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# Diametral Dimension and Köthe Spaces

*Tosun Terzioğlu*

**Abstract**

We generalize some well-known results about the diametral dimension of classical Köthe spaces.

In this note we will call a Banach space $(\ell, \| \quad \|)$ of scalar sequences *admissible*, if it satisfies the following conditions:

(i) for $a \in \ell_\infty, \quad x \in \ell$ the sequence $ax = (a_n x_n) \in \ell$ and $\|ax\| \leq \|a\|_\infty \ \|x\|$

(ii) $\|e_n\| = 1$ for $n \in \mathbf{N} = \{0, 1, 2, \ldots ...\}$.

As usual $e_n$ denotes the sequence with 1 at $n$-th place and zero elsewhere. The classical spaces $\ell_p, \ 1 \leq p \leq \infty$ and $c_0$ are the best known examples of admissible sequence spaces. One can construct another class of admissible spaces by using Orlicz functions. For an admissible sequence space $\ell$ let us define its $\alpha$-dual by $\ell^\alpha = \{u : ux \in \ell_1, \ \forall \, x \in \ell\}$. With the usual dual norm the space $\ell^\alpha$ is also admissible.

It is easy to see that the norm of an admissible space is *monotone*. More precisely, if $x \in \ell$ and $|y_n| \leq |x_n|, \ \ n \in \mathbf{N}$, then $y \in \ell$ and $\|y\| \leq \|x\|$. Further, if $|x| = (|x_n|)$, then $\| \, |x| \, \| = \|x\|$.

If $\| \cdot \|$ is a monotone norm with $\|e_n\| = 1, \quad n \in \mathbf{N}$, defined on the space $\varphi$ of sequences with finitely many non-zero terms, then the completion of $(\varphi, \| \quad \|)$ is an admissible sequence space. In this case $(e_n)$ is a basis of $\ell$. However $\ell_\infty$ is an example of a non-separable admissible space.

A set of non-negative sequences $A$ is called a *Köthe set* if

(i) $\forall \, j \in \mathbf{N} \quad \exists \, a \in A$ with $a_j > 0$;

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(ii) for $a, b \in A$ there is a $c \in A$ with $\max\{a_n, b_n\} \leq c_n$.

Let $(\ell, \| \|)$ be an admissible space. We define $K^\ell(A)$ to be the space of all sequences $x = (x_n)$ such that $xa = (x_n a_n) \in \ell$ for every $a \in A$. With the seminorms defined by $\|x\|_a = \|xa\|$, $a \in A$, $K^\ell(A)$ is a complete locally convex space and if $A$ is countable, a Fréchet space. The usual definition of a Köthe space $K(A)$ is $K^{\ell_1}(A)$, but also the spaces $K^{\ell_p}(A)$, $1 \leq p \leq \infty$ or $K^{c_0}(A)$ have an extensive literature (see for example [6], [2], [7], [8]). Dragilev [1] considered Köthe spaces, where the underlying space $\ell$ is abstractly defined. Such general Köthe spaces reappeared recently in [4] and [5]. In this note we consider the diametral dimension of $K^\ell(A)$, where $\ell$ is an admissible space. All our results are well-known in classical case, that is when $\ell = \ell_1$ ([7], [8]). The author is grateful to V. P. Zahariuta for introducing him to Dragilev's work [1].

We recall first the definition of $n$-th diameter

$$d_n(V, U) = \inf \inf\{\alpha > 0 : \quad V \subset \alpha U + L\},$$

where $V \subset \rho U$, $\rho > 0$ and $V, U$ are absolutely convex, closed neighborhoods of a locally convex space $E$. The second infimum is taken over of subspaces $L$ of $E$ with dimension not exceeding $n \in N$. The set $\Delta(E)$ of all $(\xi_n)$, such that for every $U$ there is a $V$ with $\lim \xi_n d_n(V, U) = 0$, is called the *diametral dimension* of $E$ ([8]).

