Turk J Math
32 (2008) , 409 – 427.

A Numerical Solution of Wave Equation Arising in Non-Homogeneous Cylindrical Shells*

Allaberen Ashyralyev, Mehmet Emir Köksal

Abstract

A numerical solution of wave equation arising in non-homogeneous cylindrical shells is considered. Stable numerical schemes are developed. The stability estimates for the solution of these difference schemes and first and second order difference derivatives are presented. Applying the difference schemes, the numerical methods are proposed for solving the the given initial-boundary value problem.

Key Words: Hyperbolic equation; Difference schemes; Stability.

1. Introduction

In various sciences and engineering, hyperbolic partial differential equations are used to formulate and solve problems that involve unknown functions of several variables, such as the propagation of sound, heat or wave, or more generally any process that is distributed in space, or distributed in space and time as in thermodynamics, elasticity and electromagnetic.

In recent years, many applied problems in cylindrical and spherical coordinates by using method of characteristic in mechanics and engineering science, were formulated as the mathematical model of variable types. For instance, in [1], the dynamic response of layered composites consisting of N isotropic, elastic and functionally graded cylindrical layers (nonhomogeneous) were investigated. One-dimensional transient dynamic response

AMS Mathematics Subject Classification: 65M12, 65J10.
*Workshop on Differential Equations and Its Applications, 8-10 Febr. 2007, METU, Ankara

of functionally graded spherical multilayered media was formulated as an initial-boundary value problem of which solutions were obtained by employing the method of characteristic in [2]. One-dimensional transient wave propagation in multilayered functionally graded media was investigated in papers [3] and [4]. H.J. Ding et al. [5] has developed a solution of a non-homogeneous orthotropic cylindrical shell for axisymmetric plane strain dynamic thermoplastic problems and considered numerical solution for both isotropic and orthotropic cylindrical shells. An efficient alternative to existing methods of analyzing transient analysis of unbounded media was presented in [6] and the method was extended to problems involving spherical and cylindrical computational boundaries. In [7], propagation of two-dimensional transient waves in viscoelastic cylindrical layered media was investigated by a numerical technique which combines the complex Fourier series with the method of characteristic.

In this paper, we study the numerical solution of wave equation arising in non-homogeneous cylindrical shells. Based on works [1] and [2], the problems can be formulated in the form

$$\rho\frac{\partial^2 u(t,x)}{\partial t^2} = c\frac{\partial^2 u(t,x)}{\partial x^2} + \left(\frac{dc}{dx} + \frac{c}{\widetilde{x}}\right)\frac{\partial u(t,x)}{\partial x}$$

$$+\left(\frac{d\lambda}{dx} - \frac{c}{\widetilde{x}}\right)\frac{u(t,x)}{\widetilde{x}} + f(t,\ x),\ t \in (0,\ T),\ x \in (r_i,\ r_0), \tag{1}$$

where

$$\rho = \rho_0(a + b\widetilde{x})^J,\ c = c_0(a + b\widetilde{x})^i,\ \rho_0 = (2\mu_0 + \lambda_0),\ \lambda = \lambda_0(a + b\widetilde{x})^i,\ \widetilde{x} = \frac{x - r_i}{r_0 - r_i}, \tag{2}$$

with the initial and boundary conditions

$$u(0,x) = \ u_t(0,x) = 0, \quad x \in [r_i, r_0],$$

$$cu_x(t,\ x) + \lambda u(t,\ x)|_{x=r_i} = -p_0 l(t),\ u(t,\ r_0) = 0,\ t \in [0,T],$$

where a, b, i and J are the parameters determining the dimensionless quantities, ρ_0, λ_0, μ_0 and p_0 are the parameters determining the material properties, and $l(t)$ is a prescribed continuous function of t. This problem is the more general form of the problems considered in [3] and [4].

Using the substitution $u(t,x) = v(t,x) + (x - r_0)z(t,x)$, we can rewrite the equation (1) in the following equivalent form as

$$\frac{\partial^2 v(t,x)}{\partial t^2} = \frac{c}{\rho}\frac{\partial^2 v(t,x)}{\partial x^2} + \frac{d}{dx}\left(\frac{c}{\rho}\right)\frac{\partial v(t,x)}{\partial x}$$

$$+\left(\frac{1}{\rho}\frac{c}{\widetilde{x}} + \frac{c}{\rho^2}\frac{d\rho}{dx}\right)\frac{\partial v(t,x)}{\partial x} + \frac{1}{\rho\widetilde{x}}\left(\frac{d\lambda}{dx} - \frac{c}{\widetilde{x}}\right)v + F(t,\ x),\ t \in (0,\ T),\ x \in (r_i,\ r_0), \quad (3)$$

where

$$F(t,x) = \frac{1}{\rho}\left[f(t,x) - (x - r_0)\,\rho\frac{\partial^2 z(t,x)}{\partial t^2} + (x - r_0)\,c\frac{\partial^2 z(t,x)}{\partial x^2}\right.$$

$$\left.+\left[2c + (x - r_0)\left(\frac{dc}{dx} + \frac{c}{\widetilde{x}}\right)\right]\frac{\partial z(t,x)}{\partial x} + \left[\left(\frac{dc}{dx} + \frac{c}{\widetilde{x}}\right) + \frac{(x - r_0)}{\widetilde{x}}\left(\frac{d\lambda}{dx} - \frac{c}{\widetilde{x}}\right)\right]\right] z(t,x)$$

