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On the Distribution of Random Dirichlet Series in the Whole Plane

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Abstract

For some random Dirichlet series of order(R) infinite almost surely, every horizontal line is a strong Borel line of order(R) infinite and without exceptional Little functions.

Key Words: Random Dirichlet series, Order(R), strong Borel line, little function.

1. Preliminaries

For random Dirichlet-Rademacher, Steinhaus and N series of order(R) infinite almost surely (a.s.), it was proved that a.s. every horizontal line is a Borel line of order(R) infinite and with a possible exceptional value [10], [11]. Later, in [12], by generalized Paley-Zygmund lemma in [8], it is proved that for more general random Dirichlet series of order(R) infinite a.s. every horizontal line is a Borel line of order(R) infinite and without exceptional values. In this paper, we replay exceptional values by exceptional Little functions, and prove that for the random Dirichlet series of order(R) infinite a.s., every horizontal line is a strong Borel line of order(R) infinite and without exceptional Little functions. Our method can be applied to study some random Dirichlet series of generalized Orders(R) as, [1], [5], [11], [13].

The books [2], [3], [9] are very enlightening and helpful in the related research.

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Consider random Dirichlet series

$$f(s,\omega)=\sum_{n=0}^{+\infty}a_nZ_n(\omega)e^{-\lambda_ns}, \tag{1.1}$$

and an associated Dirichlet series

$$g(s)=\sum_{n=0}^{+\infty}a_ne^{-\lambda_n}, \tag{1.2}$$

where $\{a_n\}\subset \mathrm{C}, s=\sigma+it\in \mathrm{C}, 0\le\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_n\nearrow+\infty$.

$$\overline{\lim_{n\to+\infty}}\frac{\ln n}{\lambda_n}<+\infty,\ \overline{\lim_{n\to+\infty}}\frac{\ln|a_n|}{\lambda_n}=-\infty, \tag{1.3}$$

$$\overline{\lim_{\sigma\to-\infty}}\frac{\ln^+\ln^+M_g(\sigma)}{-\sigma}=+\infty\quad(\sigma\in R), \tag{1.4}$$

$$M_g(\sigma)=\sup\{|g(\sigma+it)|\big|t\in R\}, \ln^+u=\begin{cases}\ln u & : \ of \ \ u\ge1,\\ \ \ 0 & : \ of \ \ u<1\end{cases}$$

and in the probability space $(\Omega,\mathcal{A},P),\{Z_n(\omega)\}(\omega\in\Omega)$ is a sequence of non-degenerate, symmetric and independent random variables of the same distribution and verifying

$$0<E(|Z_n(\omega)|^2)<+\infty, \tag{1.5}$$

and consequently

$$0<E(|Z_n(\omega)|)=d<+\infty. \tag{1.6}$$

Theorem 1.1 *If series* (1.1) *satisfies all the above conditions, then* $f(s,\omega)$ *is an entire function of order(R) infinite and it is almost sure (a.s.) that a.s. every horizontal line* $\{s|\mathrm{Im}s=t_0\}(t_0\in R)$ *is a strong Borel line of* $f(s,\omega)$ *of order(R) infinite and without exceptional Little functions, i.e.*$\exists A\in\mathcal{A}(P(A)=1)$ *such that* $\forall\omega\in A$, (1.4) *holds and that* $\forall\omega\in A,\forall t_0\in R,\forall\eta>0$ *and* $\forall\varphi\in H$

$$\overline{\lim_{\sigma\to-\infty}}\frac{\ln^+n(\sigma,t_0,\eta,f(s,\omega)=\varphi(s))}{-\sigma}=+\infty, \tag{1.7}$$

where

$$n(\sigma,t_0,\eta,f(s,\omega)=\varphi(s))=\sharp\{s|f(s,\omega)=\varphi(s),s\in B^*(\sigma,t_0,\eta),\varphi\in H\},$$

