

## Metric Trees, Hyperconvex Hulls and Extensions

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### Abstract

In this paper we examine the relationship between hyperconvex hulls and metric trees. After providing a linking construction for hyperconvex spaces, we show that the four-point property is inherited by the hyperconvex hull, which leads to the theorem that every complete metric tree is hyperconvex. We also consider some extension theorems for these spaces.

**Key Words:** Hyperconvex spaces, metric trees, extensions.

### 1. Introduction

The purpose of this paper is to clarify the relationship between metric trees and hyperconvex metric spaces. We provide a new so-called *linking construction* of hyperconvex spaces and show that the four-point property of a metric space is inherited by the hyperconvex hull of that space. We prove that all complete metric trees are hyperconvex. This in turn suggests a new approach to the study of extensions of operators. For a metric space  $(X, d)$  we use  $B(x; r)$  to denote the *closed* ball centered at  $x$  with radius  $r \geq 0$ .

**Definition 1.1** A metric space  $(X, d)$  is said to be *hyperconvex* if  $\bigcap_{i \in I} B(x_i; r_i) \neq \emptyset$  for every collection  $B(x_i; r_i)$  of closed balls in  $X$  for which  $d(x_i, x_j) \leq r_i + r_j$ .

This notion was first introduced by Aronszajn and Panitchpakdi in [1], where it is shown that a metric space is hyperconvex if and only if it is injective with respect to nonexpansive mappings. Later Isbell [11] showed that every metric space has an

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*AMS Mathematics Subject Classification:* 05C12, 54H12, 46M10.

injective hull, therefore every metric space is isometric to a subspace of a minimal hyperconvex space. Hyperconvex metric spaces are complete and connected [13]. The simplest examples of hyperconvex spaces are the set of real numbers  $\mathbb{R}$ , or a finite-dimensional real Banach space endowed with the maximum norm. While the Hilbert space  $l^2$  fails to be hyperconvex, the spaces  $L^\infty$  and  $l^\infty$  are hyperconvex. In [5] it is shown that  $\mathbb{R}^2$  with the “river” or “radial” metric is hyperconvex. We will show that there is a general “linking construction” yielding hyperconvex spaces. Constructions of the river and radial metrics are obtained as special cases. Moreover, in these spaces paths between points are restricted; they must pass through certain “common” points. On the other hand, the concept of a metric tree in graph theory also has a built-in restriction. A complete metric space  $X$  is a *metric tree* provided that for any two points  $x$  and  $y$  in  $X$  there is a unique arc joining  $x$  and  $y$ , and this arc is a geodesic arc. The study of metric trees, also known as T-theory or  $R$ -trees began with J. Tits [18] in 1977 and since then, applications have been found for metric trees within many fields of mathematics. For an overview of geometry, topology, and group theory applications, consult Bestvina [3]. A complete discussion of these spaces and their relation to so-called  $CAT(0)$  spaces we refer to [4]. Applications of metric trees in biology and medicine involve phylogenetic trees [17], and metric trees even find applications in computer science that involve string matching [2].

For more on metric trees we also refer the reader to [6], [9], [10] and [18]. One particularly useful characterization of metric trees is given by the “four-point condition”.

**Definition 1.2** A metric space  $(X, d)$  is said to satisfy the *four-point* property provided that for each set of four points  $x, y, u, v$  in  $X$  the following holds:

$$d(x, y) + d(u, v) \leq \max(d(x, u) + d(y, v), d(x, v) + d(y, u)).$$

The four-point condition is stronger than the triangle inequality (take  $u = v$ ), but it should not be confused with the ultrametric definition. An ultrametric satisfies the condition  $d(x, y) \leq \max(d(x, z), d(y, z))$ , and this is stronger than the four-point condition. The four-point condition is equivalent to saying two of the three numbers

$$d(x, y) + d(u, v), \quad d(x, u) + d(y, v), \quad d(x, v) + d(y, u)$$

are the same and the third one is less than or equal to that number. The study of spaces with the four-point property has a practical motivation (in numeric taxonomy), but also

has interesting theoretical aspects. If the space  $X$  is finite then  $X$  can be imagined as subspaces of usual graph-theoretic trees (with nonnegative weight on edges determining their length). In [9] it is shown that a metric space is a metric tree if and only if it is complete, connected and satisfies the four-point property. For Structure of metric trees and four point property we refer to [4]. Another interesting connection with four point property and the Traveling Salesman Problem can be found in [8]. The first section of this paper is devoted to hyperconvex spaces and hyperconvex hulls. Next we show that the four-point property is inherited by the hyperconvex hull. In the last section, we mention some known extension properties in the context of  $P_1$ -spaces, which can be rephrased now for complete metric trees.

## 2. The Linking Construction for Hyperconvex Spaces, and the Hyperconvex Hull

The understanding of hyperconvex spaces rests on how these spaces can be constructed. There is one obvious way to construct a hyperconvex space which is analogous to the direct product: take a collection of hyperconvex spaces and put the supremum metric on the Cartesian product. This new space will be hyperconvex essentially because any pairwise overlapping collection must overlap in each coordinate. In the following we will present two different constructions, each of which builds a larger space out of smaller spaces. We will take several hyperconvex spaces and join each of them by one point to a central hyperconvex space. This type of linking creates a restrictive movement in the sense that in order to pass between different points in different spaces, one must travel through the common point, and through the central hyperconvex space. A similar construction to this is also presented in [5] and [12].