with the initial and boundary conditions

$$v(0,x) = -(x - r_0)z(0,x), v_t(0,x) = -(x - r_0)z_t(0,x),\ x \in [r_i, r_0],$$

$$cv_x(t,\ x) + \lambda v(t,\ x)|_{x=r_i}. = 0,\ v(t,\ r_0) = 0,\ t \in [0,T].$$

Here, $z(t,x)$ is defined by the formula

$$c\left((x - r_0)z(t,\ x)\right)_x + \lambda x z(t,\ x)|_{x=r_i} = -p_0 l(t).$$

2. Difference schemes. Stability estimates

The discretization of the problem (3) is carried out in two steps. In the first step the grid sets

$$[r_i,\ r_0]_h = \{x : x_n = r_i + nh,\ 0 \le n \le M,\ Mh = r_0 - r_i\}$$

are defined. The Banach space $L_{2h} = L_2([r_i,\ r_0]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ defined on $[r_i,\ r_0]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{n=1}^{M-1}|\varphi_n|^2 h\right)^{1/2}$$

is introduced. To the differential operator B^x and C^x generated by the problem (3), we assign the difference operators B_h^x and C_h^x by the formulas

$$B_h^x \varphi^h(x) = \left\{ -\frac{1}{\rho(x_n)} \left(\left.\frac{dc}{dx}\right|_{x=x_n} - \frac{c(x_n)}{\rho(x_n)} \left.\frac{d\rho}{dx}\right|_{x=x_n} \right) \varphi_{\dot{x},n} - \frac{c(x_n)}{\rho(x_n)} \varphi_{x\overline{x},n} \right\}_1^{M-1}$$

and

$$C_h^x \varphi^h(x) = \left\{ \frac{1}{\rho(x_n)} \left(\frac{c(x_n)}{\rho(x_n)} \left.\frac{d\rho}{dx}\right|_{x=x_n} + \frac{c(x_n)}{\widetilde{x}_n} \right) \varphi_{\dot{x},n} + \frac{1}{\widetilde{x}_n \rho(x_n)} \left(\left.\frac{d\lambda}{dx}\right|_{x=x_n} - \frac{c(x_n)}{\widetilde{x}_n} \right) \varphi_n \right\}_1^{M-1}$$

acting in the space of grid functions $\varphi^h(x)$, satisfying the conditions $\varphi_0 = 0$, $\frac{c}{2h}(-3\varphi_0 + 4\varphi_1 - \varphi_2) + \frac{\lambda}{r_i}\varphi_0 = 0$ for $\widetilde{x} = r_i$. Here,

$$\varphi_{x\overline{x},n} = \left\{ \frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{h^2} \right\}_1^{M-1}$$

and

$$\varphi_{\dot{x},n} = \left\{ \frac{\varphi_{n+1} - \varphi_{n-1}}{2h} \right\}_1^{M-1}.$$

With the help of B_h^x and C_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{d^2u^h(t,x)}{dt^2} + B_h^x u^h(t,x) = F^h(t,x) + C_h^x u^h(t,x), \ 0 \le t \le 1, \ x \in [r_i, r_0]_h, \\ u^h(0,x) = \varphi^h(x), \ \frac{du^h(0,x)}{dt} = \psi^h(x), \ x \in [r_i, r_0]_h, \end{cases}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (3) by the difference schemes in [8]

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2} + B_h^x u_{k+1}^h(x) = F_k^h(x) + C_h^x u_{k+1}^h(t,x), \\ F_k^h(x) = F^h(t_{k+1},x),\ t_k = k\tau, 1 \le k \le N-1,\ N\tau = 1,\ x \in [r_i, r_0]_h, \\ \frac{u_1^h(x)-u_0^h(x)}{\tau} + \tau B_h^x \left(u_1^h(x) - u_0^h(x)\right) \\ = \psi^h(x) + \tau C_h^x \left(u_1^h(x) - u_0^h(x)\right),\ x \in [r_i, r_0]_h, \\ u_0^h(x) = \varphi^h(x),\ x \in [r_i, r_0]_h, \end{cases} \tag{4}$$

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2} + \frac{1}{2}B_h^x u_k^h(x) + \frac{1}{4}B_h^x u_{k+1}^h(x) + \frac{1}{4}B_h^x u_{k-1}^h(x) \\ = F_k^h(x) + \frac{1}{2}C_h^x u_k^h(x) + \frac{1}{4}C_h^x u_{k+1}^h(x) + \frac{1}{4}C_h^x u_{k-1}^h(x), \\ F_k^h(x) = F^h(t_k,x),\ t_k = k\tau,\ 1 \le k \le N-1,\ N\tau = 1,\ x \in [r_i, r_0]_h, \\ \frac{u_1^h(x)-u_0^h(x)}{\tau} + \tau B_h^x (u_1^h(x) - u_0^h(x)) \\ = \frac{\tau}{2}(F_0^h(x) + \frac{1}{4}\left(2C_h^x u_0^h(x) + C_h^x u_1^h(x) + C_h^x u_{-1}^h(x)\right) - B_h^x u_0^h(x)) \\ +\tau C_h^x (u_1^h(x) - u_0^h(x)) + \psi^h(x), \\ F_0^h = F^h(0,x),\ u_0^h(x) = \varphi^h(x),\ x \in [r_i, r_0]_h, \end{cases} \tag{5}$$

respectively. Applying the abstract theorems of paper [8], we obtain the following results.

