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Radical Anti-Invariant Lightlike Submanifolds of Semi-Riemannian Product Manifolds

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Abstract

We introduce radical anti-invariant lightlike submanifolds of a semi Riemannian product manifold and give examples. After we obtain the conditions of integrability of distributions which are involved in the definition of radical anti-invariant lightlike submanifolds, we investigate the geometry of leaves of distributions. We also obtain the induced connection is a metric connection and a radical anti-invariant lightlike submanifold is a product manifold under certain conditions. Finally, we study totally umbilical radical anti-invariant lightlike submanifolds and observe that they are totally geodesic under a condition.

Key Words: Degenerate Metric, Semi-Riemannian Product Manifold, r-Lightlike Submanifold, Locally Riemannian Product

1. Introduction

The geometry of lightlike submanifolds of a semi-Riemannian manifold was presented in [5] (see also in [6]) by K. L. Duggal and A. Bejancu. In [5], they also introduced CR-lightlike submanifolds of indefinite Kaehler as lightlike version of non-degenerate CR-submanifolds. But, they showed that such lightlike submanifolds do not contain invariant and anti-invariant submanifolds contrary to the non-degenerate CR-submanifolds. Therefore, in [8] (see also [9]), K. L. Duggal and B. Sahin introduced screen CR-lightlike submanifolds, and showed that such lightlike submanifolds include invariant lightlike

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submanifolds as well as anti-invariant (screen real) submanifolds of indefinite Kaehler manifolds. They also showed that there are no inclusion relation between CR-lightlike submanifolds and SCR-lightlike submanifolds. Therefore, K. L. Duggal and B. Sahin introduced generalized CR-lightlike submanifolds of indefinite Kaehler manifolds as a generalization of CR-lightlike and SCR-lightlike submanifolds in [10]. It is important to note that radical distribution of lightlike submanifolds mentioned above is invariant under the action of an almost complex structure of a Kaehler manifold. In other words, if we denote an almost complex structure of an indefinite Kaehler manifold by J, then $J(\mathrm{Rad}(TM))$ is a distribution on the submanifold. This tells us that screen real submanifolds of an indefinite Kaehler manifold are not lightlike version of anti-invariant submanifolds. Therefore, in [13], B. Sahin introduced transversal lightlike submanifolds such that $J(Rad(TM))$ is a distribution on transversal bundle of a lightlike submanifold. Then he studied the geometry of such submanifolds. On the other hand, lightlike submanifolds of almost para-Hermitian manifolds were investigated by Bejan in [2]. She mainly studied invariant lightlike submanifodls of para Hermitian manifolds in that paper. As an analogue of CR-lightlike submanifolds, semi-invariant lightlike submanifolds were introduced by M. Atceken and E. Kilic in [1].

Considering above information on lightlike submanifolds of indefinite Kaehler manifolds, similar research is needed for the geometry of lightlike submanifolds of semi-Riemannian product manifolds. Therefore, as a first step, in this paper, we introduce radical anti-invariant lightlike submanifolds of semi-Riemannian product manifolds and study their geometry.

The paper arranged as follows. In Section 2 and Section 3, we summarize basic materials on lightlike submanifolds and semi-Riemannian product manifolds, which will be useful throughout this paper. In Section 4, we introduce radical anti-invariant lightlike submanifolds and give examples. We prove a theorem which shows that the induced connection is a metric connection under some conditions. Then we investigate the geometry of leaves of distributions. We also obtain that a radical anti-invariant lightlike submanifolds is a product manifold of totally lightlike manifold and semi-Riemannian manifold. In Section 5, we study totally umbilical radical anti-invariant lightlike submanifolds and give an example. We also obtain that the induced connection is a metric connection under a new condition. In this section, we observe that radical anti-invariant lightlike submanifolds are foliated by totally lightlike submanifolds and semi-Riemannian manifolds.

2. Lightlike Submanifolds

We follow [5] (see also in [6]) for the notation and formulas used in this paper. Let $(\overline{M}, \overline{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold with index $q > 0$ and M be a submanifold of n-codimension of $\overline{M}$. If $\overline{g}$ is degenerate of the tangent bundle TM on M, then M is called a lightlike submanifold of $\overline{M}$. We denote by g the induced metric of $\overline{g}$ on M. For the degenerate tensor field g on M, there exists locally a vector field $\xi \in \Gamma(TM)$, $\xi \neq 0$, $g(X, \xi) = 0$, for any $X \in \Gamma(TM)$. Then for each tangent space T_xM, $x \in M$, we have

$$T_xM^{\perp} = \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_xM\},$$

which is degenerate n-dimensional subspace of $T_x\overline{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal distributions. In this case, there exists a subspace $\mathrm{Rad}(T_xM) = T_xM \bigcap T_xM^{\perp}$ which is called radical subspace and

$$\mathrm{Rad}(T_xM) = \{\xi_x \in T_xM : g(\xi_x, X) = 0, \forall x \in T_xM\}.$$

The dimension of $\mathrm{Rad}(T_xM)$ depends on $x \in M$. The submanifold M of $\overline{M}$ is said to be r-lightlike submanifold if the mapping

$$\mathrm{Rad}(TM) : x \longrightarrow \mathrm{Rad}(T_xM),$$

defines a smooth distribution on M of $\mathrm{rank}(\mathrm{Rad}(TM)) = r > 0$, where $\mathrm{Rad}(TM)$ is called the radical (null) distribution on M.