**Theorem 2.1** Suppose  $(X, d)$  is a hyperconvex space and  $(W_\alpha, d_\alpha)_{\alpha \in I}$  is a collection of hyperconvex spaces. Suppose we have functions

$$f : I \rightarrow X$$

$$g : I \rightarrow \bigcup W_\alpha$$

such that  $g(\alpha) \in W_\alpha$ . Let's abbreviate  $\tilde{W}_\alpha = W_\alpha - g(\alpha)$ . We define the metric  $\rho$  on the set

$$Z = X \cup \bigcup_{\alpha} \tilde{W}_{\alpha}$$

by

$$\rho(y, z) = \begin{cases} d(y, z) & \text{for } y, z \in X \\ d_{\alpha}(y, z) & \text{for } y, z \in \tilde{W}_{\alpha} \\ d(y, f(\alpha)) + d_{\alpha}(g(\alpha), z) & \text{for } y \in X, z \in \tilde{W}_{\alpha} \\ d_{\alpha}(y, g(\alpha)) + d(f(\alpha), f(\beta)) + d_{\beta}(g(\beta), z) & \text{for } y \in \tilde{W}_{\alpha}, z \in \tilde{W}_{\beta} \end{cases}$$

and under this metric,  $(Z, \rho)$  is hyperconvex.

**Proof.** Define the function

$$F : Z \rightarrow X$$

by

$$F(x) = \begin{cases} x & \text{for } x \in X \\ f(\alpha) & \text{for } x \in \tilde{W}_{\alpha} \end{cases}$$

Let  $\{x_i\} \subset Z$ ,  $\{r_i\} \subset \mathbb{R}^+$ . We consider two cases. In the first, all the balls “overflow” into  $X$ , which is hyperconvex. In the second, one of the balls does not overflow, and so the intersection must be found within that  $\tilde{W}_{\alpha}$ .

**Case 1.**

Let  $r_i \geq \rho(x_i, F(x_i))$  for all  $i$ , and let  $\bar{r}_i = r_i - \rho(x_i, F(x_i))$ . We claim that

$$\rho(F(x_i), F(x_j)) \leq \bar{r}_i + \bar{r}_j$$

If  $x_i, x_j \in \tilde{W}_{\alpha}$  then  $F(x_i) = F(x_j)$  and so the claim is immediate. If  $x_i, x_j$  are not in the same  $\tilde{W}_{\alpha}$  then we have

$$\rho(x_i, x_j) = \rho(x_i, F(x_i)) + \rho(F(x_i), F(x_j)) + \rho(F(x_j), x_j)$$

We know  $\rho(x_i, x_j) \leq r_i + r_j$ , so this gives us

$$\rho(x_i, F(x_i)) + \rho(F(x_i), F(x_j)) + \rho(F(x_j), x_j) \leq r_i + r_j$$

which implies

$$\rho(F(x_i), F(x_j)) \leq \bar{r}_i + \bar{r}_j$$

Since  $X$  is hyperconvex and  $F(x_i), F(x_j) \in X$  we have  $\bigcap_{i \in I} B(F(x_i), \bar{r}_i) \neq \emptyset$ . However we already have  $B(F(x_i), \bar{r}_i) \subset B(x_i, r_i)$ .

**Case 2.** Suppose we have  $x_m \in W_\alpha \setminus \{g(\alpha)\}$  with  $r_m$  such that  $r_m < \rho(x_m, F(x_m))$ . Now observe that for any  $x_i \notin W_\alpha \setminus \{g(\alpha)\}$ , we have

$$\rho(x_m, x_i) = \rho(x_m, F(x_m)) + \rho(F(x_m), x_i) \leq r_m + r_i,$$

and this together with the condition on  $r_m$  implies that  $\rho(F(x_m), x_i) < r_i$ . We now set

$$\bar{r}_i = r_i - \rho(F(x_m), x_i) \text{ and } J := \{i \in I : x_i \notin W_\alpha \setminus \{g(\alpha)\}\}.$$

Since  $\bar{r}_j > 0$  we have  $\bigcap_{j \in J} B(g(\alpha), \bar{r}_j) \neq \emptyset$  and from hyperconvexity of  $W_\alpha$  we also know  $\bigcap_{i \in I \setminus J} B(x_i, r_i) \neq \emptyset$ . Note that  $g(\alpha) \notin B(x_m, r_m)$ , therefore the intersection point cannot be  $g(\alpha)$ . Next we claim that balls of the form  $B(g(\alpha), \bar{r}_j)$  where  $j \in J$ , and  $B(x_i, r_i)$  where  $i \in I \setminus J$ , will intersect pairwise. For if we consider

$$\rho(x_j, F(x_m)) + d_\alpha(g(\alpha), x_i) = \rho(x_i, x_j) \leq r_i + r_j,$$

subtracting  $\rho(x_j, F(x_m))$  from both sides will give

$$d_\alpha(g(\alpha), x_i) \leq r_i + r_j - \rho(x_i, x_j) = r_i + \bar{r}_j.$$

Using the hyperconvexity of  $W_\alpha$ ,

$$[\bigcap_{j \in J} B(g(\alpha), \bar{r}_j)] \cap [\bigcap_{i \in I \setminus J} B(x_i, r_i)] \neq \emptyset.$$