Theorem 1. Let τ and $|h|$ be sufficiently small numbers. Then for the solution of the difference schemes (4) and (5), the following inequality is valid:

$$\max_{0 \le k \le N} ||u_k^h \|_{L_{2h}} + \max_{1 \le k \le N} \| \frac{u_k^h - u_{k-1}^h}{\tau} \|_{L_{2h}}$$

$$\le C_1[\max_{1 \le k \le N-1} \| f_k^h \|_{L_{2h}} + \| \psi^h ||_{L_{2h}} + \| \varphi_{\dot{x},n}^h ||_{L_{2h}}],$$

where C_1 is independent of τ, h, f_k^h, g_k^h and φ^h.

Theorem 2. Let τ and $|h|$ be sufficiently small numbers. Then for the solution of the difference schemes (4) and (5), the following inequality is valid:

$$\max_{1\leq k\leq N-1} \parallel \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} \parallel_{L_{2h}} + \max_{1\leq k\leq N-1} ||(u_{k+1}^h)_{x\overline{x},n} \parallel_{L_{2h}}$$

$$\leq C_2[\max_{1\leq k\leq N-1} \parallel \frac{f_k^h - f_{k-1}^h}{\tau} \parallel_{L_{2h}} + \parallel f_0^h \parallel_{L_{2h}} +||\psi_{\overline{x},n}^h \parallel_{L_{2h}} +||\varphi_{\overline{x},n}^h \parallel_{L_{2h}}],$$

where C_2 is independent of τ, h, f_k^h, g_k^h and φ^h.

The proof of these theorems are based on the discrete analogies of integral inequality and on the following formulas:

$$u_k^h = \frac{1}{2}\left(R_h^{k-1} + \widetilde{R}_h^{k-1}\right)\varphi^h + \left(R_h - \widetilde{R}_h\right)^{-1}\tau R_h\left(R_h^k - \widetilde{R}_h^k\right)\left(\psi^h + \tau C_h^x\left(u_1^h - u_0^h\right)\right)$$

$$-\frac{\tau}{2i}\sum_{s=1}^{k-1}\left(B_h^x\right)^{-1/2}\left(R_h^{k-s} - \widetilde{R}_h^{k-s}\right)\left(F_s^h + C_h^x u_{s+1}^h\right), k = 2, ..., N$$

for the solution of difference scheme (4);

$$u_k^h = \left[P_h^k + \tau P_h\left(P_h - \widetilde{P}_h\right)^{-1}\left(P_h^k - \widetilde{P}_h^k\right)\left(I + \tau^2 B_h^x\right)^{-1}\left(-\tau B_h^x + i\left(B_h^x\right)^{1/2}\right)\right]\varphi^h$$

$$+\tau P_h\left(P_h - \widetilde{P}_h\right)^{-1}\left(P_h^k - \widetilde{P}_h^k\right)\left(I + \tau^2 B_h^x\right)^{-1}\left(\psi^h + \tau C_h^x\left(u_1^h - u_0^h\right)\right)$$

$$+\frac{\tau^2}{2}P_h\left(P_h - \widetilde{P}_h\right)^{-1}\left(P_h^k - \widetilde{P}_h^k\right)\left(I + \tau^2 B_h^x\right)^{-1}F_0^h$$

$$-\frac{\tau}{2i}\sum_{s=1}^{k-1}\left(B_h^x\right)^{-1/2}\left(P_h^{k-s} - \widetilde{P}_h^{k-s}\right)\left[F_s^h + \frac{1}{4}\left(2C_h^x u_s^h + C_h^x u_{s+1}^h + C_h^x u_{s-1}^h\right)\right], k = 2, ..., N$$

for the solution of difference scheme (5); and on the estimates

$$\left\|R_h\right\|_{H\to H} \leq 1, \left\|\widetilde{R}_h\right\|_{H\to H} \leq 1, \left\|R_h\widetilde{R}_h^{-1}\right\|_{H\to H} \leq 1, \left\|\widetilde{R}_h R_h^{-1}\right\|_{H\to H} \leq 1,$$

$$\left\|\tau\left(B_h^x\right)^{1/2}R_h\right\|_{H\to H} \leq 1, \left\|\tau\left(B_h^x\right)^{1/2}\widetilde{R}_h\right\|_{H\to H} \leq 1,$$

and

$$\|P_h\|_{H\to H} \le 1, \left\|\widetilde{P}_h\right\|_{H\to H} \le 1, \left\|P_h\widetilde{P}_h^{-1}\right\|_{H\to H} \le 1, \left\|\widetilde{P}_hP_h^{-1}\right\|_{H\to H} \le 1,$$

$$\left\|\left(I \pm \frac{i\tau}{2}\left(B_h^x\right)^{1/2}\right)^{-1}\right\|_{H\to H} \le 1, \left\|\left(I \pm i\tau\left(B_h^x\right)^{1/2}\right)^{-1}\right\|_{H\to H} \le 1,$$