Let M be an m-dimensional lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $\overline{M}$ and $\mathrm{rank}(\mathrm{Rad}(TM)) = r$. Then there are four possible cases:

Case 1: r - lightlike if $r < min\{m, n\}$;
Case 2: Co - isotropic if $r = n < m$;
Case 3: Isotropic if $r = m < n$;
Case 4: Totally lightlike if $r = m = n$.

For Case 1, there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to $\mathrm{Rad}(TM)$ in TM. Therefore

$$TM = \mathrm{Rad}(TM) \perp S(TM),$$

where $\perp$ denotes the orthogonally direct sum. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/\mathrm{Rad}(TM)$. Since $S(TM)$ is a non-degenerate vector subbundle of TM in $T\overline{M}|_M$, we can write

$$T\overline{M}|_M = S(TM)\perp S(TM)^{\perp},$$

where $S(TM)^{\perp}$ is the orthogonal complementary vector subbundle to $S(TM)$ in $T\overline{M}|_M$. Denote by $S(TM^{\perp})$ a complementary vector subbundle to $\mathrm{Rad}(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $\ell tr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and to $\mathrm{Rad}(TM)$ in $S(TM)^{\perp}$, respectively. Then we have

$$\begin{aligned} tr(TM) &= \ell tr(TM)\perp S(TM^{\perp}), \\ T\overline{M}|_M &= TM \oplus tr(TM) \\ &= [\mathrm{Rad}(TM)\oplus \ell tr(TM)]\perp S(TM)\perp S(TM^{\perp}), \end{aligned}$$

where $\oplus$ is the direct sum ($\mathrm{Rad}(TM)$ and $\ell tr(TM)$ are not orthogonal each other). As we have seen from above equations, $T\overline{M} = TM \oplus tr(TM)$ and TM and $tr(TM)$ are not orthogonal. Then it follows that the geometry for lightlike submanifolds are different from the semi-Riemannian cases. It is known that the same situation is valid for affine immersions [11]. Thus the methods of affine differential geometry may be useful for the study of lightlike submanifolds.

From the above decomposition of a semi-Riemannian manifold $\bar{M}$ along a lightlike submanifold M, we can consider the following local quasi-orthonormal field of frames of $\overline{M}$ along M:

$$\{X_1, ..., X_{m-r}, \xi_1, ..., \xi_r, N_1, ..., N_r, W_1, ..., W_{n-r}\},$$

where $\{X_1, ..., X_{m-r}\}$ and $\{W_1, ..., W_{n-r}\}$ are orthonormal basis of $\Gamma(S(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively and $\{\xi_1, ..., \xi_r\}$ and $\{N_1, ..., N_r\}$ are lightlike basis of $\Gamma(\mathrm{Rad}(TM))$ and $\Gamma(\ell tr(TM))$, respectively, such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$, for any $i, j \in \{1, ..., r\}$ [5].

The Gauss and Weingarten formulas are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \forall X, Y \in \Gamma(TM), \tag{2.1}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \forall X \in \Gamma(TM), V \in \Gamma(tr(TM)), \tag{2.2}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(ltr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $ltr(TM)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. Using the projections $\ell : tr(TM) \to \ell tr(TM)$ and $s : tr(TM) \to S(TM^{\perp})$, then we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^{\ell}(X,Y) + h^s(X,Y), \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\ell}_X(N) + D^s(X,N), \tag{2.4}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^{s}_X(W) + D^{\ell}(X,W), \quad \forall X,Y \in \Gamma(TM), \tag{2.5}$$

$N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Then, by using (2.1), (2.3)–(2.5) and taking into account that $\overline{\nabla}$ is a metric connection, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^{\ell}(X,W)) = g(A_W X, Y), \tag{2.6}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X). \tag{2.7}$$

Denote the projection of TM on $S(TM)$ by $\bar{P}$, we set

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{2.8}$$

$$\nabla_X \xi = -A^*_{\xi} X + {\nabla^*}^t_X \xi, \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$. By using above equations we obtain

$$\overline{g}(h^{\ell}(X,\bar{P}Y),\xi) = g(A^*_{\xi} X, \bar{P}Y), \tag{2.10}$$

$$\overline{g}(h^*(X,\bar{P}Y),N) = g(A_N X, \bar{P}Y), \tag{2.11}$$

$$\overline{g}(h^{\ell}(X,\xi),\xi) = 0 \quad , \quad A^*_{\xi}\xi = 0. \tag{2.12}$$

In general, the induced connection ∇ on M is not metric connection. Since $\overline{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^{\ell}(X,Y),Z) + \overline{g}(h^{\ell}(X,Z),Y). \tag{2.13}$$

However, it is important to note that $\nabla^{\star}$ is a metric connection on $S(TM)$. We now recall a result which will be useful later.

Theorem 2.1 (*[5] p.161*) *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if* $\mathrm{Rad}(TM)$ *is a parallel distribution with respect to ∇.*

Now we recall the definition of totally umbilical lightlike submanifold [7].

Definition 2.1 *A lightlike submanifold* (M, g) *of a semi-Riemannian manifold* $(\overline{M}, \overline{g})$ *is called totally umbilical in* $\overline{M}$*, if there is a smooth transversal vector field* $\mathcal{H} \in \Gamma(tr(TM))$ *on* M*, called the transversal curvature vector field of* M*, such that, for all* $X, Y \in \Gamma(TM)$*,*

$$h(X, Y) = g(X, Y)\mathcal{H}. \tag{2.14}$$

It is known that M is totally umbilical if and only if on each coordinate neighborhood $\mathcal{U}$, there exist smooth vector fields $\mathcal{H}^\ell \in \Gamma(\ell tr(TM))$ and $\mathcal{H}^s \in \Gamma(S(TM^\perp))$ such that

$$h^\ell(X, Y) = g(X, Y)\mathcal{H}^\ell, \quad h^s(X, Y) = g(X, Y)\mathcal{H}^s, \tag{2.15}$$

for any $X, Y \in \Gamma(TM)$.

