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Proximinality in $L^1(I,X)$

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Abstract

Let X be a Banach space and let (I,Ω,μ) be a measure space. For $1 \le p < \infty$, let $L^p(I,X)$ denote the space of Bochner p-integrable functions defined on I with values in X. The object of this paper is to give sufficient conditions for the proximinality of $L^1(I,H) + L^1(I,G)$ in $L^1(I,X)$, where H and G are two proximinal subspaces of X which include as a special case the proximality of $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ in $L^1(I \times I)$.

Key Words: Proximinal, Banach spaces.

0. Introduction

Let (I,Ω,μ) be a measure space and let $L^p(I,X)$ denotes the space of Bochner p-integrable functions (equivalent classes) defined on (I,Ω,μ) with values in a Banach space X. It is known [2] that $L^p(I,X)$ is a Banach space under the norm

$$\|f\|_p = \left(\int \|f(t)\|^p \, d\mu(t)\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

A subspace E of a Banach space X is said to be proximinal if for each $x \in X$ there exists at least one $y \in E$ such that

$$\|x - y\| = d(x,E) = \inf\{\|x - z\| : z \in E\}.$$

The element y is called a best approximation of x in E.

If X and Y are Banach spaces, then $X \overset{\wedge}{\otimes} Y$ and $X \overset{\vee}{\otimes} Y$ denote the completions of the injective and projective tensor product of X with Y, [9] . Light and Cheney, (Theorem 2.26, [9]) ,

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proved that if G and H are finite dimensional subspaces of $L^1(S)$ and $L^1(I)$ respectively, then each element of $L^1(I \times S) = L^1(I) \overset{\vee}{\otimes} L^1(S)$ has a best approximation in the subspace $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(S)$. Deeb and Khalil, (Theorem 3.3 [1]), proved that if G and H are 1-summand subspaces of $L^1(I)$, then $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(S)$ is proximinal in $L^1(I \times I)$.

The object of this paper is to discuss the proximinality of $L^1(I, H) + L^1(I, G)$ in $L^1(I, X)$, where H and G are two proximinal subspaces of X. Further we conclude from our results the proximinality of $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ in $L^1(I \times I)$.

1. Distance Formula

For a Banach space X and two closed subspaces G and H of X, the distance formula from a point $f \in L^p(\mu, X)$ to the set $L^p(\mu, G)+ L^p(\mu, G)$ is computed by the following theorem.

Theorem 1.1. *Let (I, Ω, μ) be a measure space, X a Banach space and H, G be two closed subspaces of X. Then for each $f \in L^p(I, X)$*

$$\begin{aligned} \operatorname{dist}(f, L^p(I,H) + L^p(I,G)) &= \left(\int_I (\operatorname{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} \\ &= \left\| \operatorname{dist}(f(.), H+G) \right\|_p \end{aligned}$$

Proof. Let $f \in L^p(I, X)$ and $u \in L^p(I, H) + L^p(I, G)$. Then $u = u_1 + u_2$ where $u_1 \in L^p(I, H)$ and $u_2 \in L^p(I, G)$ and

$$\begin{aligned} \|f - (u_1 + u_2)\|_p^p &= \int_I \|f(s) - (u_1(s) + u_2(s))\|^p \, ds \\ &\geq \int_I (\operatorname{dist}(f(s), H+G))^p \, ds. \end{aligned}$$

This implies that:

$$\begin{aligned} \|f-(u_1 + u_2)\|_p &\geq \left(\int_I (\operatorname{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} \\ &= \|\operatorname{dist}(f(.), H+G)\|_p . \end{aligned} \quad (1)$$

Now, since simple functions are dense in $L^p(I, X)$, then given $\epsilon > 0$, there exists a simple function φ in $L^p(I, X)$ such that $\|f-\varphi\|_p < \epsilon$. Write $\varphi = \sum_{i=1}^{n} \varkappa_{A_i} y_i$, where $\varkappa_{A_i}$ is the characteristic function of the set A_i in Ω and $y_i \in X$. We may assume that $\sum_{i=1}^{n} \varkappa_{A_i} = 1$ and $\mu(A_i) > 0$. Since $\varphi \in L^p(I, X)$ we have $\|y_i\| \, \mu(A_i) < \infty$ for $1 \leq i \leq n$. For each $i = 1, 2, ..., n$, if $\mu(A_i) < \infty$, select $h_i \in H$ and $g_i \in G$ such that:

$$\|y_i - (h_i + g_i)\| < \operatorname{dist}(y_i, H+G) + \frac{\epsilon}{(n\mu(A_i))^{\frac{1}{p}}}.$$