$$\left\|\tau\left(B_h^x\right)^{1/2}\left(I \pm i\tau\left(B_h^x\right)^{1/2}\right)^{-1}\right\|_{H\to H} \le 1,$$

and

$$\left\|\left(B_h^x\right)^{-1/2}C_h^x\right\|_{H\to H} \le M.$$

Here,

$$R_h = \left(I + i\tau\left(B_h^x\right)^{1/2}\right)^{-1}, \widetilde{R}_h = \left(I - i\tau\left(B_h^x\right)^{1/2}\right)^{-1},$$

and

$$P_h = \left(I + i\tau\left(B_h^x\right)^{1/2} - \frac{\tau^2}{2}B_h^x\right)^{-1}, \widetilde{P}_h = \left(I - i\tau\left(B_h^x\right)^{1/2} - \frac{\tau^2}{2}B_h^x\right)^{-1}.$$

Of course, the stability inequalities can be established for the general case $B_h^x(t)$ (see [9] and [10]).

Finally, one has not been able to obtain a sharp estimate for the constants figuring in the stability estimates. Therefore, our interest in the present paper is studying the difference schemes (4) and (5) by numerical experiments. Applying these difference schemes, the numerical methods are proposed in the following section for solving the one-dimensional wave equation arising in non-homogeneous cylindrical shells. The methods are illustrated by numerical examples.

3. Numerical analysis

In the present paper, the initial-boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \alpha(x)\frac{\partial^2 u(t,x)}{\partial x^2} - \beta(x)\frac{\partial u(t,x)}{\partial x} - \delta(x)u(t,x) = f(t,\ x), \\ 0 < t < 1,\ r_i < x < r_0, u(0,x) = \ u_t(0,x) = 0,\ r_i \le x \le r_0, \\ c_0(a+br_i)^i u_x(t,r_i) + \frac{\lambda_0}{r_i}(a+br_i)^i u(t,r_i) = -p_0 l(t),\ u(t,r_0) = 0,\ 0 \le t \le 1, \\ l(t) = -\frac{c_0}{r_0-r_i}(a+br_i)^i \pi t^2,\ \alpha(x) = \frac{c_0}{\rho_0}(a+b\frac{x-r_i}{r_0-r_i})^{i-J}, \\ \beta(x) = \left[\frac{ibc_0}{\rho_0}(a+b\frac{x-r_i}{r_0-r_i})^{i-J-1} + \frac{c_0}{\rho_0}(\frac{r_0-r_i}{x-r_i})(a+b\frac{x-r_i}{r_0-r_i})^{i-J}\right], \\ \delta(x) = \left(\frac{r_0-r_i}{x-r_i}\right)\left[\frac{ib\lambda_0}{\rho_0}(a+b\frac{x-r_i}{r_0-r_i})^{i-J-1} - \frac{c_0}{\rho_0}(\frac{r_0-r_i}{x-r_i})(a+b\frac{x-r_i}{r_0-r_i})^{i-J}\right], \\ f(t,x) = \left(2+\frac{\pi^2}{(r_0-r_i)^2}\alpha(x)t^2 - \delta(x)t^2\right)\sin\pi\frac{x-r_i}{r_0-r_i} - \frac{\pi}{r_0-r_i}\beta(x)t^2\cos\pi\frac{x-r_i}{r_0-r_i} \end{cases} \tag{6}$$

with variable coefficients is considered. Note that this problem is the reformulation of the test problem (1) with the specific choice of $f(t,x)$.

First, applying difference scheme (4), we present the following first order of accuracy difference scheme for the approximate solution of problem (6)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \alpha(x_n)\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} - \beta(x_n)\frac{u_{n+1}^{k+1}-u_{n-1}^{k+1}}{2h} - \delta(x_n)u_n^{k+1} \\ = f(t_{k+1},x_n),\ t_k = k\tau,\ x_n = r_i + nh, 1 \le k \le N-1, 1 \le n \le M-1, \\ u_n^0 = 0,\ \frac{u_n^1-u_n^0}{\tau} = 0,\ 1 \le n \le M-1,\ c_0(a+br_i)^i\frac{-u_2^{k+1}+4u_1^{k+1}-3u_0^{k+1}}{2h} \\ +\frac{\lambda_0}{r_i}(a+br_i)^i u_0^{k+1} = -p_0 l(t_k),\ u_M^k = 0,\ 0 \le k \le N-1,\ f(t_{k+1},x_n) = \\ \left(2+\frac{\pi^2}{(r_0-r_i)^2}\alpha_n t_{k+1}^2 - \delta_n t_{k+1}^2\right)\sin\pi\frac{x_n-r_i}{r_0-r_i} - \frac{\pi}{r_0-r_i}\beta_n t_{k+1}^2\cos\pi\frac{x_n-r_i}{r_0-r_i}, \\ l(t_k) = -\frac{c_0}{r_0-r_i}(a+br_i)^i\pi t_k^2,\ \alpha(x_n) = \frac{c_0}{\rho_0}(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J}, \\ \beta(x_n) = \left[\frac{ibc_0}{\rho_0}(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J-1} + \frac{c_0}{\rho_0}(\frac{r_0-r_i}{x_n-r_i})(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J}\right], \\ \delta(x_n) = \left(\frac{r_0-r_i}{x_n-r_i}\right)\left[\frac{ib\lambda_0}{\rho_0}(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J-1} - \frac{c_0}{\rho_0}(\frac{r_0-r_i}{x_n-r_i})(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J}\right]. \end{cases} \tag{7}$$