For geometries of lightlike submanifolds, hypersurfaces and curves, we refer to [5] and [6].

3. Semi-Riemannian Product Manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2-dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0$, $q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \longrightarrow M_1$ and $\sigma : M_1 \times M_2 \longrightarrow M_2$ the projections which are given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$, respectively. We denote the product manifold by $\overline{M} = (M_1 \times M_2, \overline{g})$, where

$$\overline{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $X, Y \in \Gamma(T\overline{M})$ and $*$ means tangent mapping. Then we have $\pi_*^2 = \pi_*$, $\sigma_*^2 = \sigma_*$, $\pi_*\sigma_* = \sigma_*\pi_* = 0$ and $\pi_* + \sigma_* = I$, where I is identity transformation. Thus $(\overline{M}, \overline{g})$ is an $(m_1 + m_2)$-dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$ is characterized by M_1 and M_2 are totally geodesic submanifolds of $\overline{M}$.

Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$ and

$$\overline{g}(FX, Y) = \overline{g}(X, FY), \tag{3.1}$$

for any $X, Y \in \Gamma(T\overline{M})$. If we denote the Levi-Civita connection on $\overline{M}$ by $\overline{\nabla}$, then it can be seen that

$$(\overline{\nabla}_X F)Y = 0, \tag{3.2}$$

for any $X, Y \in \Gamma(T\overline{M})$, that is, F is parallel with respect to $\overline{\nabla}$ [16].

Let M be a submanifold of a Riemannian (or semi-Riemannian) product manifold $\overline{M} = M_1 \times M_2$. If $F(TM) = TM$, then M is called an invariant submanifold, if $F(TM) \subset TM^{\perp}$, then M is called an anti-invariant submanifold [17].

4. Radical Anti-Invariant Lightlike Submanifolds

In this section, we introduce radical anti-invariant lightlike submanifolds of semi-Riemannian product manifolds, give examples and study the geometry of leaves of distributions which are involved in the definition of radical anti-invariant lightlike submanifolds.

Definition 4.1 *Let M be a lightlike submanifold of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$. We say that M is a radical anti-invariant lightlike submanifold if $F(\mathrm{Rad}(TM)) = \ell tr(TM)$.*

Moreover, we say that a radical anti-invariant submanifold is proper if there exists a subbundle $D' \subset S(TM)$ such that D' is anti-invariant with respect to F, i.e. $F(D') \subset S(TM^{\perp})$ and $D' \neq S(TM)$.

Now, we denote the orthogonal complementary to D' in $S(TM)$ by D_0. Thus we have the decompositions

$$TM = D_0 \oplus D, \quad S(TM) = D_0 \oplus D', \; D = \mathrm{Rad}(TM) \oplus D'. \tag{4.1}$$

Similarly, if we denote the orthogonal complementary to $F(D')$ in $S(TM^{\perp})$ by L, we have

$$S(TM^{\perp}) = F(D') \perp L.$$

Since $S(TM)$is non-degenerate, for any $X \in \Gamma(D_0)$, we have

$$\overline{g}(FX, Z) = \overline{g}(X, FZ) = 0, \; \forall Z \in \Gamma(D'),$$

and

$$\overline{g}(FX, N) = \overline{g}(X, FN) = 0, \ \forall N \in \Gamma(\ell tr(TM)),$$

due to $FN \in \Gamma(\mathrm{Rad}(TM))$. Similarly, we get

$$\overline{g}(FX, \xi) = \overline{g}(X, F\xi) = 0, \ \forall \xi \in \Gamma(\mathrm{Rad}(TM))$$

and

$$\overline{g}(FX, W) = \overline{g}(X, FW) = 0, \ \forall W \in \Gamma(S(TM^{\perp})).$$

Hence we conclude that D_0 is an invariant distribution with respect to F. Similarly, it is easy to check that, L is an invariant distribution with respect to F.

Proposition 4.1 *There exists no proper radical anti-invariant co-isotropic, isotropic or totally lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$.*

Proof. Suppose that M is a radical anti-invariant co-isotropic submanifold of $\overline{M}$. Then $S(TM^{\perp}) = \{0\}$ implies that $D' = \{0\}$. This proves our assertion. The other assertions can be proved in a similar way. □

Example 4.1 *Let $\overline{M} = \mathbb{R}^4_2 \times \mathbb{R}^4_2$ be a semi-Riemannian product manifold with semi-Riemannian product metric tensor $\overline{g} = \pi^* g_1 \otimes \sigma^* g_2, \quad i = 1, 2,$ where g_i denote standard metric tensors of $\mathbb{R}^4_2$. Consider a submanifold M of $\overline{M}$ is given by equations*

$$x_5 = x_3, \quad x_6 = \sqrt{2}\, x_1, \quad x_7 = x_2, \quad x_8 = x_1.$$

Then the tangent bundle TM is spanned by

$$\{U_1 = \frac{\partial}{\partial x_1} + \sqrt{2}\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8}, \quad U_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_7}, \quad U_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \quad U_4 = \frac{\partial}{\partial x_4}\}.$$