This could be done since $\operatorname{dist}(u, H+G) = \inf_{g \in H+G} \|u - g\|$ for all $u \in X$. If $\mu(A_i) = \infty$, put $h_i = g_i = 0$. Let

$$w = \sum_{i=1}^{n} \varkappa_{A_i} (h_i + g_i) = \sum_{i=1}^{n} \varkappa_{A_i} h_i + \sum_{i=1}^{n} \varkappa_{A_i} g_i = w_1 + w_2 .$$

Clearly $w \in L^p(I, H) + L^p(I, G)$. Set $J = \operatorname{dist}(f, L^p(I, H) + L^p(I, G))$. Then

$$\begin{aligned}
J &\leq \|f-\varphi\|_p + \operatorname{dist}(\varphi, L^p(I,H)+L^p(I,G)) \\
&\leq \epsilon + \|\varphi - w\|_p \\
&= \epsilon + \|\varphi - (w_1 + w_2)\|_p \\
&= \epsilon + \left(\int_I \|\varphi(s) - (w_1(s) + w_2(s))\|^p \, ds \right)^{\frac{1}{p}} \\
&= \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} \|y_i - (g_i + h_i)\|^p \, ds \right)^{\frac{1}{p}} \\
&\leq \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} \left(\operatorname{dist}(y_i, H+G) + \frac{\epsilon}{(n\mu(A_i))^{\frac{1}{p}}} \right)^p ds \right)^{\frac{1}{p}} \\
&\leq \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} (\operatorname{dist}(y_i, H+G))^p \, ds \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \int_{A_i} \left(\frac{\epsilon}{(n\mu(A_i))^{\frac{1}{p}}} \right)^p ds \right)^{\frac{1}{p}} \\
&= \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} (\operatorname{dist}(y_i, H+G))^p \, ds \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \int_{A_i} \frac{\epsilon^p}{n\mu(A_i)} ds \right)^{\frac{1}{p}} \\
&= \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} (\operatorname{dist}(y_i, H+G))^p \, ds \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \frac{\epsilon^p}{n\mu(A_i)} \mu(A_i) \right)^{\frac{1}{p}} \\
&= 2\epsilon + \left(\sum_{i=1}^{n} \int_{A_i} (\operatorname{dist}(y_i, H+G))^p \, ds \right)^{\frac{1}{p}} \\
&= 2\epsilon + \left(\int_I (\operatorname{dist}(\varphi, H+G))^p \, ds \right)^{\frac{1}{p}}
\end{aligned}$$

Since $\operatorname{dist}(\varphi(s), H+G) \leq \operatorname{dist}(f(s), H+G) + \|\varphi(s) - f(s)\|$, then:

$$
\begin{aligned}
J &\leq 2\epsilon + \left(\int_I (\operatorname{dist}(f(s), H+G) + \|\varphi(s) - f(s)\|)^p \, ds \right)^{\frac{1}{p}} \\
&\leq 2\epsilon + \left(\int_I (\operatorname{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} + \left(\int_I \|\varphi(s) - f(s)\|^p \, ds \right)^{\frac{1}{p}} \\
&= 2\epsilon + \left(\int_I (\operatorname{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} + \|\varphi - f\|_p \\
&\leq 3\epsilon + \left(\int_I (\operatorname{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}}.
\end{aligned}
$$

This implies that

$$
\operatorname{dist}(f, L^p(I,H) + L^p(I,G)) \leq \left(\int_I (\operatorname{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} \tag{2}
$$

From (1) and (2) the result holds. □

The following Corollary is an application of Theorem 1.1.

Corollary 1.2. *Let H and G be subspaces of a Banach space X and (I, Ω, μ) be a measure space such that $\mu(I) < \infty$. Then $g \in L^p(I,H) + L^p(I,G)$ is a best approximation for $f \in L^p(I,X)$ if and only if for almost all $t \in I$, $g(t)$ is a best approximation in $H+G$ for $f(t)$.*

Proof. Let $g = g_1 + g_2$ be a best approximation in $L^p(I,H) + L^p(I,G)$ for $f \in L^p(I,X)$. Then $\|f - g\|_p = \operatorname{dist}(f, L^p(I,H) + L^p(I,G))$. By Theorem 1.1 we have:

$$
\left(\int_I \|f(t) - (g_1(t) + g_2(t))\|^p \, dt \right)^{\frac{1}{p}} = \left(\int_I (\operatorname{dist}(f(t), H+G))^p \, dt \right)^{\frac{1}{p}}.
$$