We have $(N+1)\times(M+1)$ system of linear equations in (7) and we write them in the matrix form

$$\begin{cases} A\,U^{k+1} + B\,U^{k} + C\,U^{k-1} = D\varphi^{k}, \quad 1 \le k \le N-1, \\ \\ U^{0} = \widetilde{0},\ U^{1} = \widetilde{0}, \end{cases} \tag{8}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ d_1 & e_1 & f_1 & \dots & 0 & 0 & 0 \\ 0 & d_2 & e_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & e_{M-2} & f_{M-2} & 0 \\ 0 & 0 & 0 & \dots & d_{M-1} & e_{M-1} & f_{M-1} \\ -3s+z & 4s & -s & \dots & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & p & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & p & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & p & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & p & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & q & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & q & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & q & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & q & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$D=\begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$U^s=\begin{bmatrix} U_0^s \\ U_1^s \\ ... \\ U_M^s \end{bmatrix}_{(M+1)\times(1)}, \text{ for } s=\ k\pm 1,\ k.$$

Here,

$$d_n=-\frac{\alpha_n}{h^2}+\frac{\beta_n}{2h},\ e_n=\frac{1}{\tau^2}+\frac{2\alpha_n}{h^2}-\delta_n,\ f_n=-\frac{\alpha_n}{h^2}-\frac{\beta_n}{2h},\ 1\leq n\leq M-1$$

$$s=\frac{1}{2h}c_0(a+br_i)^i,\ z=\frac{\lambda_0}{r_i}(a+br_i)^i,\ p=-\frac{2}{\tau^2},\quad q=\frac{1}{\tau^2}$$

$$\varphi_n^k=\begin{cases} 0,\ \ n=0, \\ f(t_{k+1},x_n),\ 1\leq n\leq M-1, \\ c_0\frac{1}{r_0-r_i}(a+br_i)^i\pi t_k^2,\ n=M, \end{cases},\ \varphi^k=\begin{bmatrix} \varphi_0^k \\ \varphi_1^k \\ ... \\ \varphi_M^k \end{bmatrix}_{(M+1)\times 1}.$$

So, we have the second order difference equation (8) with respect to k with matrix coefficients. To solve this difference equation, we apply the following procedure

$$\begin{cases} U^{k+1}=A^{-1}D\varphi^k-A^{-1}BU^k-A^{-1}CU^{k-1},\quad k=1,2,\cdots,N-1, \\ \\ U^0=\widetilde{0},\ U^1=\widetilde{0}. \end{cases}$$

Second, applying difference scheme (5), we present the following second order of accuracy difference scheme for the approximate solution of problem (6)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \alpha(x_n)\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} + \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{4h^2}\right. \\ \left.+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2}\right) - \beta(x_n)\left(\frac{u_{n+1}^k-u_{n-1}^k}{4h} + \frac{u_{n+1}^{k+1}-u_{n-1}^{k+1}}{8h} + \frac{u_{n+1}^{k-1}-u_{n-1}^{k-1}}{8h}\right) \\ -\delta\left(x_n\right)\left(\frac{1}{2}u_n^k + \frac{1}{4}\left(u_n^{k+1}+u_n^{k-1}\right)\right) = f(t_k,x_n),\ x_n = r_i + nh,\ t_k = k\tau, \\ 1 \le k \le N-1,\ 1 \le n \le M-1,\ u_n^0 = 0,\ 1 \le n \le M-1, \\ \frac{u_n^1-u_n^0}{\tau} = \frac{\tau}{2}\left(\alpha_n \frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2} + \beta_n \frac{u_{n+1}^1-u_{n-1}^1}{2h} + \delta_n u_n^1 + f(0,x_n)\right), \\ x_n = r_i + nh,\ 1 \le n \le M-1,\ c_0(a+br_i)^i \frac{-u_2^{k+1}+4u_1^{k+1}-3u_0^{k+1}}{2h} \\ +\frac{\lambda_0}{r_i}(a+br_i)^i u_0^{k+1} = -p_0 l(t_k),\ u_M^k = 0,\ 0 \le k \le N-1,\ f(t_{k+1},x_n) \\ = \left(2 + \frac{\pi^2}{(r_0-r_i)^2}\alpha_n t_{k+1}^2 - \delta_n t_{k+1}^2\right)\sin\pi\frac{x_n-r_i}{r_0-r_i} - \frac{\pi}{r_0-r_i}\beta_n t_{k+1}^2 \cos\pi\frac{x_n-r_i}{r_0-r_i}, \\ l(t_k) = -\frac{c_0}{r_0-r_i}(a+br_i)^i \pi t_k^2,\ \alpha(x_n) = \frac{c_0}{\rho_0}(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J}, \\ \beta(x_n) = \left[\frac{ibc_0}{\rho_0}(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J-1} + \frac{c_0}{\rho_0}(\frac{r_0-r_i}{x_n-r_i})(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J}\right], \\ \delta(x_n) = \left(\frac{r_0-r_i}{x_n-r_i}\right)\left[\frac{ib\lambda_0}{\rho_0}(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J-1} - \frac{c_0}{\rho_0}(\frac{r_0-r_i}{x_n-r_i})(a+b\frac{x_n-r_i}{r_0-r_i})^{i-J}\right]. \end{cases} \tag{9}$$

