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Stack-Sortable Permutations and Polynomials

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Abstract

The Catalan numbers show up in a diverse variety of counting problems. In this note we give yet another characterization of the Catalan number $C(n)$. It is characterized as the dimension of a certain space of multilinear polynomials by exhibiting a basis.

Key word and phrases: Catalan numbers, stack-sortable permutations, permutations acting on polynomials

1. Introduction

The action of permutations on polynomials is one of the most indispensable techniques of algebra and it shows up in almost all considerations. In [1], to characterize differential forms which can be factorized as $\mu \wedge \omega$ for a fixed 2-form μ by means of a number of homogeneous exterior equations, a set of generators of $Ann(\mu)$, the annihilator ideal of (μ) has been exhibited. In [2] this construction has been generalized to certain even and odd forms. However, the generating sets of even forms under consideration there are far from being minimal. In the construction of a minimal basis for $Ann(\mu)$, where $\mu = \mu_1 + \cdots + \mu_{2n}$, whose terms are exterior products of vectors in a vector space for wich $\mu_1 \cdots \mu_{2n} \neq 0$, the first step is to construct a basis for the subspace of the exterior algebra spanned by the products

$$(\mu_{\sigma(1)} - \mu_{\sigma(2)}) \cdots (\mu_{\sigma(2n-1)} - \mu_{\sigma(2n)}), \quad \sigma \in S_{2n}.$$

However, it is worthwhile to handle this problem in a more general context by considering the algebra of polynomials. This is the objective of this paper. We consider $F[x_1, \ldots, x_{2n}]$, the ring of polynomials in the undeterminates $x_1, x_2, ..., x_{2n}$ over the field F on which permutations acts canonically,

$$(\sigma f)(x_1, \ldots, x_{2n}) = f(x_{\sigma(1)}, \ldots, x_{\sigma(2n)}),$$

and exhibit a basis for $F[S_{2n}]p$, the cyclic submodule generated by

$$p(x_1, \ldots, x_{2n}) = (x_1 - x_2)(x_3 - x_4) \ldots (x_{2n-1} - x_{2n})$$

of the module $F[x_1, \ldots, x_{2n}]$ over the group algebra $F[S_{2n}]$ where S_{2n} is the symmetric group on $2n$ letters.

2. A Basis Corresponding to Stack-Sortable Permutations

In order to facilitate the presentation we consider the polynomial algebra $F[x_1, \ldots, x_n;\ y_1, \ldots, y_n]$ and the action of permutations in S_n on this algebra defined by

$$f_\sigma = (\sigma f)(x_1, \ldots, x_n; y_1, \ldots, y_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; y_1, \ldots, y_n).$$

Now, we consider the polynomial

$$p(x_1, \ldots, x_n; y_1, \ldots, y_n) = (x_1 - y_1) \ldots (x_n - y_n),$$

and the cyclic submodule $F[S_n]p$ over the group algebra $F[S_n]$. We construct a basis for this submodule by using "231-avoiding permutations", that is permutations for which $\sigma(k) < \sigma(i) < \sigma(j)$ cannot occur when $i < j < k$. Such permutations are called stack-sortable permutations. Their set will be denoted by St_n. It is well known that the number of stack-sortable permutations of degree n is equal to the Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$. For this and many other characterizations of the Catalan numbers we refer to page 219 of the book [3]. Our construction furnishes yet another characterization of Catalan numbers as the dimension of a space of polynomials.

Proposition 1 *The sets*

$$\{p_\sigma(x_1, \ldots, x_n; y_1, \ldots, y_n) \mid \sigma \in St_n\} \text{ and } \{p_\sigma(x_1, \ldots, x_n; y_1, \ldots, y_{n-1}, 0) \mid \sigma \in St_n\}$$

are linearly independent.

Proof. To each permutation σ we assign a sequence $\varphi_\sigma = (s_1, s_2, \ldots, s_n; s'_1, s'_2, \ldots s'_n)$, where $s_{\sigma(n)} = 1$, $s'_n = 0$ and for $k < n$ the terms $s_{\sigma(k)}$ and s'_k are defined by

$$s_{\sigma(k)} = \begin{cases} 1 & \text{if } \sigma(k) < \sigma(k+1) \\ 0 & \text{if } \sigma(k) > \sigma(k+1) \end{cases} \qquad \text{and} \qquad s'_k = 1 - s_{\sigma(k)}.$$

We shall compute $p_\tau(\varphi_\sigma)$ for stack-sortable permutations σ and τ; and using these values we shall obtain a linear system of equations for the coefficients a_τ in the relation $\sum_{\tau \in St_n} a_\tau p_\tau = 0$ from which it will be immediate that all these coefficients are 0. Since the last term of φ_σ is zero in our discussion we do not need any distinction of the sets given in the proposition.