$$B^*(\sigma, t_0, \eta) = \{s|\mathrm{Re}s \geq \sigma\} \cap B(t_0, \eta),$$

$$B(t_0, \eta) = \{s||\mathrm{Im}s - t_0| < \eta\},$$

$$H = \left\{ \varphi = \sum_{n=0}^{+\infty} \beta_n e^{-\lambda_n s} \middle| \overline{\lim_{n\to+\infty}} \frac{\ln n}{\lambda_n} < +\infty, \overline{\lim_{n\to+\infty}} \frac{\ln |a_n|}{\lambda_n} = -\infty, M_\varphi(\sigma) = \circ(M_f(\sigma))(\sigma \to -\infty) \right\}.$$

2. Lemmas

In order to prove Theorem 1.1 we need some lemmas.

Lemma 2.1 *Under condition* (1.3)*, series* (1.2) *converges absolutely in C. Condition* 1.4 *indicates the entire function g(s) is of order(R) infinite and*

$$(1.4) \Leftrightarrow \overline{\lim_{n\to+\infty}} \frac{\ln^+ |a_n|}{\lambda_n \ln \lambda_n} = 0. \tag{2.8}$$

Proof of the lemma is stated in [11], [12].

The following is an extension of Nevanlinna second theorem in [4] in special case (see [5], [6], [12]):

Lemma 2.2 *Let* $G(w)$ *and* $g_j(w)(j = 1, 2)$*be holomorphic in* $D(1)$ *and satisfy the limit*

$$\lim_{R\to1} \frac{\ln^+ T(R, G(w))}{-\ln(1 - R)} = +\infty \tag{2.9}$$

and

$$T(R, g_j(w)) = \circ(T(R, G(w)))(R \to 1). \tag{2.10}$$

Then

$$T(R, R(w)) \leq 3\sum_{j=1}^{2} N(\frac{R+1}{2}, G(w) = g_j(w)) + 6\sum_{j=1}^{2} T(\frac{R+1}{2}, g_j(w)) + A \ln(1-R)^{-1} + B, \tag{2.11}$$

where $t_0 \in R$ *and* A *and* B *are positive constants.*

Given $t_0 \in R$ and $\eta > 0$, consider the simple mapping

$$z = \phi_1(s) = \exp[-\frac{\pi}{2\eta}(s - it_0)], \quad w = \phi_2(z) = \frac{z-1}{z+1}. \tag{2.12}$$

Denote the inverse mappings by $s = \Phi_1(z)$ and $z = \Phi_2(w)$ and let

$$w = \phi(s) = \phi_2 \circ \phi_1(s), s = \Phi(w) = \Phi_1 \circ \Phi_2(w),$$

$$H_1 = \{z || \arg z| < \frac{\pi}{2}\}, H_2 = \{z || \arg z| < \frac{\pi}{4}\},$$

$$H^*(r) = \{z||z| \leq r\} \cap H_k(k = 1, 2), D(R) = \{w||w| < R\}(R \in (0, 1]).$$

Then $\Phi(D(1)) = B(t_0, \eta)$; and we have the following lemma [7], [12], [13].

Lemma 2.3 *For* $R \in (0, 1)$*, let*

$$r = \frac{1+R}{1-r}, \sigma = -\frac{2\eta}{\pi} \ln r.$$

Then we have

$$B^*(\sigma - \frac{2\eta}{\pi} \ln k_1, \eta_0, \frac{\eta}{2}) \cap \{s | \mathrm{Re} s = \sigma - \frac{2\eta}{\pi} \ln k_1\} \subset \Phi(D(R)) \subset B^*(\sigma, t_0, \eta)(\frac{1}{6} < k_1 < \frac{1}{2}), \tag{2.13}$$

and

$$-\frac{\pi\sigma}{2\eta} - \ln 2 < -\ln(1-R) < -\frac{\pi\sigma}{2\eta}. \tag{2.14}$$

