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Real Gromov-Witten Invariants on the Moduli Space of Genus 0 Stable Maps to a Smooth Rational Projective Space*

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Abstract

We characterize transversality, non-transversality properties on the moduli space of genus 0 stable maps to a rational projective surface. If a target space is equipped with a real structure, i.e, anti-holomorphic involution, then the results have real enumerative applications. Firstly, we can define a real version of Gromov-Witten invariants. Secondly, we can prove the invariance of Welschinger's invariant in algebraic geometric category.

Key Words: Gromov-Witten invariant, enumerative invariant, transversality, intersection multiplicity, real structure.

1. Introduction

Let $\overline{M}_k(X,\beta)$ be the moduli space of stable maps from a k-pointed arithmetic genus 0 curve to X, representing a 2nd homology class β. Let $[\Upsilon_1], \dots, [\Upsilon_k]$ be Poincaré duals to the homology classes represented by $\Upsilon_1, \dots, \Upsilon_k$, where $\Upsilon_1, \dots, \Upsilon_k$ are pure dimensional varieties in the target space X. The Gromov-Witten invariant on $\overline{M}_k(X,\beta)$ is defined as:

$$I_\beta([\Upsilon_1], \dots, [\Upsilon_k]) := \int_{\overline{M}_k(X,\beta)} ev_1^*([\Upsilon_1]) \cup \cdots \cup ev_k^*([\Upsilon_k]),$$

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*Dedicated to the originator Gang Tian

where ev_i is an i-th evaluation map. The Gromov-Witten invariant $I_\beta([\Upsilon_1], \ldots, [\Upsilon_k])$ may be non-trivial only when $\sum \mathrm{codim}(\Upsilon_i) = \dim \overline{M}_k(X, \beta)$. We say that *the Gromov-Witten invariant has an enumerative meaning* if $I_\beta([\Upsilon_1], \ldots, [\Upsilon_k])$ equals to the actual number of points in $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_k^{-1}(\Gamma_k)$, where $\Gamma_1, \ldots, \Gamma_k$ are any pure dimensional varieties in a general position such that $[\Gamma_i] = [\Upsilon_i]$, $i = 1, \ldots, k$. So, the Gromov-Witten invariant counts the number of stable maps whose i-th marked point maps into Γ_i if it has an enumerative implication. Note that the number of intersection points in $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_k^{-1}(\Gamma_k)$ doesn't vary depending on the general choices of the cycle's representatives Γ_i. And $ev_i^{-1}(\Gamma_i)$, $i = 1, \ldots, k$, meet transversally for the general choices of the cycle's representatives Γ_i. As it is well-known, the Gromov-Witten invariant has an enumerative implication if the target space is a homogeneous variety. See [4]. See [5] for an enumerative application of the Gromow-Witten invariant of blow-ups of $\mathbb{CP}^2$ at the finite number of points.

Let X be a rational projective surface. That is, X is deformation equivalent to either $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$ blown-up at the finite number of points $B := \{b_1, \ldots, b_r\}$ which we will denote by $r\mathbb{CP}^2$. The aim of this paper is to study the intersection theoretic properties on $\overline{M}_k(X, \beta)$ and real enumerative applications when X is equipped with a real structure induced by a complex conjugation map on the complex projective space. That is,

- $\mathbb{CP}^2 \to \mathbb{CP}^2$, $[a : b : c] \mapsto [\overline{a} : \overline{b} : \overline{c}]$
- $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$, $([a : b], [c : d]) \mapsto ([\overline{a} : \overline{b}], [\overline{c}, \overline{d}])$.

Let $r\mathbb{CP}^2$ come from blown-ups of $\mathbb{CP}^2$ at $B := \{b_1, \ldots, b_r\}$, where B is preserved by a complex conjugation map on $\mathbb{CP}^2$. Then $r\mathbb{CP}^2$ has an obvious real structure which is induced by a complex conjugation involution on $\mathbb{CP}^2$.

A real projective variety is a projective variety defined over $\mathbb{C}$, having a real structure, i.e., an anti-holomorphic involution. *A real part* of the real projective variety is the locus which is fixed by an anti-holomorphic involution. Let X be a real projective rational surface which is described above. Then, $\overline{M}_k(X, \beta)$ is a real projective variety. Based on the results in [10], we can study the real enumerative problems on the real part $\overline{M}_k(X, \beta)^{re}$.

Let $\Gamma_1, \ldots, \Gamma_k$ be any pure dimensional *real* projective varieties in the real rational surface X in a general position such that $[\Gamma_i] = [\Upsilon_i]$, $i = 1, \ldots, k$. Assume that the Gromov-Witten invariant $I_\beta([\Upsilon_1], \ldots, [\Upsilon_k])$ is non-trivial. Then, the number of intersection points $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_k^{-1}(\Gamma_k) \cap \overline{M}_k(X,\beta)^{re}$ varies depending on the actual choice of $\Gamma_1, \ldots, \Gamma_k$. However, if each point in $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_k^{-1}(\Gamma_k) \cap \overline{M}_k(X,\beta)^{re}$ has an intersection multiplicity one, then the number of points doesn't vary by little perturbations of cycle's representatives $\Gamma_1, \ldots, \Gamma_k$. The changes of the number of intersection points in $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_k^{-1}(\Gamma_k) \cap \overline{M}_k(X,\beta)^{re}$ happen only after some of the intersection points have intersection multiplicities greater than one. So, it is important to study transversality and non-transversality properties.

Definition 1.1 *Let X be deformation equivalent to $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$ blown-up at finite number of points.*

(a) A stable map $f: C \to X$ is called a cuspidal stable map *if C is isomorphic to $\mathbb{CP}^1$, and its image $f(C)$ contains only node singularities and a unique cuspidal singularity.*

(b) By an equi-singular locus, *we mean the set of stable maps in $\overline{M}_k(X,\beta)$ which have the same type of singularities on the image curves if the domain curve is irreducible. If the domain curve is reducible, then an* equi-singular locus *means the set of stable maps having the same number of irreducible components in the domain curve.*

Theorem 1.1 and Theorem 1.2, below, are the main results of this paper. Both Theorems are important for real enumerative applications. However, in both Theorems, we don't need to assume X has a real structure.

Theorem 1.1 *Let X be deformation equivalent to either $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$. Let Γ_i, $i = 1, \ldots, k$, be pure dimensional subvarieties in X. Let p be a point in $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_k^{-1}(\Gamma_k)$, which represents a cuspidal stable map. Assume that $\sum_{i=1}^{k} codim\Gamma_i = dim\overline{M}_k(X,\beta)$.*

(i) Assume $k = -\int_\beta \omega_X - 1$. Then the intersection multiplicity at p is 2. The cuspidal stable maps locus is a unique equi-singular locus having a codimension ≤ 1 on which a transversality always fails.

(ii) Assume $k > -\int_\beta \omega_X - 1$. Suppose that the image of the stable map represented by p meets all non-trivial Chow cycle's representatives Γ_i transversally. Then, the intersection multiplicity at p is 2. Transversality uniformly fails along the cuspidal stable maps' locus.

An easy consequence of Theorem 1.1 is the number of points in $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_k^{-1}(\Gamma_k)$ is always strictly less than the Gromov-Witten invariant $I_\beta([\Gamma_1], \ldots, [\Gamma_k])$ if any one of the points in $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_k^{-1}(\Gamma_k)$ is a cuspidal stable map.

A variety X is called a *convex variety* if $H^1(C, f^*TX)$ vanishes for all arithmetic genus 0 curves C. Let $r\mathbb{CP}^2$ be deformation equivalent to $\mathbb{CP}^2$ blown up at r points $\{b_1, \ldots, b_r\}$. Then $r\mathbb{CP}^2$ is a non-convex variety. Although we need to consider a virtual fundamental cycle, the Gromov-Witten invariant has an enumerative meaning. See [5]. That is, the intersection theory on the reducible domain curve's locus is not important for the enumerative application purpose in complex category. For a real enumerative application, we consider only a *non-exceptional divisor class* $d\cdot$[line] in $r\mathbb{CP}^2$ in this paper.

Theorem 1.2 *Let p_i, $i = 1, \ldots, 3d-1$, be points in general position in $r\mathbb{CP}^2 \setminus (\mathbb{CP}^1_1 \cup \cdots \cup \mathbb{CP}^1_r)$, where $\mathbb{CP}^1_j$ is an exceptional divisor, $j = 1, \ldots, r$. Let p be a point in $ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1}) \subset M_{3d-1}(r\mathbb{CP}^2, d \cdot [line])$, which represents a cuspidal stable map. Then, the intersection multiplicity at p is 2. A cuspidal stable maps locus is a unique equi-singular locus in $M_{3d-1}(r\mathbb{CP}^2, d \cdot [line])$ which has a codimension ≤ 1 and on which a transversality always fails.*

When the target space X is equipped with a real structure, Theorem 1.1 and Theorem 1.2 play key roles in defining real Gromov-Witten invariants which are local invariants on $X^{re} \times \cdots \times X^{re} := (X^k)^{re}$, in the case of $I_\beta([\text{point}], \ldots, [\text{point}])$. Other natural application is the proof of the invariance of the Welschinger's invariant in algebraic geometric category.

One of the possible application problems suggested by Gang Tian is the following problem. I invite the challenging readers to attempt to resolve the following real enumerative problem:

Prove or disprove: There are 11 real configuration points in $\mathbb{CP}^2$ such that all degree 4 rational nodal curves passing through those configuration points are real curves.

The result with 564 real configuration points in [11] is currently 2 the best toward this problem. See Theorem 3.6 in [12] for the degree 3 case.

This paper is organized as follows: In Section 2.1, we show that the nodal Severi variety is embedded into $\overline{M}_0(\mathbb{CP}^2, d)$ as an open locus on which the intersection theory is established. Then, by using the classical results, we classify the degeneration properties of the nodal stable maps in $\overline{M}_0(\mathbb{CP}^2, d)$. In Section 2.2.1, we go through the local calculations of the differential of the ev map on the loci whose codimensions are less than or equal to one in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$. The index of the ev map is identical to the intersection multiplicity. In Section 2.2.2, we characterize walls and chambers in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$ and the Chow 0-cycles parameter space. And then, we define the real version of Gromov-Witten invariants in I_d([point], ... , [point]) case. In Section 2.2.3, we work on the I_d([point], ... , [point],[line], ... , [line]) case. Different from the results in Section 2.2.2, walls and chambers are not characterized by the geometric properties of the stable maps. In general case, we need to add the tangency condition in our consideration. In Section 3, we work on $r\mathbb{CP}^2$ with the divisor class of $d\cdot$[line] and $\mathbb{CP}^1 \times \mathbb{CP}^1$ target space case.

Throughout the paper, we will assume the degree of stable maps is greater than or equal to 3 if the target space is $\mathbb{CP}^2$ or $r\mathbb{CP}^2$. But tangent space splitting calculations done in Section 2.2.1 hold for all degree d. We will use a notation d when we consider $d \cdot [line]$ in most cases.