Obviously we have $p_\tau(\varphi_\tau) = \pm 1$. We consider the ordering of permutations defined by

$$\tau > \sigma \iff \exists\, 1 \le k \le n \text{ such that } \tau(i) = \sigma(i) \text{ for all } i > k \text{ and } \tau(k) > \sigma(k).$$

we claim that when σ and τ are stack-sortable permutations with $\tau > \sigma$, we have

$$p_\tau(\varphi_\sigma) = p_\tau(s_1, s_2, \ldots, s_n; s'_1, s'_2, \ldots s'_n) = 0\ .$$

In fact, assuming $\tau(k) > \sigma(k)$ for some $k \le n$, and $\tau(i) = \sigma(i)$ for all $i > k$, we observe that

$$\tau(k) = \sigma(r) \text{ for some } r < k \quad \text{and} \quad \sigma(k) = \tau(l) \text{ for some } l < k.$$

Since for stack-sortable permutations σ and τ, neither inequality $\sigma(k) < \sigma(i) < \sigma(j)$ nor $\tau(k) < \tau(i) < \tau(j)$ can occur when $i < j < k$. If we had $s_{\sigma(k)} = 0$, this would mean $k < n$ and $\sigma(k) > \sigma(k+1)$, which would imply

$$\sigma(k+1) = \tau(k+1) < \sigma(k) = \tau(l) < \tau(k) \text{ for } l < k < k+1.$$

This would contradict the stack-sortability of τ. Thus $s_{\sigma(k)} = 1$ and hence $s'_k = 0$. The same argument repeated for $s_{\sigma(r)} = 1$, where $r < k$, would imply

$$\sigma(k) < \tau(k) = \sigma(r) < \sigma(r+1) \text{ for } r < r+1 \leq k.$$

This in turn shows that $r+1 = k$ cannot happen, and $r < r+1 < k$ contradicts the stack-sortability of σ. Thus we obtained

$$s_{\tau(k)} = s_{\sigma(r)} = 0 \text{ and } s'_k = 0.$$

to evaluate

$$p_\tau(x_1, \ldots, x_n; y_1, \ldots, y_n) = (x_{\tau(1)} - y_1) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n)$$

at $\varphi_\sigma = (s_1, s_2, \ldots, s_n; s'_1, s'_2, \ldots s'_n)$, we substitute $x_{\tau(k)} = s_{\tau(k)} = 0$ and $y_k = s'_k = 0$ and obtain

$$p_\tau(\varphi_\sigma) = p_\tau(s_1, s_2, \ldots, s_n; s'_1, s'_2, \ldots s'_n) = 0,$$

as claimed. To complete the proof , suppose that

$$\sum_{\tau \in St_n} a_\tau p_\tau = 0$$

and that σ is the least permutation for which $a_\sigma \neq 0$. Then we have

$$\sum_{\tau \in St_n} a_\tau p_\tau(\varphi_\sigma) = \pm a_\sigma = 0,$$

which is a contradiction. □

Lemma 2 *(i) If $n \leq 2$, every permutation is stack sortable.*

(ii) $\sum_{\tau \in St_3} sgn(\tau) p_\tau = 0$ and hence the polynomial p_σ, associated to the 3-cycle $\sigma = (123)$, is a linear combination of the p_{τ_i}, where $\tau_1 = (12), \tau_2 = (13), \tau_3 = (23), \tau_4 = (132)$ and $\tau_5 = (1)$.

(iii) If σ has a 231-pattern, say, $\sigma = [\ldots, \sigma(i), \ldots, \sigma(j), \ldots \sigma(k), \ldots]$, with $\sigma(k) < \sigma(i) < \sigma(j)$, then p_σ is a linear combination of the $p_{\sigma\tau_l}$, where $1 \leq l \leq 5$ $\tau_1 = (ij), \tau_2 = (ik), \tau_3 = (jk), \tau_4 = (ijk)$ and $\tau_5 = (kji)$.

