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Primary Finitely Compactly Packed Modules and S-Avoidance Theorem for Modules

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Abstract

In this paper we introduce the concept of primary finitely compactly packed modules, which generalizes the concept of primary compactly packed modules. We first find the conditions that make the primary finitely compactly packed modules primary compactly packed. Also, several results on the primary finitely compactly packed modules are proved. In addition, the necessary and sufficient conditions for an $R-$module M to be primary finitely compactly packed are investigated. Finally, we introduce the S-Avoidance Theorem for modules.

Key Words: Primary submodules, primary compactly packed modules, primary finitely compactly packed modules, s-prime submodules, S-Avoidance Theorem for modules.

1. Introduction

Let Mbe a unitary $R-$module, where $R$is a commutative ring with identity. A proper submodule $N$of $M$is primary if $rm \in N$for $r \in R$and $m \in M$implies that either $m \in N$or $r^n M \subseteq N$for some positive integer n. It is known that a proper submodule $N$of an $R-$module $M$is primary compactly packed (pcp) if for each family $\{P_\alpha\}_{\alpha \in \lambda}$of primary submodules of $M$with $N \subseteq \underset{\alpha \in \lambda}{\cup} P_\alpha, \exists \beta \in \lambda$ such that $N \subseteq P_\beta$. A module$M$is called pcp if every proper submodule of $M$is pcp; see [3]. We generalizes the concept of pcp modules to the concept of primary finitely compactly packed (pfcp) modules. Thus we say that a proper submodule $N$of an $R-$module $M$is pfcp if for each family $\{P_\alpha\}_{\alpha \in \lambda}$of primary submodules of$M$with $N \subseteq \underset{\alpha \in \lambda}{\cup} P_\alpha, \exists \alpha_1, \alpha_2, ..., \alpha_n \in \lambda$ such that $N \subseteq \overset{n}{\underset{i=1}{\cup}} P_{\alpha_i}$. A module

M is said to be pfcp if every proper submodule of M is pfcp.

In section 2 of this paper, we give some examples of pfcp modules and find the relation between pcp modules and pfcp modules. We also find the conditions that make a pfcp module pcp.

In section 3, we investigate some properties of pfcp modules. We also find the necessary and sufficient conditions for any R-module M to be pfcp.

In 1997 Chin Pi Lu proved the Prime Avoidance Theorem for modules, see [9]. Mohammed El- Atrash and Arwa Ashour introduced the Primary Avoidance Theorem for modules in 2005, see [3]. In Section 4 of this paper we introduce and prove the $S-$Avoidance Theorem for modules.

Throughout this paper, all rings are assumed to be commutative rings with identity and all modules will be unitary.

2. Relation Between Primary Compactly Packed Modules and Primary Finitely Compactly Packed Modules

We first recall the following definitions.

Definitions 2.1 *Let M be a unitary $R-$module, where R is a commutative ring with identity. A proper submodule N of M is primary if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n.*

A proper submodule N of an $R-$module M is primary compactly packed (pcp) if for each family $\{P_\alpha\}_{\alpha\in\lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha, \exists\beta \in \lambda$ such that $N \subseteq P_\beta$. A module M is said to be pcp if every proper submodule of M is pcp.

Now we give the following definition.

Definitions 2.2 *A proper submodule N of M is primary finitely compactly packed (pfcp) if for each family $\{P_\alpha\}_{\alpha\in\lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$, $\exists\alpha_1, \alpha_2, ..., \alpha_n \in \lambda$ such that $N \subseteq \bigcup_{i=1}^{n} P_{\alpha_i}$. A module M is said to be pfcp if every proper submodule of M is pfcp.*

Remark 2.3 *It is clear from the definitions that every pcp module is pfcp module; however, the converse is not true, as illustrated in the following first example.*

Examples 2.4 *1) Let V be a vector space with dimension greater than 2 over the field $F = Z/2Z$.*

Then every submodule of V is prime, so every submodule of V is primary. Let e_1 and e_2 be distinct vectors of a basis for V. Let $V_1 = e_1F, V_2 = e_2F, V_3 = (e_1 + e_2)F$ and let $L = V_1 + V_2$. Then $L = \{0, e_1, e_2, e_1 + e_2\}$. Thus V_1, V_2 and V_3 are primary submodules of V with the property that $L \subseteq \bigcup_{i=1}^{3} V_i$, but $L \not\subset V_i, \forall i \in \{1, 2, 3\}$. Thus $L$is pfcp, however $L$is not pcp.