2. Transversality properties on $\overline{M}_k(\mathbb{CP}^2, d)$ and their real enumerative implications

2.1. The classical rational nodal Severi variety v.s. the moduli space of stable maps $\overline{M}_0(\mathbb{CP}^2, d)$

The geometric genus of a plane nodal curve is the sum of an arithmetic genus on each component after we desingularize all nodes.

A rational nodal Severi variety $\mathcal{NS}^d$ of degree d is the set of reduced, irreducible plane curves of degree d, having only nodes as singularities, and having geometric genus zero. The rational nodal Severi variety $\mathcal{NS}^d$ is a $3d-1$ dimensional (quasi-)projective variety.

Proposition 2.1 *The rational nodal Severi variety $\mathcal{NS}^d$ of degree d is embedded into the moduli space $\overline{M}_0(\mathbb{CP}^2, d)$ of stable maps as an open sublocus.*

Proof. Let $\mathbf{c}$ be a point in $\mathcal{NS}^d$. Let C be a reduced, irreducible, rational nodal curve in $\mathbb{CP}^2$ which is represented by $\mathbf{c}$. Consider the morphism F from $\mathcal{NS}^d$ to $\overline{M}_0(\mathbb{CP}^2, d)$ defined by $\mathbf{c} \mapsto [(f, \widetilde{C})]$, where f is a normalization of C. The morphism is well-defined because the normalization is unique up to isomorphism and the isomorphism is exactly the equivalence relation of stable maps.

Let $T_{\mathbf{c}}\mathcal{NS}^d$ be a tangent space at $\mathbf{c} \in \mathcal{NS}^d$. Then, the Kodaira-Spencer map $\mathcal{K}_{\mathbf{c}} : T_{\mathbf{c}}\mathcal{NS}^d \to H^0(\widetilde{C}, N_{\mathbf{c}})$ is onto, where $N_{\mathbf{c}}$ is Coker$(df : T\widetilde{C} \to f^*T\mathbb{CP}^2)$. See [8, p.110]. $\mathcal{K}_{\mathbf{c}}$ is isomorphic because the degree of the line bundle $N_{\mathbf{c}}$ is $3d-2$. The standard K-group calculations from the tangent obstruction long exact sequence show the following isomorphism of the tangent space at $[(f, \widetilde{C})] \in \overline{M}_0(\mathbb{CP}^2, d)$:

$$T_{[(f,\tilde{C})]}\overline{M}_0(\mathbb{CP}^2, d) \cong \mathrm{Ext}^1(f^*\Omega^1_{\mathbb{CP}^2} \to \Omega^1_{\tilde{C}}, \mathcal{O}_{\tilde{C}}) \cong H^0(\widetilde{C}, N_{\mathbf{c}}) \tag{2.1}$$

Thus, the Kodaira-Spencer map with (2.1) shows that the differential $dF_{\mathbf{c}}$ at $\mathbf{c}$ is an isomorphism. The Proposition follows because $\mathcal{NS}^d$ is a quasi-projective variety. □

In fact, the rational nodal Severi variety $\mathcal{NS}^d$ is embedded into the fine moduli locus in $\overline{M}_0(\mathbb{CP}^2, d)$. The reason is there is a universal curve over $\mathcal{NS}^d$ such that the fiber over each point in $\mathcal{NS}^d$ is the normalization of the corresponding plane curve at the node singularities and the canonical morphism from the universal curve to $\mathbb{CP}^2$. See p6 in [2].

Corollary 2.2 *Let* $\mathcal{NL} \subset \overline{M}_k(\mathbb{CP}^2, d)$ *be the locus of pointed stable maps* $[(f, \mathbb{CP}^1, a_1, \ldots, a_k)]$ *which satisfies the following:*

- *The image curve* $f(\mathbb{CP}^1)$ *is reduced, irreducible, rational nodal curves*
- $f(a_1), \ldots, f(a_k)$ *do not meet the node singularities in* $f(\mathbb{CP}^1)$.

Then $\mathcal{NL}$ *is an open sublocus.*

Proof. Note that the condition in the second item is an open condition. Since the forgetful map is continuous, Proposition 2.1 implies that $\mathcal{NL}$ is an open sublocus. □

Remark 2.3 As it is well-known, the Gromov-Witten invariant counts irreducible rational nodal curves passing through $3d-1$ points in general position in $\mathbb{CP}^2$. The proof of Proposition 2.1 shows the exact correspondences between the irreducible rational nodal curves and the stable maps passing through the same set of $3d-1$ points. Therefore, the locus in Corollary 2.2 is the locus where the intersection theory is established.

Definition 2.1 A nodal stable map *is a stable map* $f : C \to \mathbb{CP}^2$ *such that* C *is isomorphic to* $\mathbb{CP}^1$ *and its image* $f(C)$ *contains only (ordinary) node singularities.*

Definition 2.2 A tac node stable map is a stable map $f : C \to \mathbb{CP}^2$ such that C is isomorphic to $\mathbb{CP}^1$ and its image $f(C)$ contains only ordinary node singularities and a unique tacnode. A triple node stable map is a stable map $f : C \to \mathbb{CP}^2$ such that C is isomorphic to $\mathbb{CP}^1$ and its image $f(C)$ contains only ordinary node singularities and a unique triple node.

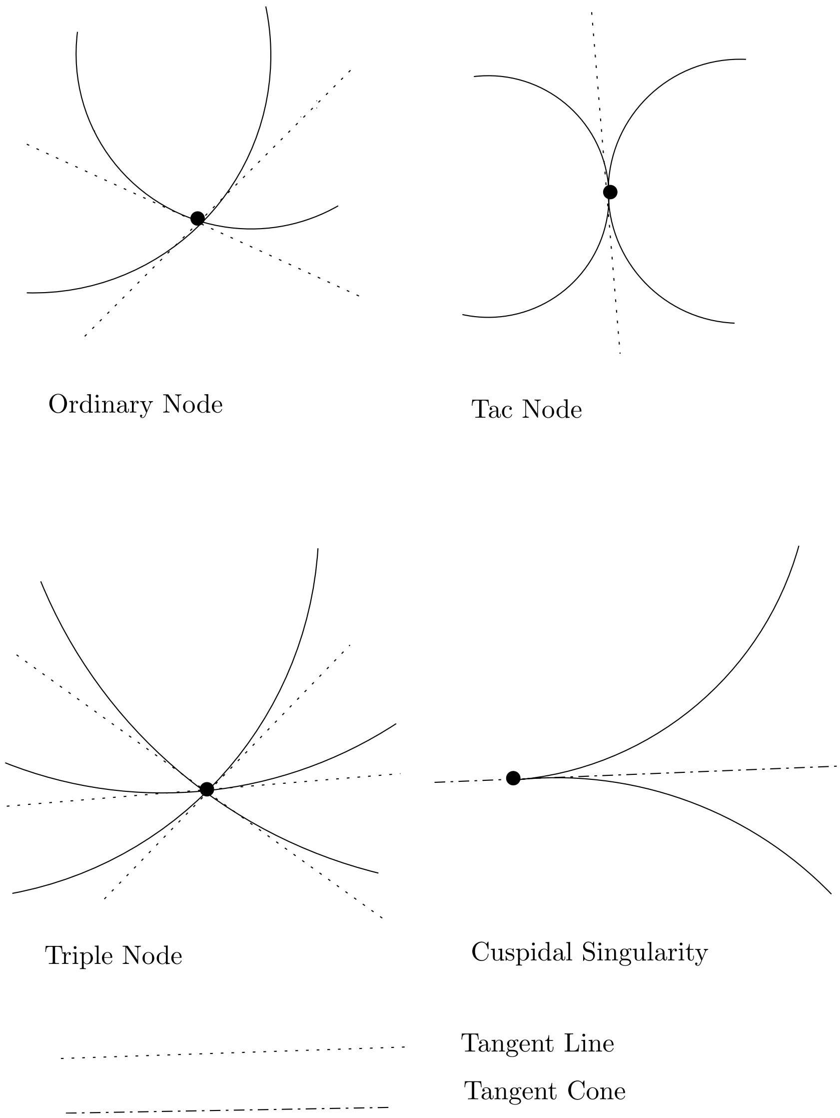

Figure 1. Type of singularities

The following Theorem characterizes the codimension one equi-singular locus. One can easily see that the same characterization holds in $\overline{M}_k(\mathbb{CP}^2, d)$.

In the proof of Theorem 2.4, the cuspidal curve locus in the partially compactified nodal Severi variety $\mathcal{NS}^d$ is the locus where each curve has exactly one cuspidal singularity and all other singularities are node singularities.

Theorem 2.4 *Let $\mathcal{NL}$ denote the nodal stable maps locus in $\overline{M}_0(\mathbb{CP}^2, d)$. The codimension one equi-singular loci in $\overline{M}_0(\mathbb{CP}^2, d) \setminus \mathcal{NL}$, are the cuspidal stable maps locus, the tacnode stable maps locus, the triplenode stable maps locus, the locus consisting of the stable maps whose domain curves consist of two irreducible components.*

Proof. The last case is well-known.

Theoren 1.4 in [3] shows that the partially compactified rational nodal Severi variety has the same classification as codimension one equi-singular loci with the classification of the codimension one equi-singular loci in $\overline{M}_0(\mathbb{CP}^2, d)$ in this Theorem. There is a universal curve on the normalized partially complactified rational nodal Severi variety, whose fiber over each point is the normalization of the corresponding plane curve at the assigned singularities. And there is a canonical morphism from the universal curve to $\mathbb{CP}^2$. See p.6 in [2]. Thus, there is a canonical embedding from the normalized partially compactified rational nodal Severi variety to the fine moduli locus in $\overline{M}_0(\mathbb{CP}^2, d)$. This canonical embedding relates a plane curve represented by a point in the partially compactified rational nodal Severi variety with the normalization of the plane curve in $\overline{M}_0(\mathbb{CP}^2, d)$. The normalization on the partially compactified rational nodal Severi variety does not change the codimension properties in the partially compactified nodal Severi variety. Thus, the Theorem follows. □

2.2. Transversality property on $\overline{M}_k(\mathbb{CP}^2, d)$ and its real enumerative implications

2.2.1. Transversality properties for the Gromov-Witten invariant I_d([point], . . . , [point]) case

The tangent space at $[(f, \mathbb{CP}^1, a_1, \dots, a_k)]$ in $\overline{M}_k(\mathbb{CP}^2, d)$ is the hyperext group $Ext^1(f^*\Omega^1_{\mathbb{CP}^2} \to \Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1})$. We calculate the hyperext group in terms of the ordinary sheaf cohomology group.

Lemma 2.5 *Let* $[(f, \mathbb{CP}^1, a_1, \ldots, a_k)]$ *be a point in* $\overline{M}_k(\mathbb{CP}^2, d)$*. Then the tangent space at* $[(f, \mathbb{CP}^1, a_1, \ldots, a_k)]$ *is*

$$T\overline{M}_k(\mathbb{CP}^2, d)\,|_{[(f,\mathbb{CP}^1,a_1,\ldots,a_k)]} \cong\ H^0(\mathbb{CP}^1, N_f) \oplus T_{a_1}\mathbb{CP}^1 \oplus \cdots \oplus T_{a_k}\mathbb{CP}^1$$

where N_f *is Coker* $(df : T\mathbb{CP}^1 \to f^*T\mathbb{CP}^2)$.