Proof. (i) is obvious, (ii) is obtained by a straightforward verification, and (iii) is an easy consequence of (ii). □

Definition 3 *An n-term sequence* $[t_1, t_2, \ldots, t_k, \ldots, t_m]$ *is said to be tidy if*

$$t_k \geq t_i \quad \text{for all} \quad i \quad , \text{and } t_j > t_i \text{ for all } j \geq k \ , \ i < k.$$

A permutation is said to be tidy if the sequence $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$ *is tidy that is if* $\sigma(i) \geq \sigma^{-1}(n)$ *for all* $i \geq \sigma^{-1}(n)$.

Lemma 4 *If* $\sigma^{-1}(n) \in \{1, n\}$, *then* σ *is tidy.*

Proof. Obvious. □

Proposition 5 *Every* p_σ *is a linear combination of the* p_τ *corresponding to tidy permutations* τ.

Proof. Let

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n) \text{ with } \sigma(k) = n$$

Considering Lemma 2 (i) we can use induction on n. By assuming the assertion is true for $n-1$, we consider the permutation $\overline{\sigma} \in S_{n-1}$ defined by

$$\overline{\sigma}(i) = \begin{cases} \sigma(i) & \text{if} \quad i < k \\ \sigma(i+1) & \text{if} \quad i \geq k \end{cases} .$$

And applying Lemma 2(iii) to

$$(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k-1)} - y_{k-1})(x_{\sigma(k+1)} - y_{k+1}) \cdots (x_{\sigma(n)} - y_n)$$

we may assume without loss of generality that

$$\overline{\sigma}(i) \geq \overline{\sigma}^{-1}(n-1) \text{ for all } \quad i \geq \overline{\sigma}^{-1}(n-1) \text{ in } \{1, 2, \ldots, n-1\}.$$

In other words, the sequence $[\sigma(1), \ldots, \sigma(k-1), \sigma(k+1), \ldots, \sigma(n)]$ is tidy with largest term $\sigma(l) = n-1$. Now, we consider several particular cases separately.

Case 1. Let $\sigma^{-1}(n-1) = n$. Then

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(n-1)} - y_{n-1})(x_{n-1} - y_n);$$

and by induction hypothesis applied to bijections from $\{1, 2, \ldots, n-1\}$ onto $\{1, 2, \ldots, n\} - \{n-1\}$, it is a linear combination of polynomials of the form

$$p_\tau = ((x_{\tau(1)} - y_1) \cdots (x_{\tau(n-2)} - y_{n-2})(x_{\tau(n)} - y_{n-1}))(x_{n-1} - y_n),$$

where $\tau(i) \geq \tau^{-1}(n)$ for all $i \geq \tau^{-1}(n)$ in $\{1, 2, \ldots, n-1\}$. Now, extending τ to a permutation of $\{1, 2, \ldots, n\}$ with $\tau(n-1) = n$, we still have $\tau(i) \geq \tau^{-1}(n)$ for all $i \geq \tau^{-1}(n)$ in $\{1, 2, \ldots, n\}$. Thus, each τ is tidy.

Case 2. Let $\sigma^{-1}(n-1) = 1$. Then we may assume $\sigma^{-1}(n) \neq n$ by Lemma 4, and the 231 pattern shows up with

$$\sigma(1) = n-1, \sigma(k) = n \text{ for some } k < n, \text{ and } \sigma(n) = r \quad \text{for some } r < n-1.$$

By using Lemma 2 (iii) we can express p_σ as a linear combination of the p_τ where $\tau \in \{\sigma \circ (1n), \sigma \circ (1k), \sigma \circ (kn), \sigma \circ (1kn), \sigma \circ (1nk)\}$. For $\tau = \sigma \circ (1n)$ we have $\tau(n) = n-1$ as in Case 1, and for all others we have either $\tau(n) = n$ or $\tau(1) = n$ as in Lemma 4.