By the mappings (2.12), the series (1.1) and $\forall \varphi(s) \in H$ are transformed into a random series of holomorphic functions in $D(1)$:

$$\Psi(w, \omega) = \sum_{n=0}^{+\infty} a_n Z_n(\omega) \exp(-\lambda_n \Phi(w)), \tag{2.15}$$

$$\psi(w) = \sum_{n=0}^{+\infty} \beta_n \exp(-\lambda_n \Phi(w)) \tag{2.16}$$

and $\Psi(w,\omega)$ and $\psi(w)$ are a random holomorphic in $D(1)$. Let

$$H' = \{\psi(w) | \psi(\phi(s)) = \varphi(s) = \sum_{n=0}^{+\infty} \beta_n \exp(-\lambda_n s) \in H\}. \tag{2.17}$$

By lemma 2.3, it obviously holds that $T(R, \psi(w)) = \circ(T(R, \Psi(w)))(R \to 1)$. We now have the following lemma.

Lemma 2.4 *For $\Psi(w,\omega)$ in $D(1)$,*

$$\overline{\lim_{R\to 1^-}} \frac{\ln^+ T(R, \Psi(w,\omega))}{-\ln(1-R)} = +\infty \quad a.s., \tag{2.18}$$

and $\forall \psi \in H'$ with a possible exceptional value ψ_ω.

$$\overline{\lim_{R\to 1^-}} \frac{\ln^+ N(R, \Psi(w,\omega) = \psi(w))}{-\ln(1-R)} = +\infty \quad a.s., \tag{2.19}$$

where

$$T(R, \Psi(w,\omega)) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |\Psi(Re^{i\theta}, \omega)| \mathrm{d}\theta,$$

$$N(R, \Psi(w,\omega) = \psi(w)) = \int_{R_0}^{R} \frac{n(u, \Psi(w,\omega) = \psi(w))}{u} \mathrm{d}u,$$

$$n(u, \Psi(w,\omega) = \psi(w)) = \sharp\{w | \Psi(w,\omega) = \psi(w), |w| < u\},$$

R_0 being a fixed number $\in (0,1)$.

Proof. By (2.13) and (2.14), we have the relation

$$M_f(\sigma - \frac{2\eta}{\pi} \ln k_1, t_0, \frac{\eta}{2}, \omega) \le M_\Psi(R,\omega) \le M_f(\sigma, \omega)$$

and

$$\frac{\ln^+ \ln^+ M_f(\sigma - \frac{2\eta}{\pi} \ln k_1, t_0, \frac{\eta}{2}, \omega)}{-\pi\sigma/2\eta} \le \frac{\ln^+ \ln^+ M_\Psi(R,\omega)}{-\ln(1-R)} \le \frac{\ln^+ \ln^+ M_f(\sigma,\omega)}{(-\pi\sigma/2\eta) - \ln 2}.$$

By Lemma 4 in [12],

$$\overline{\lim_{\sigma\to -\infty}} \frac{\ln^+ \ln^+ M_\Psi(R,\omega)}{-\ln(1-R)} = +\infty \quad a.s. \tag{2.20}$$

Since

$$\ln^+ M_\Psi(R,\omega) \geq T(R,\Psi(\omega,0)) \geq \frac{1-R}{3R-1}\ln^+ M_\Psi(2R-1,\omega),$$

(2.18) follows from (2.20), (2.19) follows from Lemma 2.3. □

Consider now some non-random holomorphic function in $D(1)$.$\forall M(\in N) > 1$. Let $\{c_j\}_{j=M+1}^{+\infty} \subset C$ such that

$$\overline{\lim_{n\to+\infty}} \frac{\ln|a_n c_n|}{\lambda_n \ln \lambda_n} = 0.$$

Then by Lemma 2.1 and Lemma 2.4,

$$G(w) = \sum_{n=M+1}^{+\infty} a_n c_n \exp(-\lambda_n \Phi(w)) \tag{2.21}$$

is holomorphic in $D(1)$ and satisfies the first condition (2.9).