Since $\operatorname{dist}(f(t), H+G) \leq \|f(t)-(y+z)\|$ for any $y \in H$ and $z \in G$, it follows that

$$\operatorname{dist}(f(t), H+G) \leq \|f(t)-(g_1\ (t)+g_2\ (t))\|.$$

Since x^p is an increasing function for $p \geq 1$, we have:

$$(\operatorname{dist}(f(t), H+G))^p \leq \|f(t)-(g_1\ (t)+g_2\ (t))\|^p.$$

Thus $\|f(t)-(g_1\ (t)+g_2\ (t))\|^p=(\operatorname{dist}(f(t), H+G))^p$ for almost all $t \in I$ and so

$$\|f(t)-(g_1\ (t)+g_2\ (t))\|=\operatorname{dist}(f(t), H+G) \text{ for almost all } t \in I.$$

Hence $g_1\ (t)+g_2\ (t)$ is a best approximation for $f(t) \in X$ for almost all $t \in I$. □

2. Proximinality in $L^1(I, X)$.

Let X be a Banach space and (I, Ω, μ) be a finite measure space. The Main result of this section is to give sufficient conditions for the proximinality of $L^1(I, H)+L^1(I, G)$ in $L^1(I, X)$, where H and G are two proximinal subspaces of X which include as a special case the proximality of $L^1(I) \overset{\wedge}{\otimes} G+H \overset{\wedge}{\otimes} L^1(I)$ in $L^1(I \times I)$. We start with the following definition.

Definition 2.1 *Two subspaces H and G of a Banach space X are said to be p-orthogonal if $\|h\ +g\|^p=\|h\|^p+\|g\|^p$ for every $h \in H$ and $g \in G$.*

It is easy to see that if H and G are p-orthogonal, then a function f is in $L^p(I, H+G)$ whenever f is in $L^p(I, H)+L^p(I, G)$.

Now we prove the following Main result

Theorem 2.2. *Let G and H be closed subspaces in X such that H and G are p-orthogonal. Then the following are equivalent:*

(1) *$L^1(I, H)+L^1(I, G)$ is proximinal in $L^1(I, X)$,*

(2) *$L^p(I, H)+ L^p(I, G)$ is proximinal in $L^P(I, X)$, $1 \leq p<\infty$.*

Proof. (1) $\rightarrow$ (2). Let $f \in L^p(I, X)$. Since (I, Ω, μ) is a finite measure space, then $f \in L^1(I, X)$. By assumption there exists $g=g_1\ +g_2\ \in L^1(I, H)+L^1(I, G)$ such that: $\|f-(g_1\ +g_2\ )\| \leq\|f-U\|$ for every $U \in L^1(I, H)+L^1(I, G)$. So

$\|f(t)-(g_1(t)+g_2(t))\| \leq \|f(t)-y\|$, for every $y=y_1+y_2 \in H+G$ and for almost all $t \in I$. Thus

$$\|f(t)-(g_1(t)+g_2(t))\| \leq \|f(t)-w(t)\|,$$

for all $w \in L^p(I,H+G)$. Since $0 \in H+G$ it follows that: $\|(g_1(t)+g_2(t))\| \leq 2\|f(t)\|$. Hence $g_1+g_2 \in L^p(I,H+G)$. Since H and G are p-orthogonal it follows that $g_1+g_2 \in L^p(I,H)+L^p(I,G)$.

(2) $\rightarrow$ (1). Consider the map, $J: L^1(I,X) \rightarrow L^p(I,X)$, where, $J(f)(t)=\|f(t)\|^{\frac{1}{p}-1} f(t)$. As in the proof of Theorem 1.1, [1], J is one-one, onto and $J(L^1(I,G)) = L^p(I,G)$, $J(L^1(I,H)) = L^p(I,H)$.