2) If M is an R-module that contains only a finite number of primary submodules, then M is pfcp module.

Theorem 2.5 *Let M be an R-module in which every finite family of primary submodules of $M$is totally ordered by inclusion; then $M$is pcp if and only if $M$is pfcp.*

Proof.

$(\rightarrow)$ Trivial

$(\leftarrow)$ Let $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$, where $P_\alpha$is primary submodule for each α. Since M is pfcp, there exist $\alpha_1, \alpha_2, ..., \alpha_n$such that $N \subseteq \bigcup_{i=1}^{n} P_{\alpha_i}$. Since the family $\{P_{\alpha_i}\}_{i=1}^{n}$ of primary submodules of M is totally ordered by inclusion, there exists $\beta \in \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\bigcup_{i=1}^{n} P_{\alpha_i} = P_\beta$. Thus M is pcp. □

We remember now the Primary Avoidance Theorem for modules, which was proved in [3].

Theorem 2.6 *(The Primary Avoidance Theorem for Modules)*

Let Mbe an R-module, $L_1, L_2, ..., L_n$ a finite number of submodules of $M$and L a submodule of M such that $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$. Assume that at most two of the $L_i's, i = 1, 2, ..., n$ are not primary and that $(L_j : M) \not\subset \sqrt{(L_k : M)}$ whenever $j \neq k$. Then $L \subseteq L_k$ for some $k \in \{1, 2, ..., n\}$.

The following Theorem follows immediately from the Primary Avoidance Theorem for modules.

Theorem 2.7 *If M is an R-module with the property that for each submodule L of M if $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ in which at most two of the $L_i's$ are not primary, and $(L_j : M) \not\subset \sqrt{(L_k : M)}$ whenever $j \neq k$, then M is pcp if and only if $M$is pfcp.*

3. Important Results on Primary Finitely Compactly Packed modules

The following Theorem was proved in [1] for pcp modules, we prove that it is also satisfied for pfcp modules.

Theorem 3.1 *If M is pfcp module which has at least one maximal submodule, then M satisfies the ACC on primary submodules.*

Proof. Let $N_1 \subseteq N_2 \subseteq ...$ be an ascending chain of primary submodules of M and let $L = \bigcup_i N_i$. If $L = M$ and H is a maximal submodule of M, then $H \subset \bigcup_i N_i$. Since M is pfcp, $\exists n_1, n_2, ..., n_k$ such that $H \subseteq \bigcup_{j=1}^{k} N_{n_j}$. Since $N_1 \subseteq N_2 \subseteq ...$ is an ascending chain, $\exists m \in \{1, 2, ..., k\}$ such that $\bigcup_{j=1}^{k} N_{n_j} = N_{nm} = N_r$ for some $r \in \{1, 2, 3, ...\}$. Since H is maximal, $H = N_r$. Since $N_r \subseteq N_{r+i} \subseteq \bigcup_i N_i, \forall i = 1, 2, ...$ and N_r is maximal, then $N_{r+i} = \bigcup_i N_i = M$, which is impossible. Thus L must be a proper submodule of M. Now since M is pfcp, $\exists n_1, n_2, ..., n_s$ such that $L \subseteq \bigcup_{j=1}^{r} N_{n_j}$. Since $N_1 \subseteq N_2 \subseteq ...$ is an ascending chain, $\exists m \in \{1, 2, ..., r\}$ such that $\bigcup_{j=1}^{r} N_{n_j} = N_{nm} = N_k$ for some $k \in \{1, 2, 3, ...\}$. Hence $N_1 \subseteq N_2 \subseteq ... \subseteq N_k = N_{k+1} = ...$ Therefore the ACC is satisfied for primary submodules. □

Since every finitely generated module and every multiplication module has a proper maximal submodule , see[2], then we have the following Corollary.