It follows that $\mathrm{Rad}(TM)$ *is spanned by $\{\xi_1 = U_2, \quad \xi_2 = U_3\}$ and $S(TM)$ is spanned by $\{U_1, \quad U_4\}$. If we choose*

$$V_1 = 2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_7}, \quad V_2 = \frac{\partial}{\partial x_3} + 2\frac{\partial}{\partial x_5},$$

then using the technics in [5], we get transversal vector fields as

$$N_1 = -\frac{1}{2}(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_7}), \quad N_2 = \frac{1}{2}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}).$$

Furthermore, $S(TM^{\perp})$ *is spanned by*

$$\{W_1 = \frac{\partial}{\partial x_1} - \sqrt{2}\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8}, \quad W_2 = \frac{\partial}{\partial x_6} + \sqrt{2}\frac{\partial}{\partial x_8}\}.$$

Then it is easy to see that $F(\mathrm{Rad}(TM)) = \ell tr(TM)$ *and* $F(D') \subset S(TM^{\perp})$*, where* $D' = Span\{U_1\}$*. Moreover, it follows that* $D_0 = Span\{U_4\}$ *is invariant. Thus* M *is a radical anti-invariant lightlike submanifold.*

We give another example in $\mathbb{R}^8_4$.

Example 4.2 *Let* M' *be a submanifold of* $\mathbb{R}^8_4 = \mathbb{R}^4_2 \times \mathbb{R}^4_2$ *given by*

$$x_i = u_i, i = 1, 2, 3, 4, \; x_5 = u_1 \sinh\theta, \; x_6 = \cosh u_3, \; x_7 = \sinh u_3, \; x_8 = u_1 \cosh\theta.$$

Then the tangent bundle TM' *is spanned by* Z_1, Z_2, Z_3 *and* Z_4*, where*

$$Z_1 = \frac{\partial}{\partial x_1} + \sinh\theta\frac{\partial}{\partial x_5} + \cosh\theta\frac{\partial}{\partial x_8}, \quad Z_2 = \frac{\partial}{\partial x_2},$$
$$Z_3 = \frac{\partial}{\partial x_4}, \quad Z_4 = \frac{\partial}{\partial x_3} + \sinh u_3\frac{\partial}{\partial x_6} + \cosh u_3\frac{\partial}{\partial x_7}.$$

Thus M' *is a 1-lightlike submanifolds with* $\mathrm{Rad}(TM') = Span\{Z_1\}$*. By direct computations, we obtain screen transversal bundle and the lightlike transversal bundle* $S(TM^{\perp}) = Span\{W_1, W_2, W_3\}$ *and* $\ell tr(TM) = Span\{N\}$*, respectively, where*

$$\begin{aligned} W_1 &= -\frac{\partial}{\partial x_3} + \sinh u_3\frac{\partial}{\partial x_6} + \cosh u_3\frac{\partial}{\partial x_7}, \quad W_2 = \cosh u_3\frac{\partial}{\partial x_6} + \sinh u_3\frac{\partial}{\partial x_7}, \\ W_3 &= \cosh\theta\frac{\partial}{\partial x_5} + \sinh\theta\frac{\partial}{\partial x_8}, \quad N = \frac{1}{2}\{-\frac{\partial}{\partial x_1} + \sinh\theta\frac{\partial}{\partial x_5} + \cosh\theta\frac{\partial}{\partial x_8}.\end{aligned}$$

Then it is easy to see that $FZ_1 = N$ *which implies* $F\mathrm{Rad}(TM') = \ell tr(TM')$*. We can see that* $D_0 = Span\{Z_2, Z_3\}$ *and* $FZ_4 = W_1$*, which shows that* D_0 *is invariant and* $D' = Span\{Z_4\}$ *is anti-invariant. Thus* M' *is a radical anti-invariant lightlike submanifold of* $\mathbb{R}^8_4$*.*

It is easy to see that M is a totally geodesic and M' is neither totally geodesic nor totally umbilical lightlike submanifold of $\mathbb{R}^8_4$ in the above examples. We now give an example which is a radical anti-invariant lightlike submanifold of a non-flat semi-Riemannian manifold.

Example 4.3 *Let* $(\overline{M}, \overline{g}) = (\mathbb{R}^2_1 \times S^2_1, \pi * \overline{g}_1 + \sigma * \overline{g}_2)$ *be semi-Riemannian product manifold, where* $\mathbb{R}^2_1$ *is the semi-Euclidean plane of the signature* $(-,+)$ *with respect to canonical basis* $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ *and* S^2_1 *is the unit pseudo sphere of Minkowski space* $\mathbb{R}^3_1$ *of the signature* $(-,+,+)$ *with respect to canonical basis* $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$*. Also* $\overline{g}_1$ *and* $\overline{g}_2$ *are inner products of* $\mathbb{R}^2_1$ *and* $\mathbb{R}^3_1$*, respectively. Consider a submanifold of* $\overline{M}$ *given by*

$$x_1 = \arcsin x, \; y = \sqrt{2}\, x, \; z = \sqrt{1-x^2}.$$

The tangent bundle TM *is spanned by*

$$U_1 = \frac{\partial}{\partial x_1} + \sqrt{1-x^2}\frac{\partial}{\partial x} + \sqrt{2}\sqrt{1-x^2}\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \; U_2 = \frac{\partial}{\partial x^2}.$$