We have again $(N+1)\times(M+1)$ system of linear equations and we write them in the matrix form

$$\begin{cases} A\,U^{k+1} + B\,U^k + C\,U^{k-1} = D\varphi^k, \quad 1 \le k \le N-1, \\ U^0 = \widetilde{0},\ EU^1 = vU^0 + w\widetilde{\psi(x_n)},\ x_n = nh,\ 1 \le n \le M-1, \end{cases} \tag{10}$$

where

$$A=\begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 & 1 \\ d_1 & e_1 & f_1 & ... & 0 & 0 & 0 \\ 0 & d_2 & e_2 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & e_{M-2} & f_{M-2} & 0 \\ 0 & 0 & 0 & ... & d_{M-1} & e_{M-1} & f_{M-1} \\ -3s+z & 4s & -s & ... & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ g_1 & p_1 & q_1 & ... & 0 & 0 & 0 \\ 0 & g_2 & p_2 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & p_{M-2} & q_{M-2} & 0 \\ 0 & 0 & 0 & ... & g_{M-1} & p_{M-1} & q_{M-1} \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C=\begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ d_1 & e_1 & f_1 & ... & 0 & 0 & 0 \\ 0 & d_2 & e_2 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & e_{M-2} & f_{M-2} & 0 \\ 0 & 0 & 0 & ... & d_{M-1} & e_{M-1} & f_{M-1} \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$D=\begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}_{(M+1)\times(M+1)}.$$

$$E=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ y_1 & \eta_1 & \mu_1 & \dots & 0 & 0 & 0 \\ 0 & y_2 & \eta_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \eta_{M-2} & \mu_{M-2} & 0 \\ 0 & 0 & 0 & \dots & y_{M-1} & \eta_{M-1} & \mu_{M-1} \\ -3s+z & 4s & -s & \dots & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}.$$

Here,

$$d_n=-\frac{\alpha_n}{4h^2}+\frac{\beta_n}{8h},\ e_n=\frac{1}{\tau^2}+\frac{\alpha_n}{2h^2}-\frac{\delta_n}{4},\ f_n=-\frac{\alpha_n}{4h^2}-\frac{\beta_n}{8h},$$

$$g_n=-\frac{\alpha_n}{2h^2}+\frac{\beta_n}{4h},\ p_n=-\frac{2}{\tau^2}+\frac{\alpha_n}{h^2}-\frac{\delta_n}{2},\ q_n=-\frac{\alpha_n}{2h^2}-\frac{\beta_n}{4h},$$

$$y_n=-\frac{\tau}{2}\left(\frac{\alpha_n}{h^2}-\frac{\beta_n}{2h}\right),\ \eta_n=\frac{1}{\tau}+\frac{\tau\alpha_n}{h^2}-\frac{\tau\delta_n}{2},\ \mu_n=-\frac{\tau}{2}\left(\frac{\alpha_n}{h^2}+\frac{\beta_n}{2h}\right),\ 1\le n\le M-1,$$

$$s=\frac{1}{2h}c_0(a+br_i)^i,\ z=\frac{\lambda_0}{r_i}(a+br_i)^i,\ v=\frac{1}{\tau},\ w=\frac{\tau}{2},\ c_0=2\lambda_0+\mu_0,$$

$$\varphi_n^k=\begin{cases} 0,\ \ n=0, \\ f(t_k,x_n),\ 1\le n\le M-1, \\ c_0\frac{1}{r_0-r_i}(a+br_i)^i\pi t_k^2,\ n=M, \end{cases},\ \varphi^k=\begin{bmatrix}\varphi_0^k \\ \varphi_1^k \\ \dots \\ \varphi_M^k\end{bmatrix}_{(M+1)\times 1}.$$

So, we have the second order difference equation (10) with respect to k with matrix coefficients. To solve this difference equation, we apply the following procedure:

$$\begin{cases} U^{k+1}=A^{-1}D\varphi^k-A^{-1}BU^k-A^{-1}CU^{k-1}, \quad k=1,2,\cdots,N-1, \\ \\ U^0=\widetilde{0},\ U^1=E^{-1}vU^0+E^{-1}w\widetilde{\psi(x_n)},\ x_n=nh,\ 1\le n\le M-1. \end{cases}$$

Now, we will give the results of the numerical analysis. Similar forms of equation (2) were used in the papers [11] and [12] in the cases $a=1$, $i=J=1$, and $a=1, i=J=2$, respectively. In the first and second examples in the present paper, numerical results are presented with the parameters $a=1$, $b=0$, $i=J=1$, and $a=5/6$, $b=1/6$, $i=J=1$, respectively. In both examples, the values $\lambda_0=0.3$, $\mu_0=0.35$, $p_0=1$, $r_i=1$, $r_0=2$ are taken for material properties.