Corollary 3.2 *Let M be a pfcp $R-$module. If M is a finitely generated or a multiplication $R-$module, then M satisfies the ACC on primary submodules.*

Theorem 3.3 *If M is an $R-$module with the property that every non empty family of primary submodules of M is totally ordered by inclusion, and suppose that M satisfies the ACC on primary submodules; then M is pfcp.*

Proof. Let N be a submodule of M with the property that $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$, where $P_\alpha$is primary submodule of $M$for each α. Then by the hypothesis $\{P_\alpha\}$ is totally ordered by inclusion and satisfies the ACC on primary submodules, therefore there exists $\beta \in \lambda$ such that $\bigcup P_\alpha \subseteq P_\beta$. Hence $N \subseteq P_\beta$ for some $\beta \in \lambda$. Thus $M$is pcp. Hence M is pfcp. Recall the following definitions (see[10]). □

Definitions 3.4 *A ring R is Bezout if every finitely generated ideal of R is principal. A module $M$is called a Bezout module if every finitely generated submodule is cyclic.*

Theorem 3.5 *Let Mbe a multiplication $R-$module. If one of the following conditions holds:*

i) R is a Bezout ring.

ii) M is a Bezout module.

iii) $M$is a cyclic module.

Then M is pfcp if and only if every primary submodule of M is pfcp.

Proof. The necessity is trivial. To prove the sufficiency, suppose that every primary submodule of M is pfcp. Let N be a proper submodule of M with the property that $N \subseteq \bigcup_{\alpha \in \lambda} Q_\alpha$, where $Q_\alpha$is primary submodule of $M$for each α. We have two cases:

Case1: $\bigcup_{\alpha \in \lambda} Q_\alpha = M$. Since N is a proper submodule of a multiplication module, then by [2], there exists a primary submodule Q that contains N.By the assumption $Q$is pfcp. Thus $N \subseteq Q \subseteq M = \bigcup_{\alpha \in \lambda} Q_\alpha$. Since Q is pfcp, $\exists \alpha_1, \alpha_2, ..., \alpha_n$ such that $Q \subseteq \bigcup_{i=1}^{n} Q_{\alpha_i}$ that is $N \subseteq \bigcup_{i=1}^{n} Q_{\alpha_i}$. Hence N is pfcp. Therefore M is pfcp.

Case 2: $\bigcup_{\alpha \in \lambda} Q_\alpha \subset M$. Then by [3], there exists a primary submodule Qsuch that $N \subseteq Q \subseteq \bigcup_{\alpha \in \lambda} Q_\alpha$ and by the hypothesis $Q$is pfcp. Thus $\exists \alpha_1, \alpha_2, ..., \alpha_r$ such that $N \subseteq Q \subseteq \bigcup_{i=1}^{r} Q_{\alpha_i}$. Thus N is pfcp. Hence M is pfcp. □

4. S-Avoidance Theorem for Modules

In this section we introduce the S-Avoidance Theorem for modules and prove some results on$s$-prime submodules.

We start with the following definitions.

Definitions 4.1

- *A proper ideal$P$ of a ring $R$is called an s-prime ideal of R if for any elements a, b $\in R$ such that $a\ b \in P$ and $b \notin P$, then $a^2 \in P$.*
- *A proper submodule N of an R-module$M$ is called an s-prime submodule of M if for any $r \in R$ and $x \notin N$ with the property that $r\ x \in N$, then $r^2\ M \subseteq N$.*

Now we prove the following result.

Proposition 4.2 *If N is an $s-$prime submodule of an R-module M, then (N:M)= { r- r $\in R$, $r\ M \subseteq N$ } is an s-prime ideal of R.*

Proof. Let $a\ b \in (N : M)$, where $a, b \in R$ and $b \notin$ *(N:M)*, then $b\ M \not\subseteq N$. Thus there exists $m \in M$ such that $b\ m \notin N$. But $a\ (\ b\ m\ ) \in N$ and N is an $s-$prime submodule of M. Thus $a^2\ M \subseteq N$. Hence $a^2 \in (N : M)$. Therefore $(N : M)$ is an $s-$prime ideal of R. □

Proposition 4.2. can be generalized as follows.

Proposition 4.3 *If N is an $s-$prime submodule of an R-module M, then $(N : M)^{1/n}$ = { r- r $\in R$, $r^n\ M \subseteq N$ } is an s-prime ideal of $R$for any positive integer n.*

Proof. Let n be a positive integer. Let $a\ b \in (N : M)^{1/n}$, where $a, b \in R$ and $b \notin (N{:}M)^{1/n}$, then $b^n\ M \not\subseteq N$. Thus there exists $m \in M$ such that $b^n\ m \notin N$. But $a^n(b^n\ m\ ) \in N$ and N is an $s-$prime submodule of M. Thus $a^{2n}\ M \subseteq N$. Hence $a^2 \in (N : M)^{1/n}$. Therefore $(N : M)^{1/n}$ is an $s-$prime ideal of R. □

Now we recall the following definition, see[5].