It follows that $\mathrm{Rad}(TM)$ *is spanned by* $\{\xi = U_1\}$ *and* $S(TM)$ *is spanned by* $\{U_2\}$*. Hence,* M *is a 1-lightlike submanifold of* $\overline{M}$ *and* $\ell tr(TM)$ *is spanned by*

$$N = -\frac{1}{2}\{\frac{\partial}{\partial x_1} - \sqrt{1-x^2}\frac{\partial}{\partial x} + \sqrt{2}\sqrt{1-x^2}\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\}.$$

Furthermore, $S(TM^{\perp})$ *is spanned by*

$$W = \sqrt{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Thus, $\{\xi, U_2, N, W\}$ *is a basis of* $T\overline{M}$*. Then it is easy that* $F\mathrm{Rad}(TM) = \ell tr(TM)$*,* $D_0 = Span\{U_2\}$ *is invariant,* $D' = \{0\}$ *and* $L = Span\{W\}$ *is invariant with respect to* F*. Thus* M *is a radical anti-invariant lightlike submanifold.*

Let M be a radical anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then, for any $X \in \Gamma(TM)$, we can write

$$FX = fX + \omega X, \tag{4.2}$$

where $fX \in \Gamma(D_0)$ and $\omega X \in \Gamma(tr(TM))$. Similarly, for any $V \in \Gamma(tr(TM))$, we can write

$$FV = BV + CV, \tag{4.3}$$

where $BV \in \Gamma(TM)$ and $CV \in \Gamma(L)$.

Now, we denote the projections on D_0, D', $\mathrm{Rad}(TM)$ in TM by Q_1, Q_2, Q_3, respectively. Then, for any $X \in \Gamma(TM)$, we have

$$FX = FQ_1X + FQ_2X + FQ_3X, \tag{4.4}$$

where $FQ_1X \in \Gamma(D_0)$, $FQ_2X \in \Gamma(S(TM^\perp))$ and $FQ_3X \in \Gamma(\ell tr(TM))$. Hence we have

$$FQ_1X = fX, \quad FQ_2X = \omega Q_2X, \quad FQ_3X = \omega Q_3X.$$

In a similar way, we denote the projections on $S(TM^\perp)$ and $\ell tr(TM)$ in $tr(TM)$ by P_1 and P_2, respectively. Then, we obtain

$$FV = BP_1V + CP_1V + FP_2V, \tag{4.5}$$

for $V \in \Gamma(tr(TM))$, where $BP_1V \in \Gamma(D')$, $CP_1V \in \Gamma(L)$ and $FP_2V \in \Gamma(\mathrm{Rad}(TM))$.

It is known that the induced connection ∇ is not a metric connection, in general. Next theorem gives necessary ant sufficient conditions for the induced connection to be a metric connection.

Theorem 4.1 *Let M be a radical anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if*

$$A_{F\xi}X \in \Gamma(D') \text{ and } D^s(X, F\xi) \in \Gamma(L),$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$.

Proof. From (3.2), we have $\overline{\nabla}_X FY = F\overline{\nabla}_X Y$, for any $X, Y \in \Gamma(TM)$. Then using (2.3), (2.4), (4.2) and (4.3), we get

$$-A_{F\xi}X + \nabla^\ell_X F\xi + D^s(X, F\xi) = F\nabla_X\xi + Fh^\ell(X,\xi) + Bh^s(X,\xi) + Ch^s(X,\xi)$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$. Applying F to this equation and using (4.2) and (4.3), we have

$$-fA_{F\xi}X + F\nabla^\ell_X F\xi + BD^s(X, F\xi) = \nabla_X\xi.$$

Thus, $\nabla_X\xi \in \Gamma(\mathrm{Rad}(TM))$ if and only if

$$A_{F\xi}X \in \Gamma(D') \text{ and } D^s(X, F\xi) \in \Gamma(L).$$

Then our assertion comes from Theorem 2.1. □

Now, using (4.3), (4.4), (4.5) and taking the tangential and transversal (resp., lightlike transversal and screen transversal) parts, we get

$$(\nabla_X FQ_1)Y = A_{FQ_2Y}X + A_{FQ_3Y}X + Fh^\ell(X,Y) + Fh^s(X,Y), \tag{4.6}$$

$$FQ_3\nabla_X Y = h^\ell(X, FQ_1Y) + \nabla^\ell_X FQ_3Y + D^\ell(X, FQ_2Y) \tag{4.7}$$

$$h^s(X, FQ_1Y) + \nabla^s_X FQ_2Y + D^s(X, FQ_3Y) = FQ_2\nabla_X Y + Ch^s(X,Y), \tag{4.8}$$

for any $X, Y \in \Gamma(TM)$.

In the rest of this section, we investigate the geometry of the distributions D_0 and D. First we have the following theorem.

Theorem 4.2 *Let M be a radical anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the distribution D_0 integrable if and only if for any $X, Y \in \Gamma(D_0)$*

$$h(X, FY) = h(Y, FX).$$

Proof. For $X, Y \in \Gamma(D_0)$, from (4.7), we obtain

$$h^\ell(X, FY) = FQ_3\nabla_X Y.$$

Hence we obtain

$$h^\ell(X, FY) - h^\ell(Y, FX) = FQ_3[X, Y]. \tag{4.9}$$

In similar way, from (4.8), we get

$$h^s(X, FY) = FQ_2\nabla_X Y + Ch^s(Y, X).$$

Thus interchanging role of X and Y, we derive

$$h^s(X, FY) - h^s(Y, FX) = FQ_2[X, Y]. \tag{4.10}$$

Then the proof follows from (4.9) and (4.10). □

Theorem 4.3 *Let M be a radical anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then*

a) The distribution D is integrable if and only if

$$A_{\omega X}Y = A_{\omega Y}X$$

for any $X, Y \in \Gamma(D)$.

a) The distribution D defines a totally geodesic foliation in M if and only if $A_{\omega Y}X \in \Gamma(D)$, for any $X \in \Gamma(TM)$, $Y \in \Gamma(D)$.