Case 3. Let $1 < \sigma^{-1}(n-1) < \sigma^{-1}(n)$, then

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(l)} - y_l) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n)$$

with $\sigma(l) = n-1$, $\sigma(k) = n$ and $\sigma(i) < l = \sigma^{-1}(n-1)$ for all $i < l$. Since $l > 1$, the induction hypothesis allows us to express

$$(x_{\sigma(l)} - y_l) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n)$$

as a linear combination of polynomials

$$(x_{\tau(l)} - y_l) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n),$$

where each τ is a tidy permutation of $\{l, \ldots, n\}$, and also its extension to $\{1, \ldots, n\}$ defined by

$$\overline{\tau}(i) = \begin{cases} \sigma(i) & \text{if} \quad i < l \\ \tau(i) & \text{if} \quad i \geq l \end{cases}$$

is tidy. Thus p_σ is a linear combination of products

$$(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(l-1)} - y_{l-1})(x_{\tau(l)} - y_l) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n),$$

each of which turns out to be a polynomial associated with a tidy permutation $\overline{\tau}$.

Case 4. Let $\sigma^{-1}(n) < \sigma^{-1}(n-1) < n$. Then

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(l)} - y_l) \cdots (x_{\sigma(n)} - y_n)$$

with $\sigma(l) = n-1$ and $\sigma(k) = n$, and the sequence

$$[\sigma(1), \ldots, \sigma(k-1), \sigma(k+1), \ldots, \sigma(l), \ldots, \sigma(n)]$$

is tidy with largest term $\sigma(l) = n-1$. Since $l < n$, we can use the induction hypothesis to write $(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(l-1)} - y_{l-1})$ as a linear combination of the products

$$(x_{\tau(1)} - y_1) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(l-1)} - y_{l-1}),$$

for which the sequence $[\tau(1), \ldots, \tau(k), \ldots, \tau(l-1)]$ is tidy and its terms are in $\{1, 2, \ldots, l-1, n\}$. Then each sequence

$$[\tau(1), \ldots, \tau(k), \ldots, \tau(l-1), \sigma(l), \ldots, \sigma(n)]$$

becomes a tidy sequence associated to a permutation in S_n; and thus p_σ becomes a linear combination of polynomials

$$(x_{\tau(1)} - y_1) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(l-1)} - y_{l-1})(x_{\sigma(l)} - y_l) \cdots (x_{\sigma(n)} - y_n)$$

associated to tidy permutations. □

Combining the above propositions we establish our main result.

Theorem 6 *The set*

$$\{p_\tau | \tau \in St_n\}$$

is an F-basis for the cyclic submodule $F[S_n]p$ and hence its dimension is the Catalan number $C(n)$.

Proof. Linear independence follows from Proposition 1. To complete the proof we use induction on n. Suppose that when $m < n$, for every permutation $\rho \in S_m$, p_ρ is a linear combination of the polynomials p_ω where ω runs over stack-sortable permutations in S_m. Take any p_σ and use Proposition 5 to express it as a linear combination of polynomials p_τ corresponding to tidy permutations τ. Now, pick-up a tidy permutation τ with $k = \tau^{-1}(n)$. Then , $\tau(i) \geq k$ for $i \geq k$, and $\tau(i) < k$ for $i < k$. We consider two cases separately.

Case1. $k = 1$, we use Proposition 5 and the induction hypothesis to write

$$(x_{\tau(2)} - z_1) \cdots (x_{\tau(n)} - z_{n-1})$$

as a linear combination of the products

$$(x_{\rho(1)} - z_1) \cdots (x_{\rho(n-1)} - z_{n-1}),$$

where each $\rho \in S_{n-1}$ is stack-sortable. By letting $z_1 = y_2, \ldots, z_{n-1} = y_n$ we see that

$$\begin{aligned}(x_n - y_1)(x_{\rho(1)} - z_1) \cdots (x_{\rho(n-1)} - z_{n-1}) &= (x_n - y_1)(x_{\rho(1)} - y_2) \cdots (x_{\rho(n-1)} - y_n) \\ &= (x_{\overline{\rho}(1)} - y_1)(x_{\overline{\rho}(2)} - y_2) \cdots (x_{\overline{\rho}(n)} - y_n),\end{aligned}$$

where the permutation $\overline{\rho}$ is defined by

$$\overline{\rho}(i) = \begin{cases} \rho(i-1) & \text{if} \quad i > 1 \\ n \quad \text{if} \quad i = 1 \end{cases}$$

and it is a stack-sortable permutation in S_n .