Proof. The long exact sequence associated to the hyperext group $Ext^1(f^*\Omega^1_{\mathbb{CP}^2} \to \Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1})$ is

$$\begin{aligned} 0 \ \to\ & Hom(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1}) \ \to\ H^0(\mathbb{CP}^1, f^*T\mathbb{CP}^2) \to \\ & \to Ext^1(f^*\Omega^1_{\mathbb{CP}^2} \to \Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1}) \to \\ & \to Ext^1(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1}) \to 0 \end{aligned} \tag{2.2}$$

The last term $Ext^1(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1})$ of the deformation of a pointed curve is isomorphic to $H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k))$. It comes from an exact sequence of the local to global spectral sequence

$$\begin{aligned} 0 \ \to\ & H^1(\mathbb{CP}^1, \underline{Ext}^0_{\mathbb{CP}^1}(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1})) \ \to \\ & \to\ Ext^1(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1}) \ \to \\ & \to\ H^0(\mathbb{CP}^1, \underline{Ext}^1_{\mathbb{CP}^1}(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1})) \ \to\ 0 \end{aligned} \tag{2.3}$$

by noting

- $H^0(\mathbb{CP}^1, \underline{Ext}^1_{\mathbb{CP}^1}(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1}))$ is isomorphic to $H^0(\mathbb{CP}^1, \underline{Ext}^1_{\mathbb{CP}^1}(\Omega^1_{\mathbb{CP}^1}, \mathcal{O}_{\mathbb{CP}^1}))$, because the first order of smoothing is supported by nodes and independent of marked points. Thus it vanishes.

- $\underline{Ext}^0_{\mathbb{CP}^1}(\Omega^1_{\mathbb{CP}^1}(a_1+\cdots+a_k), \mathcal{O}_{\mathbb{CP}^1})$ is isomorphic to $Hom(\Omega^1_{\mathbb{CP}^1}(a_1+\cdots+a_k), \mathcal{O}_{\mathbb{CP}^1})$. Hence, $H^1(\mathbb{CP}^1, \underline{Ext}^0_{\mathbb{CP}^1}(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1}))$ is isomorphic to $H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k))$.

Thus, we get a splitting of a tangent space $Ext^1(f^*\Omega^1_{\mathbb{CP}^2} \to \Omega^1_{\mathbb{CP}^1}(a_1+\cdots+a_k), \mathcal{O}_{\mathbb{CP}^1})$ which is naturally isomorphic to $\ominus Hom(\Omega^1_{\mathbb{CP}^1}(a_1+\cdots+a_k), \mathcal{O}_{\mathbb{CP}^1}) \oplus H^0(\mathbb{CP}^1, f^*(T\mathbb{CP}^2)) \oplus$

$H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k))$.

The result follows from K-group calculations associated to a long exact sequence induced from the following two short exact sequences of sheaves:

$$0 \to T\mathbb{CP}^1(-a_1 - \cdots - a_k) \to T\mathbb{CP}^1 \to T_{a_1}\mathbb{CP}^1 \oplus \cdots \oplus T_{a_k}\mathbb{CP}^1 \to 0 \tag{2.4}$$

$$0 \to T\mathbb{CP}^1 \to f^*T\mathbb{CP}^2 \to N_f \to 0 \tag{2.5}$$

More precisely,

$$\begin{aligned} & Hom(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1}) \\ \cong & H^0(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k)) \\ \cong & H^0(\mathbb{CP}^1, T\mathbb{CP}^1) \ominus (T_{a_1}\mathbb{CP}^1 \oplus \cdots \oplus T_{a_k}\mathbb{CP}^1) \\ & \oplus H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k)) \quad \text{by (2.4);} \end{aligned}$$

$H^0(\mathbb{CP}^1, f^*(T\mathbb{CP}^2)) \cong H^0(\mathbb{CP}^1, T\mathbb{CP}^1) \oplus H^0(\mathbb{CP}^1, N_f)$ by (2.5).
Thus

$$\begin{aligned} & \ominus Hom(\Omega^1_{\mathbb{CP}^1}(a_1 + \cdots + a_k), \mathcal{O}_{\mathbb{CP}^1}) \oplus H^0(\mathbb{CP}^1, f^*(T\mathbb{CP}^2)) \\ & \oplus H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k)) \\ \cong & [\ominus H^0(\mathbb{CP}^1, T\mathbb{CP}^1) \oplus (T_{a_1}\mathbb{CP}^1 \oplus \cdots \oplus T_{a_k}\mathbb{CP}^1) \\ & \ominus H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k))] \bigoplus [H^0(\mathbb{CP}^1, T\mathbb{CP}^1) \oplus H^0(\mathbb{CP}^1, N_f)] \\ & \bigoplus H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k)) \\ \cong & H^0(\mathbb{CP}^1, N_f) \oplus T_{a_1}\mathbb{CP}^1 \oplus \cdots \oplus T_{a_k}\mathbb{CP}^1. \end{aligned}$$

□

Remark 2.6 Let $\overline{M}_k$ be the Delign-Mumford moduli space of genus zero curves with k marked points. Then, $\overline{M}_k(\mathbb{CP}^2, 0)$ is isomorphic to $\overline{M}_k \times \mathbb{CP}^2$. Thus, we may consider its tangent space at $[(f, \mathbb{CP}^1, a_1, \ldots, a_k)]$ is $H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k)) \oplus T_{f(\mathbb{CP}^1)}\mathbb{CP}^2$.

This tangent space formula is identical to the formula in Lemma 2.5 in a K-theoretic point of view:

$H^0(\mathbb{CP}^1, N_f) \oplus T_{a_1}\mathbb{CP}^1 \oplus \cdots \oplus T_{a_k}\mathbb{CP}^1$
$\cong H^0(\mathbb{CP}^1, f^*T\mathbb{CP}^2) \ominus H^0(\mathbb{CP}^1, T\mathbb{CP}^1) \oplus T_{a_1}\mathbb{CP}^1 \oplus \cdots \oplus T_{a_k}\mathbb{CP}^1$ by (2.6)
$\cong H^0(\mathbb{CP}^1, f^*T\mathbb{CP}^2) \oplus H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k))$ by (2.4) since k is greater than 2 and the degree of $T\mathbb{CP}^1$ is 2
$\cong T_{f(\mathbb{CP}^1)}\mathbb{CP}^2 \oplus H^1(\mathbb{CP}^1, T\mathbb{CP}^1(-a_1 - \cdots - a_k))$ because deg$f$=0.

The evaluation map ev on $\overline{M}_k(X,d)$ is a morphism from $\overline{M}_k(X,d)$ to $X \times \cdots \times X$ which sends $[(f, C, a_1, \ldots, a_k)]$ to $(f(a_1), \ldots, f(a_k))$. The following Proposition also establishes a well-known transversality property of intersection cycles on the nodal stable maps locus in the case of $I_d([point], \ldots, [point])$.

Proposition 2.7 *The evaluation map ev on* $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$ *is a local isomorphism on the nodal stable maps locus.*

Proof. Let $\mathbf{c} := [(f, \mathbb{CP}^1, a_1, \ldots, a_{3d-1})]$ represent a nodal stable map. Then, a differential map *dev* at $\mathbf{c}$ is

$$\begin{array}{llllll} T_{\mathbf{c}}\overline{M}_{3d-1}(\mathbb{CP}^2,d) \cong & H^0(\mathbb{CP}^1, N_f) & \oplus T_{a_1}\mathbb{CP}^1 & \oplus \cdots & \oplus T_{a_{3d-1}}\mathbb{CP}^1 & \\ & (s, & v_1, & \ldots, & v_{3d-1}) & \\ & & & & & \\ \overset{dev|_{\mathbf{c}}}{\longrightarrow} & T_{f(a_1)}\mathbb{CP}^2 & \times \cdots \times & T_{f(a_{3d-1})}\mathbb{CP}^2 & & \\ \mapsto & (s\,|_{a_1} + df\,|_{a_1}(v_1), & \ldots, & s\,|_{a_{3d-1}} + df\,|_{a_{3d-1}}(v_{3d-1})), & & \end{array}$$

where N_f is coker($df : T\mathbb{CP}^1 \to f^*T\mathbb{CP}^2$). Suppose that $dev\,|_{\mathbf{c}}\,(s, v_1, \ldots, v_{3d-1}) = (0, \ldots, 0)$. Then, $s\,|_{a_1} = \cdots = s\,|_{a_{3d-1}} = 0$ and $df\,|_{a_1}(v_1) = \cdots = df\,|_{a_{3d-1}}(v_{3d-1}) = 0$, because the vectors $s\,|_{a_i}$ and $df\,|_{a_i}(v_i)$ are linearly independent. Since the degree of the line bundle N_f is $3d-2$, the global section which vanishes at $3d-1$ points represents a trivial element in $H^0(\mathbb{CP}^1, N_f)$. Moreover, v_i, $i = 1, \ldots, 3d-1$, are zero because f is an immersion. This proves that ev is injective. It also implies that ev is surjective because the domain and the target of the linear map $dev\,|_{\mathbf{c}}$ have the same dimension. □

Lemma 2.8 *Assume* $p_1, \ldots, p_{3d-1}$ *represent trivial cycles in* $\mathbb{CP}^2$ *in general position. Let* $\mathbf{c} := [(f, \mathbb{CP}^1, a_1, \ldots, a_{3d-1})]$ *be a point in* $ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1})$. *Suppose that the degree of the evaluation map* $ev : \overline{M}_{3d-1}(\mathbb{CP}^2, d) \to \mathbb{CP}^2 \times \cdots \times \mathbb{CP}^2$ *at* $\mathbf{c}$ *is* k. *Then the intersection multiplicity at* $\mathbf{c}$ *is* k.

Proof. Since the transversality property is an open condition, the Lemma follows from the definition of the local degree of the map and the intersection multiplicity at $\mathbf{c}$. □

If the evaluation map is regular at $\mathbf{c}$ in Lemma 2.8, then Lemma 2.8 implies that the cycles $ev_1^{-1}(p_1), \ldots, ev_{3d-1}^{-1}(p_{3d-1})$ meet transversally at $\mathbf{c}$.

Proposition 2.9 *Assume $p_1, \ldots, p_{3d-1}$ represent trivial cycles in $\mathbb{CP}^2$ in general position. Let $\boldsymbol{c} := [(f, \mathbb{CP}^1, a_1, \ldots, a_{3d-1})]$ be a point in $ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1})$, which represents either a nodal stable map, a tacnode stable map or a triple node stable map. Then the intersection multiplicity at $\boldsymbol{c}$ is one.*

Proof. The case of a nodal stable map follows from Proposition 2.7 and Lemma 2.8. One obtains a tacnode stable map by the normalization of the image curve twice. Each normalization is an immersion. Thus, a tacnode stable map is an immersion. A triple node stable map is the normalization of the image curve, which is an immersion. In either cases, N_f is a line bundle of degree $3d-2$. Thus, one can follow the proof of Proposition 2.7 to show that ev is an immersion at $\mathbf{c}$ if $\mathbf{c}$ represents either a tacnode stable map or a triple node stable map. The Proposition follows from Lemma 2.8. □

Remark 2.10 Despite the results in Proposition 2.9, the tacnode stable maps locus, the triple node stable maps locus are not the loci on which the intersection theory for an enumerative implication is established. The reason is those loci are codimension one loci in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$.