Proof.

a) For any $X, Y \in \Gamma(D)$, from (2.1), (2.2) and (3.2), we obtain

$$F(\nabla_X Y + h(X, Y)) = -A_{\omega Y}X + \nabla^t_X FY.$$

Taking the tangential part of this equation, we have

$$f\nabla_X Y + Bh(X, Y) = -A_{\omega Y}X.$$

Interchanging roles of X and Y, we can write

$$f\nabla_Y X + Bh(X, Y) = -A_{\omega X}Y.$$

From this last equations, we obtain

$$f[X, Y] = A_{\omega X}Y - A_{\omega Y}X.$$

Thus $[X, Y] \in \Gamma(D)$ if and only if $A_{\omega X}Y = A_{\omega Y}X$.

b) We will show that $g(\nabla_X Y, FZ) = 0$, for any $X \in \Gamma(TM)$, $Y \in \Gamma(D)$ and $Z \in \Gamma(D_0)$. Since $g(\nabla_X Y, FZ) = \overline{g}(\overline{\nabla}_X Y, FZ) = 0$, from (2.2), we have

$$g(\nabla_X Y, FZ) = -g(A_{\omega Y}X, Z).$$

Thus we have the assertion (b). □

Theorem 4.4 *Let M be a radical anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. The distribution D_0 defines a totally geodesic foliation in M if and only if $h(X, Y) \in \Gamma(L)$, for any $X, Y \in \Gamma(D_0)$.*

Proof. For any $X, Y \in \Gamma(D_0)$, $Z \in \Gamma(D')$ and $N \in \Gamma(\ell tr(TM))$, we have

$$g(\nabla_X FY, Z) = \overline{g}(h^s(X,Y), FZ), \quad g(\nabla_X FY, N) = \overline{g}(h^\ell(X,Y), FN).$$

Thus we have the assertion of this theorem. □

Theorem 4.5 *Let M be a radical anti-invariant r-lightlike submanifold of semi-Riemannian product manifold $(\overline{M}, \overline{g})$. Then M is a locally product manifold if and only if f is parallel with respect to induced connection ∇, that is, $\nabla f = 0$.*

Proof. We suppose that M is a locally product manifold. Then we have the leaves of the distributions of D_o and D are totally geodesic in M. Thus, $\nabla_Z fX \in \Gamma(D_0)$, for any $Z \in \Gamma(TM)$ and $X \in \Gamma(D_0)$. Furthermore, for any $X \in \Gamma(D_0)$, $FQ_2X = 0$ and $FQ_3X = 0$. From (4.6), we have

$$(\nabla_Z f)X = Fh^\ell(Z,X) + Fh^s(Z,X).$$

Since $(\nabla_Z f)X \in \Gamma(D_0)$, we get

$$(\nabla_Z f)X = 0.$$

For $Y \in \Gamma(D)$ and $Z \in \Gamma(TM)$, from (4.6) we have

$$(\nabla_Z f)X = A_{FQ_2Y}Z + A_{FQ_3Y}Z + Fh^\ell(Z,X) + Fh^s(Z,X).$$

Since $fY = 0$, for $Y \in \Gamma(D)$, we get

$$-f\nabla_Z Y = A_{FQ_2Y}Z + A_{FQ_3Y}Z.$$

From Theorem 4.4, we get $f\nabla_Z Y = 0$. Thus we have

$$(\nabla_Z f)Y = 0.$$

Conversely, let us suppose that $\nabla f = 0$. Then we have

$$f\nabla_X Y = \nabla_X fY,$$

for any $X, Y \in \Gamma(D_o)$ and

$$f\nabla_Z W = \nabla_Z fW = 0,$$

for any $Z, W \in \Gamma(D)$. Thus, it follows that $\nabla_X fY \in \Gamma(D_0)$ and $\nabla_Z W \in \Gamma(D)$, respectively. Hence, we conclude that leaves of the distributions D_o and D are totally geodesic in M. This completes the proof of the Theorem. □

5. Totally Umbilical Radical Anti-Invariant Lightlike Submanifolds

Now, we study totally umbilical radical anti-invariant lightlike submanifolds of a semi-Riemannian product manifold. Firstly, we give an example for a totally umbilical radical anti-invariant lightlike submanifold.