First, as we noted above one can not obtain a sharp estimate for the constants C_1 and C_2 figuring in the stability estimates of Theorems 1 and 2. We have

$$C_1 = \max_{f,u} (C_{t1}),$$

$$C_2 = \max_{f,u} (C_{t2}),$$

where

$$C_{t1} = \max_{0\le k\le N} \left\|u_k^h\right\|_{L_{2h}} + \max_{1\le k\le N} \left\|\tau^{-1}\left(u_k^h - u_{k-1}^h\right)\right\|_{L_{2h}}$$

$$\times \left[\max_{1\le k\le N-1} \left\|f_k^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \sum_{r=1}^{n} \left\|\varphi^h_{\overline{x}_r,j_r}\right\|_{L_{2h}}\right]^{-1},$$

$$C_{t2} = \left[\max_{1\le k\le N-1} \left\|\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\right\|_{L_{2h}} + \max_{1\le k\le N-1} \sum_{r=1}^{n} \left\|\left(u_{k+1}^h\right)_{\overline{x}_r,x_r,,j_r}\right\|_{L_{2h}}\right]$$

$$\times \left[\max_{1\le k\le N-1} \left\|\tau^{-1}(f_k^h - f_{k-1}^h)\right\|_{L_{2h}} + \left\|f_0^h\right\|_{L_{2h}}\right.$$

$$\left. + \sum_{r=1}^{n} \left\|\psi^h_{\overline{x}_r,j_r}\right\|_{L_{2h}} + \sum_{r=1}^{n} \left\|\varphi^h_{\overline{x}_r,x_r,,j_r}\right\|_{L_{2h}}\right]^{-1}.$$

The constants C_{t1} and C_{t2} in the case of numerical solution of initial value problem (6) are computed.

The numerical solutions are recorded for different values of $N = M$, u_n^k represents the numerical solutions of these difference schemes at (t_k, x_n). In the cases $a = 1$, $b = 0$, and $a = 5/6$, $b = 1/6$, the constants C_{t1} and C_{t2} are given in Tables 1–2 and Tables 3–4 for $N = M = 10$, 20, 30, 40 and 50.

Table 1. The values of C_{t1} in the case $a = 1$, $b = 0$.

Method	N=M=10	N=M=20	N=M=30	N=M=40	N=M=50
Difference scheme (4)	0.2314	0.1273	0.0896	0.0693	0.0566
Difference scheme (5)	0.2749	0.1390	0.0951	0.0725	0.0587

Table 2. The values of C_{t2} in the case $a = 1$, $b = 0$.

Method	N=M=10	N=M=20	N=M=30	N=M=40	N=M=50
Difference scheme (4)	0.4300	0.2091	0.1397	0.1051	0.0843
Difference scheme (5)	0.5000	0.2251	0.1468	0.1091	0.0869

Table 3. The values of C_{t1} in the case $a = 5/6$, $b = 1/6$. ¿

Method	N=M=10	N=M=20	N=M=30	N=M=40	N=M=50
Difference scheme (4)	0.2260	0.1260	0.0890	0.0690	0.0563
Difference scheme (5)	0.2686	0.1376	0.0945	0.0721	0.0584

Table 4. The values of C_{t2} in the case $a = 5/6$, $b = 1/6$.

Method	N=M=10	N=M=20	N=M=30	N=M=40	N=M=50
Difference scheme (4)	0.4217	0.2072	0.1389	0.1047	0.0840
Difference scheme (5)	0.4907	0.2231	0.1459	0.1086	0.0866

Recall that we have not been able to obtain a sharp estimate for the constants C_1 and C_2 figuring in the stability estimates. The numerical results in the Tables 1, 3 and 2, 4 give $C_{t1} \cong 0.5$ and $C_{t2} \cong 0.8$, respectively. That means the constants C_1 and C_2 figuring in the stability estimates of Theorem 1 and 2 are not large numbers and the difference schemes (4) and (5) are stable in the case of numerical solution of initial value problem (6). Further, the properties of the material represented by the parameters a and b do not effect the values of the coefficients of the stability estimates C_1 and C_2 very much.

Second, for the accurate comparison of the two different difference schemes considered, the errors of the numerical solution computed by

$$E_0 = \max_{1 \le k \le N-1} \left(\sum_{n=1}^{M-1} \left| u(t_k, x_n) - u_n^k \right|^2 h \right)^{1/2},$$

are recorded for various values of N and M, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). The errors for the first order of accuracy difference scheme are given in the Tables 5–7 for $a = 1$, $b = 0$, and $a = 5/6$, $b = 1/6$, respectively. For the same values of a and b, Tables 6 and 8 show the errors for the second order of accuracy difference scheme.

Table 5. The errors for the difference scheme (4) in the case $a = 1$, $b = 0$.

$N \backslash M$	10	20	30	40	50
10	0.1059	0.1074	0.1076	0.1077	0.1078
20	0.0588	0.0605	0.0608	0.0609	0.0609
30	0.0401	0.0419	0.0421	0.0422	0.0423
40	0.0302	0.0320	0.0322	0.0323	0.0324
50	0.0240	0.0258	0.0261	0.0262	0.0262

Table 6. The errors for the difference scheme (5) in the case $a = 1$, $b = 0$.