We calculate the kernel and the cokernel of an evaluation map ev on $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$.

Lemma 2.11 *Let $\boldsymbol{c} := [(f, \mathbb{CP}^1, a_1, \ldots, a_{3d-1})] \in \overline{M}_{3d-1}(\mathbb{CP}^2, d)$ represent a stable map such that $df\mid_{a_i} \neq 0$, $i = 1, \ldots, 3d-1$. Then, the kernel of the differential of the ev map at $\boldsymbol{c}$ is isomorphic to $H^0(\mathbb{CP}^1, N(-a_1 - \cdots - a_{3d-1}))$, where N is coker$(df : T\mathbb{CP}^1 \to f^*T\mathbb{CP}^2)$.*

Proof. Recall that dev at $\mathbf{c}$ is

$$\begin{array}{rcccc} T_{\mathbf{c}}\overline{M}_{3d-1}(\mathbb{CP}^2, d) \cong & H^0(\mathbb{CP}^1, N) & \oplus T_{a_1}\mathbb{CP}^1 & \oplus \cdots & \oplus T_{a_{3d-1}}\mathbb{CP}^1 \\ & (s, & v_1, & \ldots, & v_{3d-1}) \end{array}$$

$$\begin{array}{cccc} \xrightarrow{dev|_{\mathbf{c}}} & T_{f(a_1)}\mathbb{CP}^2 & \times \cdots \times & T_{f(a_{3d-1})}\mathbb{CP}^2 \\ \mapsto & (s\mid_{a_1} + df\mid_{a_1}(v_1), & \ldots, & s\mid_{a_{3d-1}} + df\mid_{a_{3d-1}}(v_{3d-1})). \end{array}$$

Since f is a local immersion at a_i, $df\mid_{a_i}(v_i)=0$ only if $v_i=0$. $s\mid_{a_i}$, $i=1,\dots,3d-1$, vanish only if $s\in H^0(\mathbb{CP}^1,N(-a_1-\cdots-a_{3d-1}))$. $df\mid_{a_i}(v_i)$ and $s\mid_{a_i}$ are independent vectors if both are non-trivial. Thus, the kernel of *dev* is $H^0(\mathbb{CP}^1,N(-a_1-\cdots-a_{3d-1}))\oplus 0\oplus\cdots\oplus 0\cong H^0(\mathbb{CP}^1,N(-a_1-\cdots-a_{3d-1}))$. □

Let N be coker($df: T\mathbb{CP}^1\to f^*T\mathbb{CP}^2$). If f has k singularities, then N has k skyscraper sheaves which are supported by critical points. The complement of k skyscraper sheaves in N is a locally free sheaf. We will denote this locally free sheaf by NB_k and will call it *the normal bundle of f.*

Lemma 2.12 *Let $\mathbf{c}:=[(f,\mathbb{CP}^1,a_1,\dots,a_{3d-1})]\in\overline{M}_{3d-1}(\mathbb{CP}^2,d)$ represent a stable map f such that $df\mid_{a_i}\neq 0$, $i=1,\dots,3d-1$. Suppose that f has exactly k singular points $b_1,\dots,b_k$ of degrees $d_1,\dots,d_k$ respectively. Then, the cokernel of the differential of the ev map at $\mathbf{c}$ is isomorphic to $H^1(\mathbb{CP}^1,N(-a_1-\cdots-a_{3d-1}))\cong H^1(\mathbb{CP}^1,NB_k(-a_1-\cdots-a_{3d-1}))$.*

Proof. We have the following short exact sequences of sheaves:

$$0\to T\mathbb{CP}^1\to f^*T\mathbb{CP}^2\to N\to 0 \tag{2.6}$$

$$0\to\oplus_{i=1}^k\tau_i^{d_i-1}\to N\to NB_k\to 0. \tag{2.7}$$

Lemma 2.5 and (2.7) shows that the tangent space at $\mathbf{c}$ is

$$T_{\mathbf{c}}\overline{M}_{3d-1}(\mathbb{CP}^2,d)\cong\bigoplus_{i=1}^k\tau_i^{d_i-1}\bigoplus H^0(\mathbb{CP}^1,NB_k)\bigoplus\bigoplus_{i=1}^{3d-1}T_{a_i}\mathbb{CP}^1.$$

Lemma 2.11 and (2.7) implies that the kernel of *dev* is

$$H^0(\mathbb{CP}^1,N(-a_1-\cdots-a_{3d-1}))\cong\bigoplus_{i=1}^k\tau_i^{d_i-1}.$$

Thus,

$$\mathrm{coker} dev \cong \frac{\bigoplus_{i=1}^{3d-1} T_{f(a_i)}\mathbb{CP}^2}{dev(H^0(\mathbb{CP}^1, NB_k)) \oplus dev(\bigoplus_{i=1}^{3d-1} T_{a_i}\mathbb{CP}^1)}.$$

Let B and C be subvector spaces of the vector space A such that $B \cap C = \{0\}$. Then one can easily check the elementary isomorphism $\dfrac{A}{B \oplus C} \cong \dfrac{C^{\perp} \oplus C}{B \oplus C} \cong \dfrac{C^{\perp}}{B}$, where $C^{\perp}$ is an orthogonal complement of C in A. Let N_{a_i} be the orthogonal complement of $dev(T_{a_i}\mathbb{CP}^1)$. Since $dev(H^0(\mathbb{CP}^1, NB_k)) \cap dev(\bigoplus_{i=1}^{3d-1} T_{a_i}\mathbb{CP}^1) \cong \{0\}$, we get

$$\mathrm{coker} dev \cong (\bigoplus_{i=1}^{3d-1} N_{a_i})/dev(H^0(\mathbb{CP}^1, NB_k)) \tag{2.8}$$

$$\cong H^0(\mathbb{CP}^1, \bigoplus_{i=1}^{3d-1} \nu_i)/dev(H^0(\mathbb{CP}^1, NB_k)), \tag{2.9}$$

where ν_i is a skyscraper sheaf N_{a_i} supported by a_i, $i = 1, \ldots, 3d-1$. Consider the short exact sequence of sheaves $0 \to NB_k(-a_1 - \cdots - a_{3d-1}) \to NB_k \to \bigoplus_{i=1}^{3d-1} \nu_i \to 0$. Let's take a long exact sequence of sheaf cohomologies. Then we have

$$0 \to H^0(\mathbb{CP}^1, NB_k(-a_1 - \cdots - a_{3d-1})) \to H^0(\mathbb{CP}^1, NB_k) \to$$

$$\to H^0(\mathbb{CP}^1, \bigoplus_{i=1}^{3d-1} \nu_i) \to H^1(\mathbb{CP}^1, NB_k(-a_1 - \cdots - a_{3d-1})) \to$$

$$\to H^1(\mathbb{CP}^1, NB_k) \to \cdots .$$

Note that

- $H^0(\mathbb{CP}^1, NB_k(-a_1 - \cdots - a_{3d-1}))$ vanishes because the degree of NB_k is less than $3d-1$.
- The long exact sequence induced by an exact sequence of sheaves

$$0 \to T\mathbb{CP}^1 \to f^*T\mathbb{CP}^2 \to TN \to 0$$

shows that $H^1(\mathbb{CP}^1, N)$ vanishes because $\mathbb{CP}^2$ is a convex variety, i.e., $H^1(\mathbb{CP}^1, f^*T\mathbb{CP}^2) = 0$. Since $H^1(\mathbb{CP}^1, N)$ is isomorphic to $H^1(\mathbb{CP}^1, NB_k)$, $H^1(\mathbb{CP}^1, NB_k)$ vanishes.

Thus, we get the desired isomorphism. □

Proposition 2.13 *The cuspidal stable maps locus forms a degree 2 critical points set of the evaluation map ev.*

Proof. Let $\mathbf{c} := [(f, \mathbb{CP}^1, a_1, \ldots, a_{3d-1})]$ represent a cuspidal stable map. Lemma 2.11 implies the kernel of $dev_{\mathbf{c}} := \tau$, where τ is a skyscraper sheaf supported by the cuspidal singularity. cf. (2.7). Lemma 2.12 implies the cokernel of $dev := H^1(\mathbb{CP}^1, NB_k(-a_1 - \cdots - a_{3d-1}))$. The Kodaira-Serre duality shows that $H^1(\mathbb{CP}^1, NB(-a_1 - \cdots - a_{3d-1}))$ is isomorphic to $H^0(\mathbb{CP}^1, \omega_{\mathbb{CP}^1} \otimes [NB(-a_1 - \cdots - a_{3d-1})]^*)$, where $\omega_{\mathbb{CP}^1}$ is a dualizing sheaf on $\mathbb{CP}^1$ and $[NB(-a_1 - \cdots - a_{3d-1})]^*$ is a dual vector bundle of $[NB(-a_1 - \cdots - a_{3d-1})]$. There is a canonical residue morphism of degree 2 from the local slice of the direction τ to the direction $H^1(\mathbb{CP}^1, NB_k(-a_1 - \cdots - a_{3d-1}))$ induced by the ev map. The morphism is

$$\begin{aligned} \tau \simeq \tau^* \to \quad & H^0(\mathbb{CP}^1, \omega_{\mathbb{CP}^1} \otimes [NB(-a_1 - \cdots - a_{3d-1})]^*)^* \\ v dz \mapsto \quad & \frac{1}{2\pi i} \int_\gamma \frac{v^2}{z} dz \end{aligned}$$

because the local index of f is 2. Thus, the Proposition follows. □

Remark 2.14 From the results in this section, we get the following equivalent conditions:

- dev has a critical point at $[(f, \mathbb{CP}^1, a_1, \ldots, a_{3d-1})]$;
- f is not an immersion;
- $H^0(\mathbb{CP}^1, N(-a_1 - \cdots - a_{3d-1}))$ does not vanish;
- $H^1(\mathbb{CP}^1, N(-a_1 - \cdots - a_{3d-1}))$ does not vanish,

where N is Coker $(df : T\mathbb{CP}^1 \to f^*T\mathbb{CP}^2)$.

Let C be a pointed reducible curve which has two, pointed irreducible components $(C_1, z_1, \dots, z_r)$, $(C_2, w_1, \dots, w_s)$. Let $q_1 \in C_1$, $q_2 \in C_2$ be (pre)gluing points. Then a pointed stable map $(f, C, z_1, \dots, z_r, w_1, \dots, w_s)$ can be written as $((f_1, (C_1, q_1), z_1, \dots, z_r), (f_2, (C_2, q_2), w_1, \dots, w_s))$.