Example 5.1 *Let $\overline{M} = \mathbb{R}^4_2 \times \mathbb{R}^3_1$ be a semi-Riemannian product manifold with semi-Riemannian metric tensor $\overline{g} = \pi^* g_1 + \sigma^* g_2$, where g_1 and g_2 denote standard metric tensors of $\mathbb{R}^4_1$ and $\mathbb{R}^3_1$, respectively. Consider in $\overline{M}$ a submanifold M given by the equations with $x_2 \neq 0$,*

$$x_4 = \arcsin x_2, \quad x_5 = x_3, \quad x_6 = x_1, \quad x_7 = \sqrt{1 - x_2^2}.$$

Then TM is spanned by

$$U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_6}, \quad U_2 = \sqrt{1 - x_2^2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} - x_2\frac{\partial}{\partial x_7}, \quad U_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}.$$

Thus M is a 2-lightlike submanifold with $\mathrm{Rad}(TM) = Span\{\xi_1 = U_1, \xi_2 = U_3\}$. *Also, $S(TM) = Span\{U_2\}$ and $S(TM^\perp) = Span\{W_1, W_2\}$, where*

$$W_1 = \sqrt{1 - x_2^2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_7}, \quad W_2 = \frac{\partial}{\partial x_2} + \sqrt{1 - x_2^2}\frac{\partial}{\partial x_4}.$$

Hence $S(TM) = D'$, $D_0 = \{0\}$. By direct computation the lightlike transversal bundle $\ell tr(TM)$ is spanned by

$$N_1 = -\frac{1}{2}\frac{\partial}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_6}, \quad N_2 = \frac{1}{2}\frac{\partial}{\partial x_3} - \frac{1}{2}\frac{\partial}{\partial x_5}.$$

Then it is easy see $F\xi_1 = -2N_1$, $F\xi_2 = 2N_2$ and $FU_2 = W_1$. Thus M is a radical anti-invariant lightlike submanifold of $\overline{M}$. By direct calculations, we obtain

$$\overline{\nabla}_X \xi_1 = \overline{\nabla}_X \xi_2 = 0,$$

for any $X \in \Gamma(TM)$ and

$$\overline{\nabla}_{U_2} U_2 = -\frac{\sqrt{1 - x_2^2}}{x_2} U_2 + \frac{1}{x_2} W_2.$$

Then using Gauss formula, we have $h^\ell(U_2, U_2) = 0$ and $h^s(U_2, U_2) = g(U_2, U_2)\mathcal{H}^s$, where $\mathcal{H}^s = \frac{1}{2x_2^3} W_2$. Thus M is a totally umbilical radical anti-invariant lightlike submanifold.

Now, let M be a totally umbilical radical anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. For any $X, Y \in \Gamma(D_0)$, from (2.3), we have

$$h^\ell(X, FY) + h^s(X, FY) = \omega \nabla_X Y + Ch^s(X, Y).$$

From (4.4), we have

$$h^s(X, FY) = FQ_2 \nabla_X Y + Ch^s(X, Y),$$

$$h^\ell(X, FY) = \omega Q_3 \nabla_X Y,$$

or

$$g(X, FY)H^\ell = \omega Q_3 \nabla_X Y. \tag{5.1}$$

For $X = FY$, we get

$$g(Y, Y)H^\ell = \omega Q_3 \nabla_{FY} Y.$$

For $Z \in \Gamma(D_0)$, we have

$$\nabla_Z FZ = fQ_1 \nabla_Z Z + Fh^\ell(Z, Z) + Bh^s(Z, Z), \tag{5.2}$$

$$h^s(Z, FZ) = FQ_2 \nabla_Z Z + Ch^s(Z, Z). \tag{5.3}$$

Corollary 5.1 *Let M be a totally umbilical radical anti-invariant submanifold of a semi-Riemann product manifold $\overline{M}$. Then the induced connection is a metric connection if and only if $h^*(X, Y) = 0$, for $X, Y \in \Gamma(D_0)$.*

Proof. If the induced connection ∇ is a metric connection, then from Theorem 2.2 in page 159 of [5], $h^\ell = 0$. Hence using (5.1) we get $\omega Q_3 \nabla_X Y = 0$, that is $\nabla_X Y \in \Gamma(S(TM))$. Thus we have $h^*(X, Y) = 0$, this implies $\nabla_X Y \in \Gamma(S(TM))$ for $X, Y \in \Gamma(D_0)$. Thus proof follows from (5.1). □

Corollary 5.2 *Let M be a totally umbilical proper radical anti-invariant submanifold of a semi-Riemannian product manifold such that $\dim D_0 > 2$. If f is paralel then M is totally geodesic.*

Proof. If f is parallel, then from (5.2), we have

$$\nabla_Z fZ = fQ_1\nabla_Z Z + Fh^\ell(Z,Z) + Bh^s(Z,Z),$$

for any $Z \in \Gamma(D_0)$. Hence

$$Fh^\ell(Z,Z) = 0, \quad Bh^s(Z,Z) = 0.$$

Since F is non-singular and D_0 is non-degenerate, $\mathcal{H}^\ell = 0$ and $\mathcal{H}^s \in \Gamma(L)$. On the other hand, if $(\nabla_X f)Y = 0$, then $\nabla_X fY = fQ_1\nabla_X Y$. Thus we have $\nabla_X fY \in \Gamma(D_0)$. Hence D defines a totally geodesic foliation. Thus from (5.3), we have

$$h^s(X,Y) = Ch^s(X,Y).$$

Since $\dim(D_0) > 2$, we can choose orthonormal vector fields X and FY such that $g(X,Y) \neq 0$. Then we obtain

$$Ch^s(X,Y) = 0, \quad C\mathcal{H}^s = 0, \quad \mathcal{H}^s \in \Gamma(FD').$$

Since $FD' \cap L = \{0\}$, we have $\mathcal{H}^s = 0$. This implies M is totally geodesic. □

Theorem 5.1 *Let M be a totally umbilical radical anti-invariant r-lightlike submanifold of semi-Riemannian product manifold $\overline{M}$. Then the radical distribution* $\mathrm{Rad}(TM)$ *is always integrable.*

