

W. K. Nicholson and Sanchez Campos [15, Proposition 28] point out that, if R is a left morphic ring (see [15]), then $Z_l(R) \subseteq J(R)$.

If every left $R-$ monic $f : R \longrightarrow R$ is epic, then $Z_l(R) \subseteq J(R)$. In fact, if $a \in Z_l(R)$, then $l(1-a) = 0$. Hence the left $R-$ map $f : R \longrightarrow R$ via $f(x) = x(1-a), x \in R$ is monic. Hence f is an epic, and so $1 = b(1-a), b \in R$. This implies that $a \in J(R)$ because $Z_l(R)$ is an ideal of R. Hence we have the following corollary.

Corollary 2.3 *Let R be a ring whose simple singular left $R-$ module is nil$-$ injective, if R satisfies one of the following conditions, then $Z_l(R) = 0$.*

(1) R is a left $C2$ ring.

(2) R is a right Kasch ring.

(3) R is a semiperfect ring.

(4) R is a left $np-$ injective ring.

(5) R is a left morphic ring.

(5) R is a left $Jcp-$ injective ring.

(6) Every left $R-$ monic $f : R \longrightarrow R$ is epic.

According to [6], a ring R is said to be idempotent reflexive if $aRb = 0$ always implies that $bRa = 0$ for all $a, b \in R$. So idempotent reflexive rings are left $GMC2$. Since there exists a left selfinjecive ring which is not idempotent reflexive, there exists a left $GMC2$

ring which is not idempotent reflexive because all left selfinjective rings are left $GMC2$ (In fact, if R is a left mininjective ring, then for any $a \in R$ and left minimal idempotent e with $aRe = 0$, we have $eRa = 0$. Otherwise there exists a $b \in R$ such that $eba \neq 0$. Since $l(e) = l(eba)$ and R is a left mininjective ring, $eR = rl(e) = rl(eba) = ebaR$. Hence $eRe = ebaRe = 0$, which is a contradiction. Hence $eRa = 0$, which implies R is a left $GMC2$ ring.). So the following theorems are significant because they are the proper generalization of [6, Proposition 7].

Recall that an element $a \in R$ is called a left weakly regular if $a \in RaRa$. So the following theorem is significant because it is a generalization of Xue [19, Proposition 2] and Chen and Ding [3, Lemma 4.1].

Theorem 2.4 *Let R be a left $GMC2$ ring. Then*

(1) If $a \in R$ is not a left weakly regular element, then every maximal left ideal M of R containing $RaR + l(a)$ must be essential in ${}_RR$.

(2) If every simple singular left $R-$ module is nil$-$ injective, then for any nonzero nilpotent element $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and $RaR + l(a^n) = R$. Therefore $N(R) \cap J(R) = 0$. Consequently, R is NI if and only if R is reduced if and only if R is 2-prime.

(3) If every simple singular left $R-$ module is Gnp$-$ injective, then for any nonnilpotent element $a \in R$, there exists a positive integer n such that $RaR + l(a^n) = R$. Therefore $N_1(R) \cap J(R) = 0$.

(4) If every simple singular left $R-$ module is nil$-$ injective, then for any $0 \neq a \in R$, $(Ra)^2 \neq 0$. Therefore R is a semiprime ring.

(5) If simple singular left $R-$ modules are nil$-$ injective and Gnp$-$ injective, then for any nonzero element $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and $RaR + l(a^n) = R$. Therefore $J(R) = 0$.

Proof. (1) Assume that $a \in R$ is not a left weakly regular element. Then $RaR+l(a)$ is contained in some maximal left ideal M. If M is not essential, then $M = l(e), e^2 = e \in R$. Then $aRe = 0$. Since R is left $GMC2$ and e is a left minimal idempotent, $eRa = 0$. Hence $e \in l(a) \subseteq M = l(e)$, which is a contradiction. This implies M is essential.