Lemma 2.15 *The tangent space splitting at* $[(f, C, z_1, \dots, z_r, w_1, \dots, w_s)] := [((f_1, (C_1, q_1), z_1, \dots, z_r), (f_2, (C_2, q_2), w_1, \dots, w_s))]$ *is*

$$H^0(C_1, N_1) \oplus H^0(C_2, N_2) \oplus T_{z_1}C_1 \oplus \cdots \oplus T_{z_r}C_1 \oplus T_{w_1}C_2 \oplus \cdots \oplus T_{w_s}C_2 \oplus (T_{q_1}C_1 \otimes T_{q_2}C_2)$$
$$\oplus T_{q_1}C_1 \oplus T_{q_2}C_2 \ominus T_{f(q)}\mathbb{CP}^2$$

where N_i is a Coker($df_i : TC_i \to f^*T\mathbb{CP}^2$), $i = 1, 2$, and $T_{f(q)}\mathbb{CP}^2$ is a skyscraper sheaf supported at $f(q)$, q is a node in C, and the degree of f_i, $i = 1, 2$, is non-trivial.

Proof. We repeat calculations similar to what was done in Lemma 2.5. From the long exact sequence associated to the hyperext group $Ext^1(f^*\Omega^1_{\mathbb{CP}^2} \to \Omega^1_C(z_1 + \cdots + w_s), \mathcal{O}_C)$,

$$0 \to Hom(\Omega^1_C(z_1 + \cdots + z_r + w_1 + \cdots + w_s), \mathcal{O}_C) \to H^0(C, f^*T\mathbb{CP}^2) \to$$
$$\to Ext^1(f^*\Omega^1_{\mathbb{CP}^2} \to \Omega^1_C(z_1 + \cdots + z_r + w_1 + \cdots + w_s), \mathcal{O}_C) \to$$
$$\to Ext^1(\Omega^1_C(z_1 + \cdots + z_r + w_1 + \cdots + w_s), \mathcal{O}_C) \to 0,$$

we get the following tangent space splitting at $(f, C, z_1, \dots, z_r, w_1, \dots, w_s)$:

$$\ominus Hom(\Omega^1_C(z_1 + \cdots + z_r + w_1 + \cdots + w_s), \mathcal{O}_C) \oplus \tag{2.10}$$
$$\oplus H^0(C, f^*T\mathbb{CP}^2) \oplus Ext^1(\Omega^1_C(z_1 + \cdots + z_r + w_1 + \cdots + w_s), \mathcal{O}_C).$$

A standard fact we will use in the following calculations is $Hom(\Omega^1_C, \mathcal{O}_C)$, $\underline{Ext}^0(\Omega_C, \mathcal{O}_C)$ are the sheaf of derivations that come from the pushforward of the sheaf of vector fields on $\widetilde{C} := C_1 \cup C_2$ vanishing at the inverse images q_1, q_2 of the node in C. Let $\pi : \widetilde{C} \to C$ be a normalization map.

We calculate the splitting of each term first.

For $\ominus Hom(\Omega^1_C(z_1+\cdots+z_r+w_1+\cdots+w_s), \mathcal{O}_C)$ term, we use the short exact sequences of sheaves:

$$0 \to TC_1(-z_1 - \cdots - z_r - q_1) \to TC_1 \to T_{z_1}C_1 \oplus \cdots \oplus T_{z_r}C_1 \oplus T_{q_1}C_1 \to 0 \qquad (2.11)$$

$$0 \to TC_2(-w_1 - \cdots - w_s - q_2) \to TC_2 \to T_{w_1}C_2 \oplus \cdots \oplus T_{w_s}C_2 \oplus T_{q_2}C_2 \to 0 \qquad (2.12)$$

to get the K-group equation

$$\begin{aligned}
&Hom(\Omega^1_C(z_1 + \cdots + z_r + w_1 + \cdots + w_s), \mathcal{O}_C) \\
=&H^0(C, TC(-z_1 - \cdots - w_s)) \\
=&H^0(C, \pi_*(T\widetilde{C}(-z_1 - \cdots - w_s - q_1 - q_2))) \\
=&H^0(\widetilde{C}, T\widetilde{C}(-z_1 - \cdots - w_s - q_1 - q_2)) \\
=&H^0(C_1, TC_1(-z_1 - \cdots - z_r - q_1)) \oplus H^0(C_2, TC_2(-w_1 - \cdots - w_s - q_2)) \\
=&H^0(C_1, TC_1) \ominus T_{z_1}C_1 \ominus \cdots \ominus T_{z_r}C_1 \ominus T_{q_1}C_1 \oplus H^1(C_1, TC_1(-z_1 - \cdots - z_r - \\
&q_1)) \oplus H^0(C_2, TC_2) \ominus T_{w_1}C_2 \ominus \cdots \ominus T_{w_s}C_2 \ominus T_{q_2}C_2 \oplus H^1(C_2, TC_2(-w_1 \\
&- \cdots - w_s - q_2))
\end{aligned}$$

by (2.11) and (2.12).

For $H^0(C, f^*T\mathbb{CP}^2)$, we use the short exact sequence of sheaves

$$0 \to f^*T\mathbb{CP}^2 \to f_1^*T\mathbb{CP}^2 \oplus f_2^*T\mathbb{CP}^2 \to T_{f(q)}\mathbb{CP}^2 \to 0,$$

to get a K-group equation

$$H^0(C, f^*T\mathbb{CP}^2) = H^0(C_1, f_1^*T\mathbb{CP}^2) \oplus H^0(C_2, f_2^*T\mathbb{CP}^2) \ominus T_{f(q)}\mathbb{CP}^2,$$

because $H^1(C, f^*T\mathbb{CP}^2)$ vanishes by Lemma 10 in [4].

For $Ext^1(\Omega^1_C(z_1 + \cdots + z_r + w_1 + \cdots + w_s), \mathcal{O}_C)$, we use an exact sequence from the local to global spectral sequence in Lemma 2.5 to get

$$Ext^1(\Omega^1_C(z_1 + \cdots + z_r + w_1 + \cdots + w_s), \mathcal{O}_C)$$

$$= H^1(C, \underline{Ext}^0(\Omega_C(z_1 + \cdots + w_s), \mathcal{O}_C)) \oplus H^0(C, \underline{Ext}^1(\Omega_C(z_1 + \cdots + w_s), \mathcal{O}_C))$$

$$= H^1(C, \pi_*(T\widetilde{C}(-z_1 - \cdots - w_s - q_1 - q_2))) \oplus H^0(C, \underline{Ext}^1(\Omega_C(z_1 + \cdots + w_s), \mathcal{O}_C))$$

$$= H^1(C_1, TC_1(-z_1 - \cdots - z_r - q_1)) \oplus H^1(C_2, TC_2(-w_1 - \cdots - w_s - q_2)) \oplus T_{q_1}C_1 \otimes T_{q_2}C_2.$$

The Lemma follows by putting all terms into (2.10). □

Remark 2.16 1. Let f_2 be a degree 0 map. Calculations similar to what was done in Remark 2.6 by using Lemma 2.15 shows that the tangent space splitting at

$$[(f, C, z_1, \ldots, z_r, w_1, \ldots, w_s)] := [((f_1, (C_1, q_1), z_1, \ldots, z_r), (f_2, (C_2, q_2), w_1, \ldots, w_s))],$$

is

$$H^0(C_1, N_1) \oplus T_{z_1}C_1 \oplus \cdots \oplus T_{z_r}C_1 \oplus T_{q_1}C_1 \oplus H^1(C_2, TC_2(-w_1 - \cdots - w_s - q_2)) \oplus (T_{q_1}C_1 \otimes T_{q_2}C_2).$$

2. One can extend the result in Lemma 2.15 to the general case. Let $\mathbf{c} := [(f, C, a_1, \ldots, a_k)]$ be a point in $\overline{M}_k(\mathbb{CP}^2, d)$. Let $\pi : \widetilde{C} := \mathbb{CP}^1_1 \cup \cdots \cup \mathbb{CP}^1_l \to C$ be a normalization map, where $\mathbb{CP}^1_i$ is biholomorphic to $\mathbb{CP}^1$. Let $g_1, \ldots, g_r$ be singular points on C, $r := l - 1$. Let's denote elements in $\pi^{-1}(g_i)$ by g_i^1, g_i^2. Let N_i be coker$(df_i : T\mathbb{CP}^1_i \to T\mathbb{CP}^2)$, where $f_i := f \mid_{\mathbb{CP}^1_i}$. Then, the tangent space $T_{\mathbf{c}}\overline{M}_k(\mathbb{CP}^2, d)$ at $\mathbf{c}$ is

$$\bigoplus_{i=1}^{l} H^0(\mathbb{CP}^1_i, N_i) \oplus \bigoplus_{i=1,\ldots,k} T_{a_i}\mathbb{CP}^1_{q(a_i)} \oplus \Big(\bigoplus_{i=1,\ldots,r} T_{g_i^1}\mathbb{CP}^1_{q(g_i^1)} \otimes T_{g_i^2}\mathbb{CP}^1_{q(g_i^2)}\Big) \oplus$$
$$\oplus \bigoplus_{i=1,\ldots,r}^{j=1,2} T_{g_i^j}\mathbb{CP}^1_{q(g_i^j)} \ominus \Big(\bigoplus_{i=1}^{r} T_{f(g_i)}\mathbb{CP}^2\Big).$$

See [10] for details of calculations.

The equivalent conditions in Remark 2.14 are no longer true if we consider reducible stable maps. The rank of the cokernel of an evaluation map is determined by the number of marked points on each irreducible domain curve and the mapping properties of the stable map f on each irreducible component.

Proposition 2.17 *Let*

$$\boldsymbol{c} := [(f, C, z_1, \dots, z_r, w_1, \dots, w_s)] \\ := [((f_1, (C_1, q_1), z_1, \dots, z_r), (f_2, (C_2, q_2), w_1, \dots, w_s))]$$

represent a reducible stable map, where C_i, $i = 1, 2$, is isomorphic to $\mathbb{CP}^1$ and f_i is an immersion of degree d_i, $i = 1, 2$ if f is not trivial on the component. Then:

*(i) If any of f_i is a degree 0 map, then the cokernel of dev at **c** has a rank bigger than one.*

*(ii) If r or s is strictly bigger than $3d_1 - 2$ or $3d_2 - 2$, respectively, then the cokernel of dev at **c** has a rank bigger than two.*

*(iii) If r or s is $3d_1 - 2$ or $3d_2 - 2$, respectively, then the cokernel of dev at **c** has a rank one.*

*(iv) If r or s is $3d_1 - 1$ or $3d_2 - 1$, respectively, then the evaluation map ev at **c** is regular.*

Remark 2.18 Bezout's theorem implies that any deformed image curves determined by vectors in $H^0(C_1, N_1)$ and $H^0(C_2, N_2)$ always meet if f is non-trivial, when the target space dimension is two. Thus, in this particular dimension, we can calculate the rank of the cokernel of the *ev* map by considering the vectors other than the vectors in $V := T_{q_1}C_1 \oplus T_{q_2}C_2 \ominus T_{f(q)}\mathbb{CP}^2$.