Case 2. $k = \tau^{-1}(n) > 1$. Then we have

$$p_\tau = (x_{\tau(1)} - y_1) \cdots (x_{\tau(k-1)} - y_{k-1})(x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n)$$

with

$$\{\tau(1), \ldots, \tau(k-1)\} = \{1, \ldots, k-1\} \text{ and } \{\tau(k), \ldots, \tau(n)\} = \{k, \ldots, n\}$$

and applying the induction hypothesis to the restrictions of τ to $\{1, \ldots, k-1\}$ and $\{k, \ldots, n\}$ we can write $(x_{\tau(1)} - y_1) \cdots (x_{\tau(k-1)} - y_{k-1})$ as a linear combination of the products $(x_{\eta(1)} - y_1) \cdots (x_{\eta(k-1)} - y_{k-1})$ with $\eta \in St_{k-1}$. Moreover, by using Case1 we can write $(x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n)$ as a linear combination of products $(x_{\varsigma(k)} - y_k) \cdots (x_{\varsigma(n)} - y_n)$ where each ς is a stake sortable permutation of $\{k, \ldots, n\}$ with $\varsigma(k) = n$. Then for p_τ we obtain a linear combination of products

$$\begin{aligned}&(x_{\eta(1)} - y_1) \cdots (x_{\eta(k-1)} - y_{k-1})(x_{\varsigma(k)} - y_k) \cdots (x_{\varsigma(n)} - y_n) \\ =\ &(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n),\end{aligned}$$

where the sequence $[\eta(1), \ldots, \eta(k-1), \varsigma(k), \ldots, \varsigma(n)]$ defines the stack- sortable permutation σ in S_n as

$$\sigma(i) = \begin{cases} \eta(i) & \text{if} \quad i < k-1 \\ \varsigma(i) & \text{if} \quad i \geq k, \end{cases}$$

and thus each product under consideration becomes

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n) \text{ for some } \sigma \in St_n.$$

□

Corollary 7 *Let*

$$p(x_1, \ldots, x_{2n}) = (x_1 - x_2)(x_3 - x_4) \ldots (x_{2n-1} - x_{2n}).$$

Then the set

$$\{p_\sigma = (x_{\sigma(1)} - x_{n+1}) \cdots (x_{\sigma(n)} - x_{2n}) | \sigma \in St_n\}$$

forms a basis for the cyclic submodule $F[S_{2n}]p$.

Proof. Letting $y_k = x_{n+k}$ for $k = 1, \ldots, n$ it is sufficient to note that

$$(x_i - x_j)(y_k - y_l) = (x_i - y_k)(x_j - y_l) - (x_i - y_l)(x_j - y_k),$$

because this allows us to write every element of $F[S_{2n}]p$ as a linear combination of polynomials in the form $(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(n)} - y_n)$, where σ is a permutation of $\{1, \ldots, n\}$.

□

Corollary 8 *Let*

$$p(x_1, \ldots, x_{2n}) = (x_1 - x_2)(x_3 - x_4) \ldots (x_{2n-1} - x_{2n}) x_{2n+1}.$$

Then the set

$$\{p_\sigma = (x_{\sigma(1)} - x_{n+2}) \cdots (x_{\sigma(n)} - x_{2n+1}) x_{\sigma(n+1)} | \sigma \in St_{n+1}\}$$

forms a basis for the cyclic submodule $F[S_{2n+1}]p$ *and hence its dimension is* $C(n)$.

Proof. We let $y_k = x_{n+k+1}$ for $k = 1, \ldots, n$ and $y_{n+1} = 0$ and we see by the theorem that each

$$p_\sigma = (x_{\sigma(1)} - x_{n+2}) \cdots (x_{\sigma(n)} - x_{2n+1})(x_{\sigma(n+1)} - 0),$$

which is considered to be the evaluation of the polynomial

$$(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(n)} - y_n)(x_{\sigma(n+1)} - y_{n+1}),$$

which at $y_{n+1} = 0$ can be expressed as a linear combination of the polynomials

$$(x_{\tau(1)} - y_1) \cdots (x_{\tau(n)} - y_n)(x_{\tau(n+1)} - y_{n+1}), \ \tau \in St_n$$

evaluated at $y_{n+1} = 0$ as desired. Linear independence follows from Proposition 1. □

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