(2) Assume that $a^n \neq 0, a^{n+1} = 0$. If a^n is a left weakly regular element, then we are done. Otherwise, by (1), there exists a maximal essential left ideal M containing $Ra^nR + l(a^n)$. Thus R/M is a simple singular left $R-$ module, so is $nil-$ injective. Hence the left $R-$ morphism $f : Ra^n \longrightarrow R/M$ defined by $f(ra^n) = r + M$ extends to

R, so there exists a $c \in R$ such that $1 - a^n c \in M$. Since $a^n c \in Ra^n R \subseteq M$, $1 \in M$, which is a contradiction. Hence $R = Ra^n R + l(a^n) = RaR + l(a^n)$.

(3) Consider the chain $RaR + l(a) \subseteq RaR + l(a^2) \subseteq \cdots$. Let $\cup_{i=1}^{\infty}[RaR + l(a^i)] = I$. If $I \neq R$, then I is contained in a maximal essential left ideal M of R. Then R/M is left $Gnp-$ injective. So there exists a positive integer n such that such that the left $R-$ morphism $Ra^n \longrightarrow M$ defined by $ra^n \longmapsto r + M$ extends to R. By a similar way as in the previous process, we obtain a contradiction. Therefore $\cup_{i=1}^{\infty}[RaR + l(a^i)] = R$, then we can easy to show that $RaR + l(a^m) = R$ for some positive integer m.

(4) If $(Ra)^2 = 0$, then by (2), we have $RaR + l(a) = R$. Hence $a \in RaRa = 0$, which is a contradiction. Thus $(Ra)^2 \neq 0$.

(5) Follows from (2) and (3). □

A left $R-$ module M is $YJ-$ injective if and only if M is $nil-$ injective and $Gnp-$ injective. Hence we have the following corollary.

Corollary 2.5 *Let R be a strongly left min-abel ring whose simple singular left $R-$ modules are $YJ-$ injective, then for any nonzero element $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and $RaR + l(a^n) = R$. Therefore $J(R) = 0$.*

It is well known that rings whose simple left $R-$ modules are $YJ-$ injective are always semiprimitive [11, Lemma 1].

Proposition 2.6 *(1) If every simple left $R-$ module is $nil-$ injective, then R is a semiprime ring.*

(2) If R is an NI ring whose every simple left $R-$ module is $nil-$ injective, then R is a reduced ring.

(3) If R is a left $MC2$ ring whose every simple singular left $R-$ module is $nil-$ injective, then R is a reduced ring if and only if R is an NI ring.

Proof. (1) It is clear from [5, Proposition 3.4].

(2) Assume that $0 \neq a \in R$ with $a^2 = 0$. Thus $l(a) \subseteq M$, where M be a maximal left ideal of R. Then by a similar way as in the previous process, there exists a $b \in R$ such that $1 - ab \in M$. Since $ab \in N(R)$, $1 - ab$ is invertible, which is a contradiction. Therefore R is a reduced ring.

(3) Assume that R is an NI ring, then R is strongly left min-abel ring, so is left $GMC2$ ring. By Theorem 2.4, R is a reduced ring. □

N. K. Kim and J. Y. Kim [7, Theorem 4] shows that if R is a ZI ring whose every simple singular left $R-$ module is $YJ-$ injective, then R is a reduced weakly regular ring. Then by Theorem 1.2 and [7, Proposition 8], we generalize the above result as follows.

Theorem 2.7 *Let R be a left $MC2$ ring whose every simple singular left $R-$ module is $YJ-$ injective. Then the following conditions are equivalent:*

(1) R is a reduced ring.

(2) R is a ZI ring.

(3) R is a 2-prime ring.

(4) R is an NI ring.

In this case, R is a weakly regular ring. And if R is also a $MELT$ ring, then R is a strongly regular ring.

Recall that a ring R is said to be left weakly $\pi-$ regular if for every $x \in R$, there exists a positive integer n, depending on x, such that $x^n \in Rx^nRx^n$. By Theorem 2.4 and the proof process of [2, Lemma 3.1], we have the following proposition which generalizes [6, Theorem 10].