Sketch of the Proof of Proposition 2.17. Suppose that f_1 is a trivial map. The stability condition implies that C_1 contains at least 2 marked points. The differential of the i-th evaluation map ev_i is zero on C_1. The maximum dimensional contribution to the rank of the *ev* map from C_2 is at most $(3d - 3) + (3d - 3)$. (i) follows from this.

For (ii), (iii), (iv), we need to calculate the dimensional contributions from $T_{q_1}C_1 \otimes T_{q_2}C_2$ and

$$\begin{aligned} & H^0(C_1, N_1) \oplus H^0(C_2, N_2) \oplus T_{z_1}C_1 \oplus \cdots \oplus T_{z_r}C_1 \oplus T_{w_1}C_2 \oplus \cdots \oplus T_{w_s}C_2 \\ & \quad \overset{dev}{\to} T_{f(z_1)}\mathbb{CP}^2 \quad \times \cdots \times T_{f(z_r)}\mathbb{CP}^2 \times T_{f(w_1)}\mathbb{CP}^2 \times \cdots \times T_{f(w_s)}\mathbb{CP}^2 \\ & \qquad (s, t, v_1, \ldots, v_r, v_1', \ldots, v_s') \mapsto (s\mid_{z_1} + df\mid_{z_1}(v_1), \ldots, t\mid_{w_s} + df\mid_{w_s}(v_s')). \end{aligned}$$

We use the following facts:

- $T_{q_1}C_1 \otimes T_{q_2}C_2$ contributes to the rank of *dev* by one.
- The vectors $s\mid_{z_i}$ and $df\mid_{z_i}(v_i)$, $i = 1, \ldots, r$, $t\mid_{w_j}$ and $df\mid_{w_j}(v_j')$, $j = 1, \ldots, s$ are linearly independent.

The reason for the first item is $T_{q_1}C_1 \otimes T_{q_2}C_2$ generates the first order deformation of smoothing node. During the smoothing node deformation, the stable map also changes because the stable maps are the same if they agree to infinite order at any point. Since the deformation space generating smoothing node deformation is smooth, the image of any non-trivial vector in $T_{q_1}C_1 \otimes T_{q_2}C_2$ by *dev* is non-trivial.

The result follows from straightforward dimension counts. □

Let $V_1, \ldots, V_k$ be the intersection cycles in the variety V. Let $\mathbf{p}$ be a point in $V_1 \cap \cdots \cap V_k$. We will say *'the transversality property is not established at* $\mathbf{p}$*'* if the intersection multiplicity at $\mathbf{p}$ is not finite. One can easily check that the loci stated in Proposition 2.17 (i)-(iii) are examples of that. Note that the image of the loci in Proposition 2.17 (iii) is the subloci of codimension 2 in $\mathbb{CP}^2 \times \cdots \times \mathbb{CP}^2$ in general points, which is different from the rank of *dev* along these loci.

Theorem 2.19 *Let* $\mathbf{p}$ *be a point in* $ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1})$ *which represents a cuspidal stable map, where* p_i *is a Chow 0-cycle's representatives,* $i = 1, \ldots, 3d-1$. *Then the intersection multiplicity at* $\mathbf{p}$ *is 2. That is, a transversality uniformly fails along the cuspidal stable maps locus. A cuspidal stable maps locus is a unique equi-singular locus which has a codimension* ≤ 1 *on which a transversality always fails.*

Proof. Proposition 2.9 implies that if $\mathbf{p}$ represents a nodal stable map, a tacnode stable map, a triple node stable map, then the intersection multiplicity at $\mathbf{p}$ is one. Proposition 2.13 and Lemma 2.8 implies that if $\mathbf{p}$ represents a cuspidal stable map, then the intersection multiplicity at $\mathbf{p}$ is 2. The classification of the codimension ≤ 1 equisingular loci in Corollary 2.2 and Theorem 2.4 implies the last statement of the Theorem. □

2.2.2. Real Version of the Gromov-Witten invariants: $I_d([point], \ldots, [point])$ case

If the target space X is equipped with a real structure τ, then it induces a real structure $\overline{M}_n(X, \beta)$ defined by $[(f, C, a_1, \ldots, a_n)] \mapsto [(\overline{f}, \overline{C}, \overline{a}_1, \ldots, \overline{a}_n)]$, where $\overline{f}(x) = \tau \circ f(\overline{x})$, $\overline{C} = (C_1, \overline{q}_1) \cup \cdots \cup (C_m, \overline{q}_m)$ if $C = (C_1, q_1) \cup \cdots \cup (C_m, q_m)$ and C_i is isomorphic to $\mathbb{CP}^1$. The evaluation map *ev* is a real map, that is, it commutes with the real structure of the moduli space and of the target space. See [10]. Let's denote the real part of the moduli space by $\overline{M}_n(X, \beta)^{re}$. Then, the *ev* map sends $\overline{M}_n(X, \beta)^{re}$ to the real part of the target space.

Theorem 2.19 doesn't have the enumerative implications in the complex Gromov-Witten theory. However, Theorem 2.19 is very important when we define the real version of local invariants. The cuspidal stable maps locus comparts the moduli space $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$. The image of the cuspidal stable maps locus by the *ev* map comparts the $3d - 1$ fold product of $\mathbb{RP}^2$. The notions of chambers and walls in the moduli space $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$ and $\mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2$ arise.

Definition 2.3 *(Chambers and Walls in the moduli space $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$.)*
Walls *in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$ are the codimension one loci on which transversality uniformly fails.* Chambers *in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$ are the connected components in the complement of walls.*

Remark 2.20 Theorem 2.19 implies that $\mathbf{c} := [(f, C, a_1, \ldots, a_{3d-1})]$ belongs to the wall if and only if $\mathbf{c}$ represents a cuspidal stable map.

We will call the $3d - 1$ fold product $\mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2$ of $\mathbb{RP}^2$ as a *real Chow 0-cycles parameter space.*

Definition 2.4 Walls *in the real Chow 0-cycles parameter space are the codimension one regions of the image of the cuspidal stable maps locus in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$ by the evaluation map ev.* Chambers *are the connected components of the complement of walls.*

Remark 2.21 If $\mathbf{p} := (p_1, \ldots, p_{3d-1}) \in \mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2$ is a general point in a chamber, then each point in $\mathbf{c} \in ev^{-1}(\mathbf{p}) := ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1})$ represents a nodal stable map. $\mathbf{p}$ is in the wall if one of the points in $ev^{-1}(\mathbf{p})$ represents a cuspidal stable map and all other points represent nodal stable maps.

Obviously, the number of points in $ev^{-1}(\mathbf{p})$ doesn't change for general points in a chamber. Therefore, the local version of the real Gromov-Witten invariants in the following Definition is well-defined.

Definition 2.5 *Let $\mathcal{C}$ be a chamber in the Chow 0-cycles parameter space.* The real Gromov-Witten invariant *of the chamber $\mathcal{C}$ is the number of points in $ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1}) \cap \overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$.*

The following Corollary shows the differences of the real Gromov-Witten invariants in adjacent chambers are exactly two. And walls in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$, $\mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2$ are the place we gain or loose two real solutions.

Corollary 2.22 *Let $\mathbf{c} := [(f, \mathbb{CP}^1, a_1, \ldots, a_{3d-1})]$ be a point in the wall in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$. Let v be a real vector in coker$(dev \mid_{\mathbf{c}})^{re}$ and $p : [-1, 1] \to \mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2$ be a path satisfying the followings:*

- *The path p is tangential to the vector v at $p(0)$.*
- *$p([-1, 0))$ is sitting in one chamber and $p((0, 1])$ is sitting in an adjacent chamber.*

Let $\mathcal{H}$ be a neighborhood of $\mathbf{c} \in p^{-1}(t) \cap \overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re} \cap \mathcal{H}$, where $\mathbf{c}$ represents a cuspidal stable map. Then, $p^{-1}(t) \cap \overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re} \cap \mathcal{H}$ consists of two elements for any general point $t \in [-1, 0)$ and $p^{-1}(t) \cap \overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re} \cap \mathcal{H}$ is an empty set for any general point $t \in (0, 1]$, or vice versa.

Proof. Proposition 2.13 shows that the local model of the ev map along the local slice perpendicular to the cuspidal stable maps locus is $z^2 - t$ or $z^2 + t$. The result follows by restricting ourselves to the real part of the moduli space $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$. □

When we cross the wall in $\mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2$, the inverse images by the ev map vary within the chambers in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$, except one pair of points which vary in adjacent chambers in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$ and meet at the wall in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$ and disappear.

In a real algebraic point of view, we have two notions of node singularities.

Definition 2.6 A non-isolated node *is a node whose local equation is* $z^2 - w^2 = 0$ *or* $zw = 0$ *by real coordinate changes.* An isolated node *is a node whose local equation is* $z^2 + w^2 = 0$ *by real coordinate changes.*

The following example shows that real singularities induce different global topological invariants in the real part of image curves. However, they do not have a topological implication in complex curves.

Example 2.23 Let T^{isol} be a rational real nodal curve of degree 3 which has an isolated node, and let $T^{non-isol}$ be a rational real nodal curve of degree 3 which has a non-isolated node. Then they are isomorphic in complex sense.

Let T be a real torus, i.e., a smooth torus in $\mathbb{CP}^2$ represented by a degree 3 real polynomial. Then T has two generators α, β in the fundamental group of T. The self-automorphism sending α to β and β to α induces an isomorphism between T^{isol} and $T^{non-isol}$ because if T^{isol} is gotten by trivializing a generator α, then $T^{non-isol}$ is from trivializing a generator β. The Euler characteristics $\chi(T^{isol})$, $\chi(T^{non-isol})$ are both one. However, the Euler characteristic of the real part of T^{isol} is 1 and that of the real part of $T^{non-isol}$ is -1.

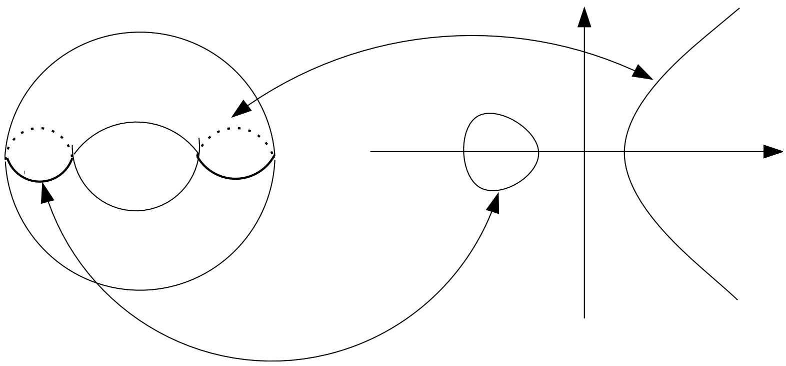

Figure 2. Smooth torus.

Remark 2.24 A cuspidal singularity is from the trivialization of two consecutive generators in a smooth torus of a genus g. If trivializing one generator created an isolated node, then trivializing the adjacent generator created a non-isolated node. The topological transition in the real part of the image curves happens only after we pass through the cuspidal stable maps locus, that is, wall in the real part of the moduli space $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$. The difference in the number of isolated nodes is exactly one. The number of isolated nodes and the number of non-isolated nodes are topological invariants within a chamber.