$N \backslash M$	10	20	30	40	50
10	0.0063	0.0052	0.0054	0.0058	0.0061
20	0.0053	0.0021	0.0015	0.0014	0.0013
30	0.0045	0.0016	0.0010	0.0008	0.0007
40	0.0040	0.0013	0.0008	0.0006	0.0004
50	0.0037	0.0012	0.0007	0.0005	0.0004

Table 7. The errors for the difference scheme (4) in the case $a = 5/6$, $b = 1/6$.

$N \backslash M$	10	20	30	40	50
10	0.1063	0.1078	0.1081	0.1082	0.1078
20	0.0591	0.0608	0.0611	0.0612	0.0612
30	0.0403	0.0421	0.0424	0.0425	0.0425
40	0.0303	0.0321	0.0324	0.0325	0.0325
50	0.0241	0.0259	0.0262	0.0263	0.0263

Table 8. The errors for the difference scheme (5) in the case $a = 5/6$, $b = 1/6$.

$N \backslash M$	10	20	30	40	50
10	0.0056	0.0052	0.0055	0.0059	0.0062
20	0.0049	0.0019	0.0015	0.0014	0.0013
30	0.0043	0.0015	0.0009	0.0007	0.0006
40	0.0039	0.0012	0.0007	0.0005	0.0004
50	0.0037	0.0011	0.0006	0.0004	0.0003

Thus, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme. Further, the rate of decrease of the error with increasing N and M is higher for the second order of accuracy difference scheme (5) than the first order of accuracy difference scheme (4). These results are common to both examples obtained by variation of parameters a and b. As in the case of the computation of the coefficients of the stability estimates, the errors of the first and second order of accuracy difference schemes do not differ very much for the two examples. For example for $N = M = 20$, the errors computed for the first order of accuracy difference schemes are 0.0605 and 0.0608 for the examples 1 and 2, respectively. Similarly for the second order of accuracy difference scheme, these errors are 0.0021 and 0.0019. Obviously in both examples the second order of accuracy difference scheme is more accurate almost 30 times.

Acknowledgments

The authors would like to thank the reviewers and to Assist. Prof. Dr. Ibrahim Abu-Alshaikh for their helpful suggestions.

References

[1] Abu-Alshaikh, I.: One-dimensional wave propagation in functionally graded cylindrical layered media, Mathematical Methods in Engineering, Springer, 111-121, (2007).

[2] Abu-Alshaikh, I., Koksal, M.E.: One-dimensional transient dynamic response in functionally gradient spherical multilayered media, Proceedings of Dynamical Systems and Applications, 1-20, (2004).

[3] Abu-Alshaikh, I., Kokluce, B.: One-dimensional transient dynamic response in functionally graded layered media, Journal of Engineering Mathematics, 54, 1, 17-30, (2006).

[4] Kokluce, B., Abu-Alshaikh, I., Yildiz, T.: Numerical solution of one-dimensional wave equation in multilayered functionally graded materials, Journal of Engineering and Natural Sciences, 4, 31-40, (2005).

[5] Ding, H.J., Wang, H.M., Chen, W.Q.: A solution of a non-homogeneous orthotropic cylindrical shell for axisymmetric plane strain dynamic thermoelastic problems, Journal of Sound and Vibration, 263, 815-829, (2003).

[6] Murthy, N., Tassoulas, G., Tassoulas, J.L.: Characteristics methods for transient analysis of wave propagation in unbounded media, Computer Methods in Applied Mechanics and Engineering, 164, 1-2, 187-206, (1998).

[7] Abu Alshaikh, I., Turhan, D., Mengi, Y.: Transient waves in viscoelastic cylindrical layered media, European Journal of Mechanics - A/Solids, 21, 5, 811-830, (2002).

[8] Ashyralyev, A., Sobolevskii, P.E.: A note on the difference schemes for hyperbolic equations, Abstract and Applied Analysis, 6, 2, 63-70, (2001).

[9] Ashyralyev, A., Koksal, M.E.: On the second order of accuracy difference scheme for hyperbolic equations in a Hilbert space, Numerical Functional Analysis and Optimizations, 26, 6-7, 739-772, (2005).

[10] Ashyralyev, A., Koksal, M.E.: Stability of a accuracy difference scheme for hyperbolic equations in a Hilbert space, Discrete Dynamics in Nature and Society, 2007,1-25, (2007).

[11] Liu, G.R., Han, X., Lam, K.Y.: Stress Waves in functionally gradient materials and its use for material characterization, Compos. Part B: Eng., 30, 383-394, (1999).

[12] Han, X., Liu, G.R.: Effects of SH waves in a functionally graded plate, Mesh. Res. Comm., 29, 327-338, (2002).

Received 13.03.2007

Allaberen ASHYRALYEV
Department of Mathematics
Fatih University
34500, B. Çekmece, İstanbul-TURKEY

Mehmet Emir KÖKSAL
Halic University,
Department of Applied Mathematics,
34394 Mecidiyeköy, İstanbul-TURKEY
Department of Mathematics
Gebze Institute of Technology
41400, Gebze, Kocaeli-TURKEY
e-mail: emir_koksal@hotmail.com