Definitions 4.4 *Let L, L_1, L_2 , ... , L_n be submodules of an $R-$module M. We call a covering $L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n$ efficient if no L_k is superfluous (i.e. we cant find k such that $L \subseteq L_k, k \in \{1, 2, ...n\}$). Analogously we shall say that $L = L_1 \cup L_2 \cup \ldots \cup L_n$ is an efficient union if non of the L_k's may be excluded.*

Remark 4.5

- *Any cover or union consisting of submodules of M can be reduced to an efficient one called an efficient reduction by deleting any unnecessary submodules.*
- *A covering of a submodule by two submodules of a module is never efficient. Thus $L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n$ may be possibly an efficient covering only when $n =$ 1 or $n > 2$, see [7].*

The next result was proved for ideals in [7] and Lu in [9] pointed out that the same result is also remains valid if ideals are replaced with subgroups of any group as in the following Lemma.

Lemma 4.6 *Let $L = L_1 \cup L_2 \cup \ldots \cup L_n$ be an efficient union of submodules of an $R-$module M for $n >$1. Then* $\bigcap_{\substack{j=1 \\ j \neq k}}^{n} L_j = \bigcap_{j=1}^{n} L_j$ *for all k, 1$\leq k \leq n$.*

Now we can prove the following Proposition.

Proposition 4.7 *Let $L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n$ be an efficient covering of submodules of an $R-$module M where $n >$1. If $(L_j : M) \nsubseteq (L_k : M)^{1/n}$ for $n =$1,2 and 4, for every $j \neq k$, then no L_k for $k \in \{$ 1,2,...,n $\}$ is an $s-$prime submodule of M.*

Proof. Since $L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n$ is an efficient covering ,

$L = (L \cap L_1) \cup (L \cap L_2) \cup \ldots \cup (L \cap L_n)$ is an efficient union. Hence for every $k \in \{$ 1,2,...,n $\}$, there exists an element $e_k \in L - L_k$. Moreover by Lemma 4.6. ,

$\bigcap_{\substack{j=1 \\ j \neq k}}^{n} (L \cap L_j) \subseteq (L \cap L_k)$. Now if $j \neq k$, then $(L_j : M) \nsubseteq (L_k : M)^{1/n}$ for every $n =$1,2 and 4. Thus there exists an element $s_j \in (L_j : M)$ but $s_j \notin (L_k : M)^{1/n}$ for every $n =$1,2 and 4. Suppose that L_k is an $s-$prime submodule of M for some $k \in \{1, 2, ..., n\}$,then by Proposition 4.3 $(L_k : M)^{1/n}$ is an $s-$prime ideal of R for any positive integer n.Therefore $s = \prod_{\substack{j=1 \\ j \neq k}}^{n} s_j \in (L_j : M)$, but $s \notin (L_k : M)^{1/2}$. Consequently, $se_k \in L \cap L_j$ for each $j \neq k$. We will prove that $se_k \notin L \cap L_k$. Suppose for contrary that $se_k \in L \cap L_k$, then$se_k \in L_k$.Since $e_k \notin L_k$ and L_k is an $s-$prime submodule of M, then $s^2 M \subseteq L_k$. Thus $s \in (L_k : M)^{1/2}$ which is a contradiction. Thus $se_k \notin L \cap L_k$. Therefore $\bigcap_{\substack{j=1 \\ j \neq k}}^{n} (L \cap L_j) \not\subset (L \cap L_k)$, but this contradicts Lemma 4.6.

Hence no L_k is $s-$prime. □

Now we are ready to introduce and prove the $S-$Avoidance Theorem for modules.

Theorem 4.8 *(The $S-$Avoidance Theorem for Modules)*

Let M be an R-module, $L_1, L_2, ..., L_n$ a finite number of submodules of M and let L be a submodule of M such that $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$. Assume that at most two of the $L_i's, i = 1, 2, ..., n$ are not $s-$prime and that $(L_j : M) \nsubseteq (L_k : M)^{1/n}$ for every n =1,2 and 4 for every $j \neq k$. Then $L \subseteq L_k$ for some $k \in \{1, 2, ..., n\}$.