There is no canonical way to give an orientation on the real part of the moduli space $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$ because the real part $\mathbb{RP}^2$ of the target space is non-orientable. Therefore, we cannot count curves on the real part of the moduli space $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$ intersection theoretically. An alternative approach was taken by J-Y Welschinger in [13]. The

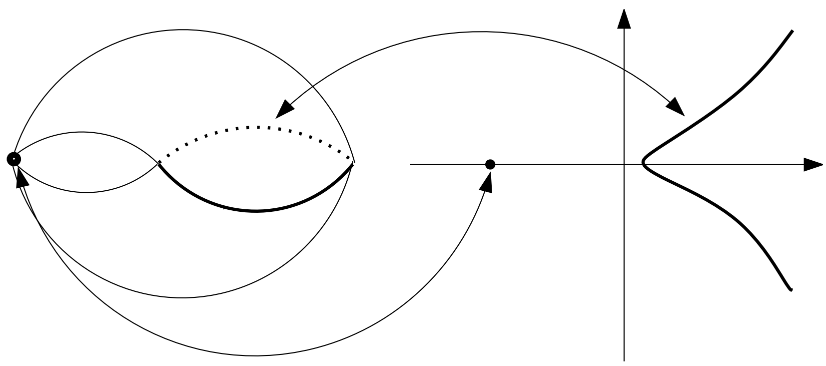

Figure 3. Singular torus having an isolated node.

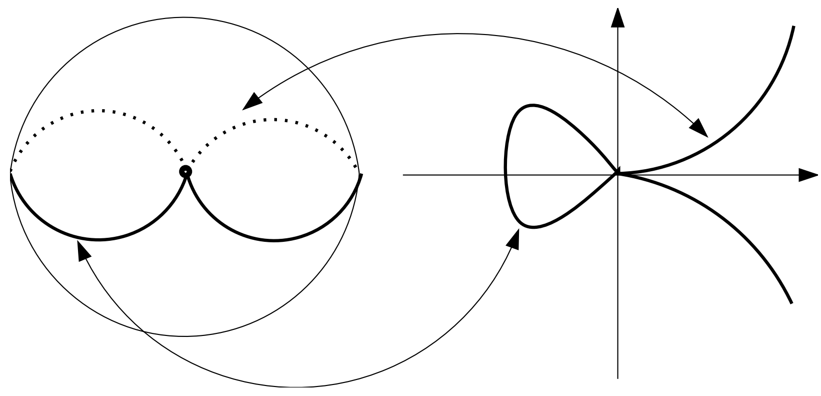

Figure 4. Singular torus having a non-isolated node.

Welschinger's invariant provides the global minimum bound. The results we have so far presented show the invariance in an algebraic category when the target space is $\mathbb{RP}^2$. However, in general target space cases, it isn't necessary that every non-trasversality result produces minimum bound results even for the $I_\beta([point], \ldots, [point])$ case.

Proposition 2.25 *Let $(p_1, \ldots, p_{3d-1})$ be any general element in $\mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2$. Then the real Gromov-Witten invariants on any chamber is greater than or equal to*

$$| \sum_k (-1)^k \textit{the number of real stable maps in } ev_1^{-1}(p_1) \cap \cdots \cap ev_{3d-1}^{-1}(p_{3d-1}) \cap$$

$$\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re} \textit{ having } k \textit{ isolated nodes } |.$$

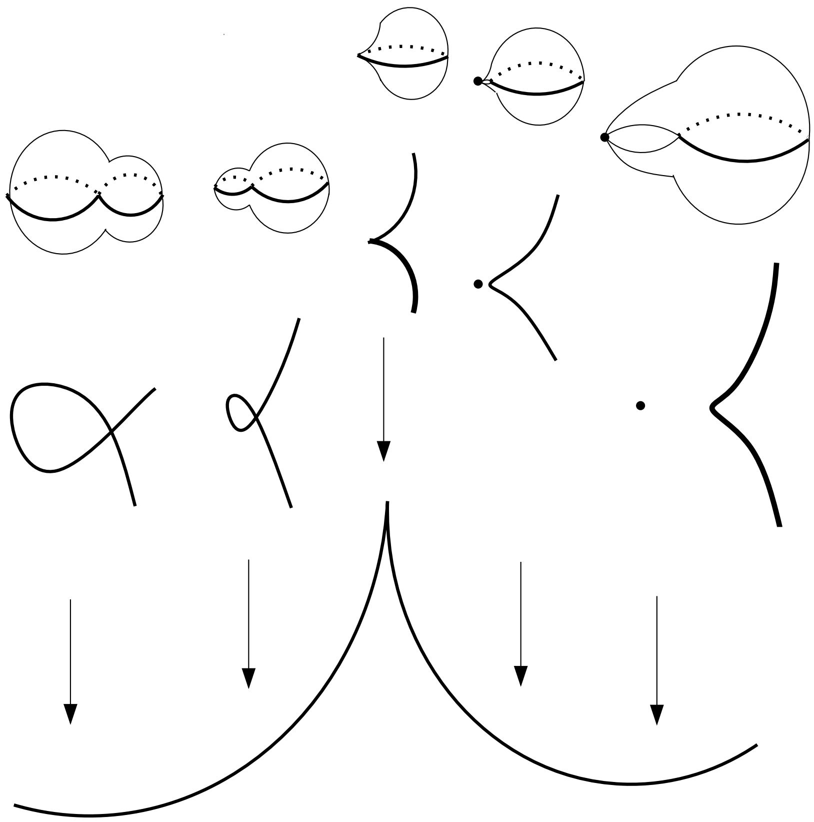

Figure 5. Deformation of elliptic curves around a cuspidal curve.

Proof. The result follows immediately from Corollary 2.22 and Remark 2.24 □

2.2.3. Transversality properties for the Gromov-Witten invariant and its real enumerative implications: the case of I_d([point], ... , [point],[line], ... , [line])

The set of divisors linearly equivalent to d[line], $d > 0$, is in one-to-one correspondence with $(H^0(\mathbb{CP}^2, \mathcal{O}(d)) \setminus \{0\})/\mathbb{C}^*$, which is isomorphic to $\mathbb{CP}^{N(d)}$, where $N(d) = \frac{d\cdot(d+3)}{2}$. We will denote it by $\mathbb{CP}(d)$. The divisor class we will consider in this section will be the combination of [point], [line], i.e., $d = 0$ or 1. Let's denote the product of the Chow cycles parameter space $\mathbb{CP}(d_1) \times \cdots \times \mathbb{CP}(d_k)$ by $\mathbb{CP}(d_1, \ldots, d_k)$, where $d_i = 0$ or 1.

Remark 2.26 Let l be the number of non-zero d_i and m be the number of trivial d_i in $\mathbb{CP}(d_1, \ldots, d_k)$. The Gromov-Witten invariant is enumerative, only when $2 \cdot m + l$ equals the dimension of $\overline{M}_k(\mathbb{CP}^2, d)$. One can easily check that $3d - 1$ is the minimum number of marked points which produce the non-trivial Gromov-Witten invariant. If $k > 3d - 1$, then there are exactly $3d - 1$ number of $d_i = 0$ and $k - (3d - 1)$ number of $d_j = 1$.

Various transversality results in section 2.2.2 are extendible as follows. However, the following Theorem does not characterize the intersection theoretic properties as in the case of I_d([point], ... , [point]).

Theorem 2.27 *Let the Gromov-Witten invariant $I_d(d_1, \ldots, d_k)$, $d_i = 0$ or 1, be enumerative. Let $\boldsymbol{p}$ be a point in $ev_1^{-1}(\Lambda_1) \cap \cdots \cap ev_k^{-1}(\Lambda_k)$, where Λ_i is a Chow 0- or 1-cycle's representative, $i = 1, \ldots, k$.*
(i) Suppose that $\boldsymbol{p}$ represents a cuspidal stable map and the stable map represented by a point $\boldsymbol{p}$ meets all Chow 1-cycle's representatives in $\{\Lambda_1, \ldots, \Lambda_k\}$ transversally. Then, the intersection multiplicity at $\boldsymbol{p}$ is 2.
(ii) Transversality uniformly fails along the cuspidal stable maps locus.
(iii) Suppose that $\boldsymbol{p}$ represents a stable map which is either a nodal stable map or a triple node stable map or a tac node stable map. Assume that the stable map represented by a point $\boldsymbol{p}$ meets all Chow 1-cycle's representatives in $\{\Lambda_1, \ldots, \Lambda_k\}$ transversally. Then, the intersection multiplicity at $\boldsymbol{p}$ is 1.

Proof. (i) By Remark 2.26, we may rearrange the cycles so that we have $\mathbb{CP}(d_1, \ldots, d_k)$ $= \mathbb{CP}(0, \ldots, 0, d_{3d}, \ldots, d_k)$, $d_i = 1$ for $3d \leq i \leq k$.

Let $\mathbf{p} := [(g, \mathbb{CP}^1, a_1, \dots, a_k)]$ be a cuspidal stable map which is in $ev_1^{-1}(\Lambda_1) \cap \cdots \cap ev_k^{-1}(\Lambda_k)$ and satisfies the assumptions. Then:

- $g(\mathbb{CP}^1)$ passes through Λ_i and $g(a_i) = \Lambda_i$, $1 \leq i \leq 3d-1$.
- $g(\mathbb{CP}^1)$ meets Λ_j transversally and $g(a_j) \in \Lambda_j$, $3d \leq j \leq k$.

Clearly, the curve $g(\mathbb{CP}^1)$ meets the perturbed Chow 1-cycles Λ_j' of Λ_j, $3d \leq j \leq k$, transversally. And $\mathbf{p}$ varies in a unique way according to the deformation of marked points whose image meet Λ_j' if Λ_i, $i = 1, \dots, 3d-1$, is fixed. Proposition 2.13 and Lemma 2.8 show that if we perturb the points Λ_i, $1 \leq i \leq 3d-1$, then we get two nodal, pointed stable maps which pass through the perturbed points. These pointed stable maps still meet the Chow 1-cycles Λ_j, $3d \leq j \leq k$ transversally because the transversality property is an open condition. Thus, the intersection multiplicity at $\mathbf{p}$ is two. (ii) is an immediate consequence of (i).

(iii) Lemma 2.8 and Proposition 2.9 show that the intersection multiplicity at $\mathbf{p}$ is one because of the transversality assumption in *(iii)*. □

In I_d([point], ... , [point]) case, the moduli space $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$ and the Chow 0-cycles parameter space were related by the ev map. Wall and chamber notions in $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$ and $\mathbb{RP}^2 \times \cdots \times \mathbb{RP}^2$ can be characterized by topological and geometric significance. That is no more the case if we consider the case of I_d([point], ... , [point],[line], ... , [line]). We don't have a map which relates a moduli space with the Chow cycles parameter space. Regardless of Theorem 2.27, the characterization of the intersection theoretic properties cannot be extended to the case of I_d([point], ... , [point],[line], ... , [line]). Nevertheless, there are notions of walls and chambers in the Chow cycles parameter space.