Now, let $f \in L^1(I,X)$, $f(t) \neq 0$. Since $J(f) \in L^p(I,X)$, then by assumption there exists $f_1+f_2 \in L^p(I,H)+L^p(I,G)$ such that

$$\|J(f)-J(f_1+f_2)\|_p \leq \|J(f)-J(u_1+u_2)\|_p$$

for all $u_1 \in L^p(I,H)$ and $u_2 \in L^p(I,G)$. Since f_1+f_2, u_1+u_2 are in $L^p(I,H)+L^p(I,G)$, it follows that f_1+f_2, $u_1+u_2 \in L^p(I,H+G)$. Hence $f_1+f_2 = J(g_1+g_2)$ and $u_1+u_2 = J(h_1+h_2)$, where $g_1+g_2, h_1+h_2 \in L^1(I,H+G)$. But H and G are p-orthogonal. Hence $g_1+g_2, h_1+h_2 \in L^1(I,H)+L^1(I,G)$. Thus

$$\|J(f)-J(g_1+g_2)\|_p \leq \|J(f)-J(h_1+h_2)\|_p$$

for all $h_1 \in L^1(I,H)$, $h_2 \in L^1(I,G)$. This implies that

$$\|J(f)(t)-J(g_1+g_2)(t)\| \leq \|J(f)(t)-(y_1+y_2)\|,$$

for every $y_1 \in H$, $y_2 \in G$ for almost all $t \in I$. Thus

$$\|J(f)(t)-J(g_1+g_2)(t)\| \leq \left\|J(f)(t)-\|f(t)\|^{\frac{1}{p}-1}(y_1+y_2)\right\|.$$

Multiply both sides by $\|f(t)\|^{1-\frac{1}{p}}$ we get:

$$\left\|f(t)-\|f(t)\|^{1-\frac{1}{p}}\|g_1(t)+g_2(t)\|^{\frac{1}{p}-1}(g_1(t)+g_2(t)\right\| \leq \|f(t)-(y_1+y_2)\|$$

for every $y_1 \in H$, $y_2 \in G$. Since $J(g_1+g_2)(t)$ is a best approximation of $J(f)(t)$ in $H+G$ (Corollary 1.2) and $0 \in H+G$ it follows that $\|J(g_1+g_2)(t)\| \leq 2\|J(f)(t)\|$ and hence $J(g_1+g_2) \in L^P(I,H+G)$. This implies that $g_1 + g_2 \in L^1(I,H+G)$. Thus w, $w(t) = \|f(t)\|^{1-\frac{1}{p}} \|g_1(t)+g_2(t)\|^{\frac{1}{p}-1} (g_1(t) + g_2(t)$ is in $L^1(I,H+G)$. Hence $w \in L^1(I,H)+L^1(I,G)$ and $\|f-w\|_1 \leq \|f-(\theta_1+\theta_2)\|_1$ for every $\theta_1 \in L^1(I,H)$, $\theta_2 \in L^1(I,G)$. □

Theorem 2.3. *Let G and H be two reflexive subspaces of a Banach space X such that G and H are p-orthogonal. Then $L^p(I,H)$+ $L^p(I,G)$ is proximinal in $L^P(I,X)$.*

Proof. Since H and G are reflexive then, by Theorem 2.13, [9], the subspaces $L^1(I,G)$ and $L^1(I,H)$ are proximinal in $L^1(I,X)$. By Theorem 1.1, [8], $L^p(I,G)$ and $L^p(I,H)$ are proximinal in $L^p(I,X)$. Since H and G are reflexive then, $L^p(I,G)$ and $L^p(I,H)$ are reflexive for $1 < p < \infty$ ([2], p, 82, 98) and hence their intersection is reflexive ([5], p, 126). By Theorem 16.12, [5], $L^p(I,H)/(L^p(I,G) \cap L^p(I,H))$ is reflexive. But $(L^p(I,G)+L^p(I,H))/L^p(I,G)$ is isomorphic to $L^p(I,H)/(L^p(I,G) \cap L^p(I,H))$, ([5], p. 123). Consequently $(L^p(I,G)+L^p(I,H))/L^p(I,G)$ is reflexive ([3], p. 9). Hence by Corollary 2.1, [10], $(L^p(I,G)+L^p(I,H))/L^p(I,G)$ is proximinal and so by Theorem 2.1, [1], $L^p(I,G)+L^p(I,H)$ is proximinal in $L^p(I,X)$. □

Corollary 2.5 *Let H and G be two reflexive subspaces of $L^1(I)$ such that H and G are p-orthogonal. Then $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ is proximinal in $L^1(I \times I)$.*

Proof. Since H and G are two reflexive p-orthogonal subspaces of $L^1(I)$, by Theorem 2.3 and 2.1 $L^1(I,G)+L^1(I,H)$ is proximinal in $L^1(I \times I)$. By Theorem 1.15, [9], $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ is proximinal in $L^1(I \times I)$. □

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