Let $A$ be a Köthe set, $a, b \in A$ with $a_n \leq b_n$ for all $n \in \mathbf{N}$. We will always assume $a_n / b_n = 0$ if $b_n = 0$. Let

$$U_a = \{x \in K^\ell(A) : \|x\|_a = \|xa\| \leq 1\}.$$

With this notation we have the following estimate which is already known in the case $\ell = \ell_1$ (cf. [8]).

**Proposition 1** *Let $J \subset \mathbf{N}$ with $|J| = n + 1$ and $a_n > 0$ for all $n \in J$. Let $I \subset \mathbf{N}$ with $|I| \leq n$. Then*

$$\inf \left\{ \frac{a_j}{b_j} : j \in J \right\} \leq d_n(U_b, U_a) \leq \sup \left\{ \frac{a_i}{b_i} : i \notin I \right\}$$

**Proof.** Let us define $P_I : K^\ell(A) \to K^\ell(A)$ by

$$P_I(x) = \sum_{n \in I} x_n e_n.$$

If $x \in U_b$, then by monotonicity of the norm of $\ell$ we have $(x - P_I(x)) \in U_b$ also. It is easily seen that

$$\|x - P_I(x)\|_a \leq \sup\{a_i/b_i : i \notin I\} \quad \|x - P_I(x)\|_b.$$

In particular, we have

$$U_b \subset \sup\{a_i/b_i : i \notin I\} U_a + P_I(K^\ell(A)).$$

This inclusion implies the night hand side. Let $s = \inf\{a_i/b_i : i \in J\}$ and assume there is a subspace $L$ of dimension $n$ and $0 < s_0 < s$ such that

$$U_b \subset s_0\, U_a + L.$$

Let $M = \{x \in K^\ell(A) : x = P_J(x)\}$. If $x \in sU_a \cap M$, by monotonicity we have $\|x\|_q \leq 1$. So

$$U_a \cap M \subset (s_0/s) U_a + L.$$

Note that $\| \ \|_a$ is a norm an $M$ and $P_J(U_a) \subset U_a$ by monotonicity. So this gives

$$U_a \cap M \subset (s_0/s) U_a \cap M + P_J(L).$$

Take any $x \in M$ with $\|x\|_a = 1$. ¿From the above inclusion we find $y_1 \in P_J(L)$ with $\|x - y_1\|_a \leq s_0/s < 1$. Repeating this we find $y_k \in P_J(L)$ with

$$\|x - y_1 - \cdots - y_k\| \leq (s_0/s)^k.$$

Hence we have $x \in P_J(L)$ or $M = P_J(L)$, but dimension of $M$ is $(n+1)$. ¿From this contradiction we get the left-hand side. □

We recall that a locally convex space $E$ is a Schwartz space if and only if for each $U$ there is a $V$ with $\lim d_n(V, U) = 0$.

For $k \in N$ let $P_k(x) = \sum_{n=0}^{k} x_n e_n, \qquad x \in K^\ell(A)$. With this notation we have the following result.

**Proposition 2** *The following are equivalent*

*(i)* $K^\ell(A)$ *is a Schwartz space*

*(ii) for every $a \in A$ there is $b \in A$ with $\lim a_n/b_n = 0$*

*(iii) for each $a \in A$ there is $b \in A$ such that for every $\epsilon > 0$ we can find $k_0$ with*

$$\|x - P_k(x)\|_a \leq \epsilon \|x\|_b, \qquad x \in K^\ell(A), \quad k \geq k_0.$$

**Proof.** Assuming $K^\ell(A)$ is a Schwartz space for $a \in A$ we choose $b \in A$ with $a_n \leq b_n$ and $\lim d_n(U_b, U_a) = 0$. Define a linear map $T : K^\ell(A) \to \ell$ by $T(x) = xa$. Then $T(U_a)$ is contained in the closed unit ball $B$ of $\ell$. Therefore $d_n(T(U_b), B) \leq d_n(U_b, U_a)$ and hence $T(U_b)$ is a relatively compact subset of $\ell$. If $\alpha_n = a_n/b_n$, the sequence $(\alpha_n e_n)$ lies in $T(U_b)$ and therefore has a subsequence which converges to some limit, which can only be zero. This means $\lim \alpha_n = 0$.