Recall the following well-known real transversality principle; Small real perturbations of a transverse intersection preserve transversality as well as the number of real and complex points in the intersection.

Based on the real transversality principle, we can extend the notion of the real version of the Gromov-Witten invariants on the cycles parameter space as follows.

Definition 2.7 Wall $\mathcal{W}$ *is a codimension one locus in the cycles parameter space* $\mathbb{CP}(d_1, \dots, d_k)^{re}$*,* $d_i = 0$ *for* $i = 1, \dots, 3d-1$ *and* $d_j = 1$ *for* $j > 3d-1$*, such that* *all elements except one in* $ev_1^{-1}(\Lambda_1) \cap \cdots \cap ev_k^{-1}(\Lambda_k) \cap \overline{M}_k(\mathbb{CP}^2, d)^{re}$ *have an intersection multiplicity one, where* $\Lambda_1, \ldots, \Lambda_k$ *are real ordered cycles represented by an element* $(\lambda_1, \ldots, \lambda_k)$ *in* $\mathcal{W}$.

A Chamber *in* $\mathbb{CP}(d_1, \ldots, d_k)^{re}$ *is a connected component in* $\mathbb{CP}(d_1, \ldots, d_k)^{re} \setminus \mathcal{W}$.

For a general element $(\lambda_1, \ldots, \lambda_k)$ in the chamber, the intersection multiplicity of any point in $ev_1^{-1}(\Lambda_1) \cap \cdots \cap ev_k^{-1}(\Lambda_k) \cap \overline{M}_k(\mathbb{CP}^2, d)^{re}$ is one. The following real version of the Gromov-Witten invariant on the chamber is well-defined by the real transversality principle:

Definition 2.8 *Let* $(\lambda_1, \ldots, \lambda_k)$ *be a general element in a given chamber* $\mathcal{C} \subseteq \mathbb{CP}(d_1, \ldots, d_k)^{re}$ *which represents the ordered cycles' representatives* $\Lambda_1, \ldots, \Lambda_k$, $k \geq 3d - 1$. *Then the Real Gromov-Witten invariant on* $\mathcal{C}$
$:= \sharp\{p \mid p \in ev_1^{-1}(\Lambda_1) \cap \cdots \cap ev_k^{-1}(\Lambda_k) \cap \overline{M}_k(\mathbb{CP}^2, d)^{re}\}$.

In the I_d([point], ... , [point],[line], ... , [line]) case, gains or losses of transverse intersection points in $ev_1^{-1}(\Lambda_1) \cap \cdots \cap ev_k^{-1}(\Lambda_k) \cap \overline{M}_k(\mathbb{CP}^2, d)^{re}$ are from two factors:

- nature of singularities in stable maps: See Theorem 2.27
- tangency conditions: marked points deformation in the neighborhood of the stable maps whose marked points go to the tangential intersection point with the complex lines.

In the latter case, the tangential order equals to the intersection multiplicity at that point. The non-transversal property caused by the tangency condition prevents the Welschinger's invariant from being extended to the I_d([point], ... , [point],[line], ... , [line]) case.

Thorough characterization of walls and chambers is possible in the I_d([point], ... , [point],[line]) case. The non-invariance of the Welschinger type invariant becomes obvious.

Example 2.28 The case of I_d([point], ... , [point],[line]):

Consider $\mathbb{CP}(d_1, \ldots, d_{3d})^{re}$, where $d_i = 0$ if $i \leq 3d - 1$ and $d_{3d} = 1$. Theorem 2.27 shows that π^{-1}(wall) constitutes a part of walls in $\mathbb{CP}(d_1, \ldots, d_{3d})^{re}$, where $\pi : \mathbb{CP}(d_1, \ldots, d_{3d})^{re} \to \mathbb{CP}(d_1, \ldots, d_{3d-1})^{re}$.

Let l be a line in $\mathbb{CP}^2$. Consider the following subset of $\overline{M}_0(\mathbb{CP}^2, d)$:

$$P_l := \{[(f, \mathbb{CP}^1, a_1, \ldots, a_{3d-1})] \in \overline{M}_0(\mathbb{CP}^2, d) \mid \sharp(f(\mathbb{CP}^1) \cap l) = d-1 \text{ and } f \text{ is an immersion}\}.$$

The image of the stable map represented by each element in P_l meets l tangentially with an intersection multiplicity 2 at one point, and transversally at all other points. Proposition 2.1 and [8, p114, Lemma (3.45)] imply that P_l is a codimension one subvariety in $\overline{M}_0(\mathbb{CP}^2, d)$. Let $F : \overline{M}_{3d-1}(\mathbb{CP}^2, d) \to \overline{M}_0(\mathbb{CP}^2, d)$ be a forgetful map and $ev := ev_1 \times \cdots \times ev_{3d-1} : \overline{M}_{3d-1}(\mathbb{CP}^2, d) \to \mathbb{CP}^2 \times \cdots \times \mathbb{CP}^2 := \mathbb{CP}(d_1, \ldots, d_{3d-1})$ be a product of i-th evaluation maps. Then one can easily see that $\mathcal{WNS}_{tan:l} := ev \circ F^{-1}(P_l)$ is a codimension one locus in $\mathbb{CP}(d_1, \ldots, d_{3d-1})$ because ev is a local isomorphism along $F^{-1}(P_l)$ and F is a submersion. Consider the following subset $\mathcal{WT}$:

$$\mathcal{WT} := \{\mathcal{WNS}_{tan:l} \times [l] \subset \mathbb{CP}(d_1, \ldots, d_{3d}) \mid [l] \in \mathbb{CP}(d_{3d})\}$$

$\mathcal{WT}$ may be considered as a codimension one fibration in the trivial fibration $\mathbb{CP}(d_1, \ldots, d_{3d})$ over $\mathbb{CP}(d_{3d})$. Obviously, the real part of $\mathcal{WT}$ is non-empty. And the real part $\mathcal{WT}^{re}$ constitutes the rest of the walls in $\mathbb{CP}(d_1, \ldots, d_{3d})^{re}$.

3. Transversality properties on $\overline{M}_{3d-1}(r\mathbb{CP}^2, d)$ and $\overline{M}_k(\mathbb{CP}^1 \times \mathbb{CP}^1, (a, b))$, and their real enumerative implications

Let $r\mathbb{CP}^2$ be $\mathbb{CP}^2$ blown-up at r points. $r\mathbb{CP}^2$ is a non-convex variety. That is, $H^1(\mathbb{CP}^1, f^*Tr\mathbb{CP}^2) \neq 0$ for some stable maps f to $r\mathbb{CP}^2$. Nevertheless, if we consider the divisor class of $d\cdot$[line], then the real Gromov-Witten invariants can be defined as in the case of $\overline{M}_{3d-1}(\mathbb{CP}^2, d)^{re}$.

Lemma 3.1 *Let $p : r\mathbb{CP}^2 \to \mathbb{CP}^2$ be a natural blow-down map. Consider a morphism* $\boldsymbol{P}$: $M_0(r\mathbb{CP}^2, d) \to M_0(\mathbb{CP}^2, d)$ *defined by* $[(f, \mathbb{CP}^1)] \mapsto [(p \circ f, \mathbb{CP}^1)]$. *Then,* $\boldsymbol{P}$ *is an embedding.*

Proof. Consider the short exact sequence of sheaves:

$$0 \to T\mathbb{CP}^1 \to f^*Tr\mathbb{CP}^2 \to Nr\mathbb{CP}^2 \to 0$$

The same calculation we have done in Lemma 2.5 shows that the tangent space at $[(f, \mathbb{CP}^1)]$ is isomorphic to $H^0(\mathbb{CP}^1, Nr\mathbb{CP}^2)$. Since $f(\mathbb{CP}^1)$ represents the divisor class $d{\cdot}[\text{line}]$, $f(\mathbb{CP}^1)$ never intersects with the exceptional divisors in $r\mathbb{CP}^2$. Thus, the degree of the locally free (sub)sheaf of $Nr\mathbb{CP}^2$ and N is the same, where $0 \to T\mathbb{CP}^1 \to f^*T\mathbb{CP}^2 \to N \to 0$. It implies that

$$T_{[(f,\mathbb{CP}^1)]}\overline{M}_0(r\mathbb{CP}^2, d) \simeq H^0(\mathbb{CP}^1, Nr\mathbb{CP}^2) \simeq H^0(\mathbb{CP}^1, N) \simeq T_{[(p\circ f,\mathbb{CP}^1)]}\overline{M}_0(\mathbb{CP}^2, d).$$

The Lemma follows. □

The morphism in Lemma 3.1 doesn't change the type of singularities in the image curves $f(\mathbb{CP}^1)$ and $p \circ f(\mathbb{CP}^1)$. Thus, the codimensions of the tacnode stable maps locus, triple node stable maps locus and cuspidal stable maps locus in $M_0(r\mathbb{CP}^2, d)$ are one. Lemma 3.1 implies that the index of the morphism $ev \circ \mathbf{P}$ is the same as the index of the ev map defined on the moduli space $\overline{M}_{3d-1}(r\mathbb{CP}^2, d)$ before compactification. Therefore, the same intersection theoretic properties stated in Lemma 3.1, Theorem 2.19 and Theorem 2.27 hold on $\overline{M}_{3d-1}(r\mathbb{CP}^2, d)$. By [6, Lemma 2.2], the Gromov-Witten invariant of $\overline{M}_{3d-1}(r\mathbb{CP}^2, d)$ and that of $\overline{M}_{3d-1}(\mathbb{CP}^2, d)$ are the same. [5, Lemma 4.3] shows that the enumerative meaning of the Gromov-Witten invariant on $\overline{M}_{3d-1}(r\mathbb{CP}^2, d)$ comes from the number of nodal stable maps in $M_{3d-1}(r\mathbb{CP}^2, d)$ which pass through the general points in $r\mathbb{CP}^2 \setminus (\mathbb{CP}^1_1 \cup \cdots \cup \mathbb{CP}^1_r)$, where $\mathbb{CP}^1_i$, $i = 1, \ldots, r$, are exceptional divisors. Walls and chambers can be constructed on $M_{3d-1}(r\mathbb{CP}^2, d)$ and $3d-1$-fold product of $r\mathbb{CP}^2 \setminus (\mathbb{CP}^1_1 \cup \cdots \cup \mathbb{CP}^1_r)$ in the same way we did in section 2.2.2. Since $f(\mathbb{CP}^1)$ and $p \circ f(\mathbb{CP}^1)$ are the same type of stable maps, i.e., f, $p \circ f$ have the same type singularities, the Welschinger's invariant becomes invariant.

$\mathbb{CP}^1 \times \mathbb{CP}^1$ is a convex variety. Let's assume $a \neq 0$ and $b \neq 0$. All results in the case of $\overline{M}_{2(a+b)-1}(\mathbb{CP}^1 \times \mathbb{CP}^1, a \times b)$ can be reproduced by repeating the same arguments we have done in section 2.

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