Lemma 2.5 *There exists at most a point* $(c'_0, c'_1, \cdots, c'_M,) \in C^{M+1}$ *and a Little function* $\psi'(w) \in H'$ *such that*

$$\overline{\lim_{R\to1^-}} \frac{\ln^+ N(R, G_1(w,c) = \psi(w))}{-\ln(1-R)} < +\infty, \tag{2.22}$$

where

$$G_1(w,c) = \sum_{n=0}^{M} a'_n c_n \exp(-\lambda_n \Phi(w)) + G(w), \tag{2.23}$$

$$c = (c'_0, c'_1, \cdots, c'_M, c'_{M+1}, c'_{M+2}, \cdots) \in C^{+\infty}.$$

Proof. We cannot find another point $(c''_0, c''_1, \cdots, c''_M) \neq (c'_0, c'_1, \cdots, c'_M)$in C^{M+1} and another $\psi''(w) \neq \psi'(w)$ in H' such that we would have (2.22′) and (2.23′) obtained from (2.19) and 2.23 by replacing $(c'_0, c'_1, \cdots, c'_M)$ and $\psi'(w)$ by $(c''_0, c''_1, \cdots, c''_M)$ and $\psi''(w)$. In this case, there would be two different holomorphic functions in $D(1)$,

$$g_1(w) = \psi'(w) - \sum_{n=0}^{M} a_n c'_n \exp(-\lambda_n \Phi(w))$$

and

$$g_2(w) = \psi''(w) - \sum_{n=0}^{M} a_n c_n'' \exp(-\lambda_n \Phi(w)),$$

which would satisfy the second condition (2.10). By Lemma 2.2, this is impossible. □

Denote by E_∞ the set of all $c \in C^{+\infty}$ which satisfy the above conditions and set

$$E_{\infty,M} = \{(c_{M+1}, c_{M+2}, \cdots)|c \in E_\infty\} \subset C^{+\infty}.$$

Now we can improve Lemma (2.4) as follows.

Lemma 2.6 *For* $\Psi(w,\omega)$ *in* $D(1)$, $\forall \psi(w) \in H'$,

$$\overline{\lim_{R\to 1^-}} \frac{\ln^+ n(R, \Psi(w,\omega) = \psi(w))}{-\ln(1-R)} = +\infty \quad a.s. \tag{2.24}$$

Proof. We calculate at first the probability of the event

$$S = \left\{\omega \middle| \exists\psi \in H' \text{ such that } \overline{\lim_{R\to 1^-}} \frac{\ln^+ N(R, \Psi(w,\omega) = \psi(w))}{-\ln(1-R)} < +\infty\right\}.$$

Let

$$S_\infty = \{(Z_0(\omega), (Z_0(\omega), \cdots)|\omega \in S\} \subset E_\infty.$$

Consider the probability space $(C, \mathcal{B}_n, \mu_n)$ generated by the random variables $Z_n(\omega)$ and let

$$\mu_\infty = \prod_{n=0}^{\infty} \mu_n, \tilde{\mu}_M = \prod_{n=0}^{M} \mu_n, \mu_{\infty,M} = \prod_{n=M+1}^{\infty} \mu_n,$$

$$z = (z_0, z_1, \cdots), \tilde{z}_M = (z_0, z_1, \cdots, z_M) \quad and \quad z_{\infty,M} = \{z_{M+1}, z_{M+2}, \cdots\}.$$