In the proof of the previous result we have observed

$$\|x - P_k(x)\|_a \leq \sup\{a_j/b_j : j \geq k\} \; \|x\|_b.$$

So if $\lim a_n/b_n = 0$, this gives (iii). Finally (iii) implies (i) is immediate. □

The diametral dimension $\Delta(E)$ of any locally convex space $E$ is equal to $c_0$ if $E$ is not a Schwartz space. On the other hand, if $K^\ell(A)$ is a Schwartz space for $a \in A$ we find $b \in A$ with $\lim a_n/b_n = 0$. So if $N_0 = \{n : a_n > 0\}$ and $|N_0| = k$ then $d_n(U_b, U_a) = 0$ for $n > k$. If $|N_0| = \infty$, we find a map $\rho : N \to N_0$ such that $(a_{\rho(n)}/b_{\rho(n)})$ is non-increasing and by Prop. 1 we easily obtain

$$d_n(U_b, U_a) = a_{\rho(n)}/b_{\rho(n)}.$$

We have several corollaries of this result. Recalling that the diametral dimension of a non-Schwartz locally convex space is always equal to $c_0$, we have the first result from above discussion.

**Proposition 3** *$\Delta(K^\ell(A))$ is independent of the admissible sequence space $\ell$.*

Let $A$ be a countable Köthe set, where $0 < a_n^k \leq a_n^{k+1}$. We call $K^\ell(A)$ *regular* ([2]) if

$$a_{n+1}^k / a_{n+1}^{k+1} \leq a_n^k / a_n^{k+1}.$$

**Proposition 4** *If $K^\ell(A)$ is a regular Köthe space, then either $K^\ell(A)$ is a Schwartz space or $K^\ell(A)$ is isomorphic to $\ell$.*

**Proof.** We have $d_n(U_{k+1}, U_k) = a_n^k / a_n^{k+1}$ by Prop.1. If $K^\ell(A)$ is not a Schwartz space, then for some $k_0$ we have $\inf_n d_n(U_k, U_{k_0}) > 0$ for all $k > k_0$. So for each $k \geq k_0$ there is a $\rho_k > 0$ with $\rho_k a_n^k \leq a_n^{k_0}$, $n \in N$. This shows that $K^\ell(A)$ is isomorphic to $\ell$ by a diagonal transformation. □

If $K^\ell(A)$ is nuclear, for $a \in A$ we find $b \in A$ with

$$\sum_{n=0}^{\infty} d_n(U_b, U_a) < \infty.$$

([3], [7]). By what we have already shown this implies $\sum a_n/b_n < \infty$. Conversely, if for each $a \in A$ we can find $b \in A$ such that $(a_n/b_n) \in \ell_1$ then in particular $K^\ell(A)$ is a Schwartz space and so $\lim_{k\to\infty} P_k(x) = x$, $x \in K^\ell(A)$ by Prop.2. If $y \in \varphi$, by monotonicity we have

$$\sup |y_n| a_n \leq \|y\|_p \leq \sum_{n \in N} |y_n| a_n, \qquad a \in A$$

and so $K^1(A) \subset K^\ell(A) \subset K^{c_0}(A)$. On the other hand we have

$$\sum |x_n| a_n \leq \rho \sup |x_n| b_n$$

and therefore $K^\ell(A) = K(A)$. This result was also proved by Dragilev in [1].

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Tosun TERZİOĞLU
Sabancı University,
34956 Orhanlı, İstanbul-TURKEY
e-mail: tosun.terzioglu@sabanciuniv.edu

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