Theorem 2.8 *Let R be a left $GMC2$ left Goldie ring. If R satisfies one of the following conditions, then R is a finite product of simple left Goldie rings.*

(1) Every simple singular left $R-$ module is $YJ-$ injective or

(2) Every simple singular left $R-$ module is nil$-$ injective and R is left weakly $\pi-$ regular,

Remark: (a) Left mininjective rings were first introduced in [17].

(b) Left $np-$ injective rings were first introduced in [12].

Acknowledgment

I would like to thank the referee for his/her helpful suggestions and comments.

References

[1] Kim, E. S. Kim, Y. J. and Park, Y. S.: Children ideals and children rings, Kyungpook Math. J. 38, 151-155 (1998).

[2] Michler, G. O. and Villamayor, O. E.: On rings whose simple modules are injective, J. Alg. 25, 185-201 (1973).

[3] Chen, J. L. and Ding, N. Q.: On generalizations of injectivity, In International Symposium on Ring Theory (Kyongju, 1999),Birkhauser, Boston, Boston, MA, 85-94 (2001).

[4] Wei, J. C.: The rings characterized by minimal left ideal, Acta. Math. Sinica. English. Series. 21:3, 473-482 (2005).

[5] Wei, J. C. and Chen, J. H.: *nil*- injective rings, in International Electronic J. Algebra, 2, 1-21 (2007).

[6] Kim, J. Y.: Certain rings whose simple singular modules are $GP-$ injective, Proc. Japan. Acad. 81, 125-128 (2005).

[7] Ding. N. Q. and Chen, J. L.: Rings whose simple singular modulesare $YJ-$ injective, Math Japonica. 40, 191-195 (1994).

[8] Kim, N. K. Nam, S. B. and Kim, J. Y.: On simple singular $GP-$ injective modules, Comm in Alg. 27:5, 2087-2096 (1999).

[9] Roger, Yue Chi Ming.: A note on semi-prime rings, Monatshefte für Math. 101, 173-182 (1986).

[10] Roger, Yue Chi Ming.: Generalizations of continuous and $p-$ injective modules, Anal. St. Univ. Al.I. CUZA.IASI. 52, 351-364 (2006).

[11] Roger, Yue Chi Ming.: On $p-$ injectivity and generalizations, Riv. Mat. Univ. Parma. 5:5, 183-188 (1996).

[12] Roger, Yue Chi Ming.: On quasi-Frobeniusean and Artinian rings, Publications De *L*'institut Math ématique. 33, 239-245 (1983).

[13] Roger, Yue Chi Ming.: On semi-prime and reduced rings, Riv. Mat. Univ. Parma. 12, 167-175 (1986).

[14] Lam, T. Y. and Dugas, A. S.: Quasi-duo rings and stable range descent, J. Pure and Appl. Alg. 195, 243-259 (2005).

[15] Nicholson, W. K. and Sanchez Campos: Rings with the dual of the isomorphism theorem, J. Alg. 271, 391-406. (2004).

[16] Nicholson, W. K. and Watters, J. F.: Rings with projective socle, Proc. Amer. Math. Soc. 102, 443-450 (1988).

[17] Nicholson, W. K. and Yousif, M. F.: Mininjective ring, J. Alg. 187, 548-578 (1997).

[18] Nicholson. W. K. and Yousif, M. F.: Weakly continuous and C2- rings, Comm in Alg. 29:6, 2429-2446 (2001).

[19] Xue, W. M.: A note on $YJ-$ injectivity, Riv. Mat. Univ. Parma. 6:1, 31-37 (1998).

[20] Lee, Y. and Hun, C.: A Note on $\Pi-$ regular rings, Kyungpook Math. J. 38, 157-161 (1998).

Junchao WEI
School of Mathematics,
Yangzhou University,
Yangzhou, 225002, P. R. CHINA
e-mail jcweiyz@yahoo.com.cn

Received 03.10.2006