We have, by Lemma 2 (iii) in [12],

$$\begin{aligned}
P(S) &= \int_{\Omega} 1_S P(d\omega) = \int_{C^{+\infty}} 1_{s_\infty} \mu(dz) \le \int_{C_\infty} 1_{E_\infty}(dz) \\
&= \int_{E_{\infty,M}} \int_{C^{M+1}} 1_{(z_0=c_0',\cdots,z_M=c_M')} \mu(d\tilde{z}_M) \\
&\le \int_{E_{\infty,M}} \prod_{n=0}^{M} P(\{Z_n(\omega)=c_n'\}) \mu_{\infty,M} \\
&< \beta^{M+1}.
\end{aligned}$$

Take $M \nearrow +\infty$. We obtain $P(S)=0$, i.e. $\forall \alpha \in \mathrm{C}$,

$$\overline{\lim_{R\to 1^-}} \frac{\ln^+ N(R, \Psi(w,\omega)=\psi(w))}{-\ln(1-R)} = +\infty. \tag{2.25}$$

By (2.25) we obtain that $\forall k>0, \forall \alpha \in \mathrm{C}$,

$$\int_0^1 N(u, \Psi(w,\omega)=\psi(w))(1-u)^k du = +\infty. \tag{2.26}$$

Otherwise $\exists k>0, \forall \epsilon.0$, for $R \in (0,1)$ and $1-R$ sufficiently small,

$$\epsilon = \int_R^1 N(u, \Psi(w,\omega)=\psi(w))(1-u)^k du \ge N(u, \Psi(w,\omega)=\psi(w)) \int_R^1 (1-u)^k du$$

$$= \frac{1}{k+1}(1-R)^{k+1} N(R, \Psi(w,\omega)=\psi(w)). \tag{2.27}$$

But by (2.25), $\exists R_m \nearrow 1$ such that $(1-R_m)^{k+1} N(R_m, \Psi(w,\omega)=\psi) > 1$. Hence (2.27) is a contradiction and we obtain (2.26).

From (2.26) it follows that $\forall k>1$ and hence $\forall k>0, \forall \psi(w) \in H'$,

$$\int_0^1 n(u, \Psi(w,\omega)=\psi(w)(1-u)^k du = +\infty. \tag{2.28}$$

For $\forall k > 1, \frac{1}{2} < R_0 < R < 1$,

$$\begin{aligned} k\int_{R_0}^{R} N(u, \Psi(w,\omega) &= \psi(w))(1-u)^{k-1}du = (1-R_0)^k N(R_0, \Psi(w,\omega) = \psi(w)) \\ &- (1-R)^k N(R, \Psi(w,\omega) = \psi(w)) \\ &+ \int_{R_0}^{R} n(u, \Psi(w,\omega) = \psi(w))(1-u)^k \frac{du}{u}. \end{aligned}$$

By (2.26), as $R \nearrow 1$, the integral in the right-hand side of the above equality diverges to $+\infty$. We have

$$\frac{1}{2}\int_{R_0}^{R} n(u, \Psi(w,\omega) = \psi(w))(1-u)^k \frac{du}{u} \leq \int_{R_0}^{R} n(u, \Psi(w,\omega) = \psi(w))(1-u)^k du$$

and (2.28) follows immediately.

If (2.24) were not true, there would exist $k > 0$ and $\psi \in h'$ such that the integral in (2.28) would converge, which is impossible. The lemma is proved. □

3. Proof of the theorem 1.1

The first part this Theorem is contained in main Theorem in [12]. Now we prove the second part. By lemma 2.3, given $t_0 \in R$ and $\eta > 0$, we have, $\forall \psi \in H'$.

$$\frac{\ln^+ n(R, \Psi(w,\omega) = \psi(w))}{-\ln(1-R)} \leq \frac{\ln^+ n(\sigma, t_0, \eta, f(s,\omega) = \varphi(s))}{-\pi\sigma/2\eta - \ln 2}$$

and (1.7) follows from (2.24).

In order to complete the proof of t_0 and η, we consider a sequence $\{\eta_m\}, \eta_m \searrow 0$.and a sequence of all rational numbers $\{t_k\}$ and apply the previous result. □

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