Proof. For the given covering $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$, let

$L \subseteq L_{i1} \cup L_{i2} \cup \dots \cup L_{im}$ be its efficient reduction. Then $1 \leq m \leq n$ and $m \neq 2$.

If $m > 2$, then there exists at least one L_{ij} to be $s-$prime. In view of Proposition 4.7. this is impossible. Hence $m = 1$,namely $L \subseteq L_k$ for some $k \in \{1, 2, ..., n\}$. □

Now, we remember the following definition.

Definitions 4.9 *Let $L_1, L_2, ..., L_n$ be submodules of an $R-$module M. Let $L_1 + e_1, L_2 + e_2, ..., L_n + e_n$ be ncosets in M. We call a covering $L \subseteq (L_1 + e_1) \cup (L_2 + e_2) \cup ... \cup (L_n + e_n)$ efficient if no coset is superfluous (i.e., we cant find k such that $L \subseteq L_k + e_k, k \in \{1, 2, ...n\}$).*

Remark 4.10 *If $e_k = e$ for every $k \in \{1, 2, ...n\}$, then the above covering in Definitions 4.9. is equivalent to $L - e \subseteq L_1 \cup L_2 \cup ... \cup L_n$ and this is a coset efficiently covered by a union of submodules.*

The following Lemma was proved by C.Gottlieb in 1994, see [5].

Lemma 4.11 *Let $L \subseteq (L_1 + e_1) \cup (L_2 + e_2) \cup ... \cup (L_n + e_n)$ be*

efficient covering of a submodule L by cosets, where $n \geq 2$. Then

$L \cap (\bigcap_{j=1, j\neq k}^{n} L_j) \subseteq L_k$, *but* $L \not\subset L_k$ *for all* k.

Proposition 4.12 *Let* $L + e \subseteq L_1 \cup L_2 \cup ... \cup L_n$ *be an efficient covering with* $n \geq 2$. *If* $(L_j : M) \nsubseteq (L_k : M)^{1/n}$ *for every* n *=1,2 and 4, for every* $j{\neq}k$, *then no* L_k *for* $k \in \{$ *1,2,...,n* $\}$ *is an* $s-$*prime submodule of* M.

Proof. By Lemma 4.11. $L \cap (\bigcap_{j=1, j\neq k}^{n} L_j) \subseteq L_k$, but $L \not\subset L_k$. Put $I = (\bigcap_{j=1, j\neq k}^{n} L_j : M)$.

Then $IL \subseteq (L \cap (\bigcap_{j=1, j\neq k}^{n} L_j)) \subseteq L_k$. Suppose L_k is an $s-$prime submodule of M for some k, then we have the following two cases:

Case 1: either $L \subseteq L_k$, which is impossible; or

Case 2: $I = (\bigcap_{j=1, j\neq k}^{n} L_j : M) = \bigcap_{j=1, j\neq k}^{n} (L_j : M) \subseteq (L_k : M)^{1/2}$, and this implies that

$(L_j : M) \subseteq (L_k : M)^{1/n}$ for some n =1,2 or 4 for some *j≠k,* because as in

Proposition 4.3. $(N : M)^{1/n}$ is an s-prime ideal of $R$for any positive integer n. However, this case is also impossible.

Hence no L_k is an $s-$prime submodule of M. □

Theorem 4.13 *Let* $L + e \subseteq L_1 \cup L_2 \cup ... \cup L_n$*be a covering such that at most two of the* $L_i's, i = 1, 2, ..., n$ *are not* $s-$*prime and that* $(L_j : M) \nsubseteq (L_k : M)^{1/n}$ *for every*

n *=1,2 and 4 for every j≠k. Then the submodule* $L + eR \subseteq L_k$ *for some* $k \in \{1, 2, ..., n\}$.

Proof. For the given covering $L + e \subseteq L_1 \cup L_2 \cup ... \cup L_n$ let

$L + e \subseteq L_{i1} \cup L_{i2} \cup ... \cup L_{im}$ be its efficient reduction. Then $1 \leq m \leq n$. and $m \neq 2$. If $m \succ 2$, then there exists at least one L_{ij}, $1 \leq j \leq m$ to be $s-$prime. In view of Proposition 4.12. this is impossible. Hence $m = 1$, namely $L + e \subseteq L_k$ for some $k \in \{1, 2, ..., n\}$.

This implies that $L + eR \subseteq L_k$ as $e = 0 + e \in L + e \subseteq L_k$. □

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