Proof. From (2.3) we have,

$$g([\xi,\xi'],Z) = \overline{g}(\overline{\nabla}_\xi \xi', Z) - \overline{g}(\overline{\nabla}_{\xi'}\xi, Z).$$

for all $\xi, \xi' \in \Gamma(\mathrm{Rad}(TM))$ and $Z \in \Gamma(D_0)$. Using (3.1), we get

$$g([\xi,\xi'],Z) = -\overline{g}(F\xi', \overline{\nabla}_\xi FZ) + \overline{g}(F\xi, \overline{\nabla}_{\xi'} FZ).$$

Hence we obtain

$$g([\xi,\xi'],Z) = -\overline{g}(F\xi', h^\ell(\xi, FZ)) + \overline{g}(F\xi, h^\ell(\xi', FZ)).$$

Since $h^\ell(\xi, FZ) = g(\xi, FZ)\mathcal{H}^\ell = 0$, we have

$$g([\xi,\xi'],Z) = 0.$$

In similar way, for any $Y \in \Gamma(D')$, we have

$$\begin{aligned} g([\xi,\xi'],Y) &= \overline{g}(\overline{\nabla}_{\xi}\xi',Y) - \overline{g}(\overline{\nabla}_{\xi'}\xi,Y) \\ &= -\overline{g}(\xi^{'},\overline{\nabla}_{\xi}Y) + \overline{g}(\xi,\overline{\nabla}_{\xi'}Y) \\ &= -\overline{g}(\xi',h^{\ell}(\xi,Y)) + \overline{g}(\xi,h^{\ell}(\xi',Y) \\ &= 0. \end{aligned}$$

This completes the proof. □

Theorem 5.2 *Let M be a totally umbilical radical anti-invariant r-lightlike submanifold of semi-Riemannian product manifold $\overline{M}$. Then the screen distribution $S(TM)$ is always integrable.*

Proof. For any $N \in \Gamma(\ell tr(TM))$, then there exists a $\xi \in \Gamma(\mathrm{Rad}(TM))$ such that $N = F\xi$. For any $Y \in \Gamma(S(TM))$, from (4.4), we have $FY = fY + \omega Q_2 Y$, where $fY \in \Gamma(D_0)$ and $\omega Q_2 Y \in \Gamma(FD')$. Since $\overline{\nabla}$ is a Levi-Civita connection, from (2.3) we get

$$\overline{g}(\overline{\nabla}_X \omega Q_2 Y, \xi) = -\overline{g}(\omega Q_2 Y, h(X,\xi)),$$

for any $X, Y \in \Gamma(S(TM))$. From (2.15) we have $h(X,\xi) = 0$. Thus we have

$$\overline{g}(\overline{\nabla}_X \omega Q_2 Y, \xi) = 0. \tag{5.4}$$

Therefore, from (5.4) we get

$$\overline{g}(\overline{\nabla}_X Y, F\xi) = \overline{g}(h^{\ell}(X, fY), \xi). \tag{5.5}$$

Thus from (2.3), (2.15) and (5.5) we obtain

$$\begin{aligned} \overline{g}([X,Y],N) &= \overline{g}([X,Y],F\xi) \\ &= \overline{g}((g(X,fY) - g(fX,Y))\mathcal{H}^{\ell},\xi). \end{aligned}$$

Since $g(fX,Y) = g(X,fY)$, we have

$$\overline{g}([X,Y],N) = 0,$$

which proves our assertion.

□

Remark 5.1 *From Theorem 5.1 and 5.2, it follows that a totally umbilical proper radical anti-invariant lightlike submanifold is foliated by a totally lightlike manifold and a semi-Riemannian manifold.*

6. Conclusion

An important class of submanifolds of complex manifolds in Riemannian geometry is anti-invariant submanifolds. One can see from the definition of a radical anti-invariant lightlike submanifold of a semi-Riemannian product manifold, such lightlike submanifolds are a lightlike version of anti-invariant submanifolds of Riemannian product manifolds. Thus, this new class has potential for further research. Let us mention about one of our next research problems. First, we note that warped product manifolds ([12]) are defined as follows: Let (M_1, g_1) and (M_2, g_2) be two semi-Riemannian manifolds with index q_1 and q_2, respectively, and $f : M_1 \to (0, \infty)$ is a warping function. The warped product $\overline{M} = M_1 \times_f M_2$ is a product manifold $M_1 \times M_2$, endowed with the metric $\overline{g} = \pi * g_1 + (f \circ \pi)^2 \sigma * g_2$, where π and σ are projections on M_1 and M_2, respectively. Thus $(\overline{M}, \overline{g})$ is a semi-Riemannian manifold with index $q_1 + q_2$. It is known that some spacetimes in physic can be modelled as a warped product manifold. In [4], K. L. Duggal introduced warped product lightlike manifolds and gave some interesting examples of such lightlike manifolds. In [14], B. Sahin consider warped product lightlike submanifolds and showed that such lightlike submanifolds have some nice geometric properties, he also gave some examples for those submanifolds. On the other hand, B.Y. Chen introduced CR-warped product submanifolds of Kaehler manifolds and obtained an inequality for squared norm of the second fundamental form in terms of warping function in [3]. Warped product semi-invariant submanifolds which are analogue of CR-warped product submanifolds were also studied in [15] by B. Sahin. Considering the definitions of radical anti-invariant lightlike submanifolds, warped product lightlike manifolds (in the sense of K. L. Duggal) and warped product lightlike submanifolds, one can conclude that similar submanifolds (lightlike version of warped product semi-invariant submanifolds) can be defined and studied for radical anti-invariant lightlike submanifolds.

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