Finally, noting  $B(g(\alpha), \bar{r}_j) \setminus \{g(\alpha)\} \subset B(x_j, r_j)$ , we have  $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$ . This concludes the proof.  $\square$

Next we show a way to construct a hyperconvex space from a given normed space by defining a different metric on this space. We take an appropriate subspace having a

hyperconvex metric, and then decompose the normed space into subspaces linked with all rays connecting points outside the subspace with their closest point. First, we need the following lemma which illustrates that if we have a subspace of a normed space for which the closest point exists and is unique, then one can partition the remaining points of the space into equivalence classes, by defining two points to be equivalent if they lie on the same ray from the subspace.

**Lemma 2.1** Suppose  $X$  is a normed space and  $Z$  is a subspace such that the closest point in  $Z$  to any  $x \in X$  exists and is unique. For any  $p \in X \setminus Z$ , denote by  $h(p)$  the closest point in  $Z$  to  $p$ . Define

$$\lambda_p = \mu(p - h(p)) + h(p)$$

where  $\mu \in [0, \infty)$  which is the the ray pointing from  $h(p)$  in the direction of  $p$ . Then, if  $p \in \lambda_q$ ,  $p = t_0(q - h(q)) + h(p)$  this implies  $h(p) = h(q)$ .

**Proof:** Suppose  $t_0 < 1$ . Let  $z \in Z$ ,  $z \neq h(q)$ . We know, by the minimality and uniqueness of  $h(q)$ , that  $d(h(q), q) < d(z, q)$ . Also, since  $p$  lies on a line segment between  $q$  and  $h(q)$ , we know that

$$d(h(q), q) = d(h(q), p) + d(p, q).$$

Therefore

$$d(h(q), q) = d(h(q), p) + d(p, q) < d(z, q) \leq d(z, p) + d(p, q)$$

by the triangle inequality. Then, subtracting, we have:  $d(h(q), p) < d(z, p)$ , which proves that  $h(q)$  is the closest point to  $p$  as well.

For the case  $t_0 > 1$ , suppose that for some  $z \in Z$ ,  $d(z, p) < d(h(q), p)$ .

Denote

$$\beta = \frac{d(q, h(q))}{d(p, h(q))}.$$

This means that  $q = h(q) + \beta[p - h(q)]$ . Then, set :

$$z^* = (1 - \beta)h(q) + \beta z.$$

$z^* \in Z$  by convexity. So, we can calculate:

$$d(z^*, q) = \| (1 - \beta)h(q) + \beta z - [h(q) + \beta(p - h(q))] \| = \| \beta(z - p) \| =$$

$$\beta d(z, p) < \beta d(h(q), p) = d(h(q), q).$$

This says that  $d(z^*, q) < d(h(q), q)$ , which is a contradiction. So, there exist no such point  $z$ ;  $h(q)$  is the best approximation to  $q$ .

Notice that this means  $p \in \lambda_q$  implies  $\lambda_p = \lambda_q$ . i.e., a point belongs to only one ray, so we can speak of the unique  $\lambda_p$  for a point  $p \in X \setminus Z$ . Then, the relation  $p \sim q$  if  $\lambda_p = \lambda_q$  is an equivalence relation.  $\square$

**Theorem 2.2** Suppose  $X$  is a normed space and  $Z$  is a subspace such that the closest point to any  $x \in X$  exists and is unique. Suppose also that  $Z$  has a different metric with which  $Z$  is hyperconvex. One can construct a metric on  $X$  so that it is hyperconvex.

**Proof.** Consider the equivalence classes of rays  $[\lambda_\alpha]_{\alpha \in I}$  described in the above lemma. We have the functions  $f : I \rightarrow Z$  which takes  $\alpha \mapsto h(p)$  for a point  $p \in \lambda_\alpha$  and  $g : I \rightarrow \bigcup_{\alpha \in I} \lambda_\alpha$  which takes  $\alpha \mapsto h(p)$ . We assumed that  $Z$  is hyperconvex under some metric  $\delta$ . Each of the  $\lambda_\alpha$  is a hyperconvex metric space under the norm restricted to  $\lambda_\alpha$ , since  $\lambda_\alpha$  is isometric to  $[0, \infty)$ . By Theorem 2.1 we have a hyperconvex space  $Z \cup \bigcup_{\alpha \in I} (\lambda_\alpha \setminus p_\alpha)$ . However, this is the normed space  $X$ , with the metric

$$d(x, y) = \begin{cases} \|x - y\|, & \text{if } x, y \in \lambda_\alpha \setminus h(x); \\ \delta(x, y), & \text{if } x, y \in Z; \\ \delta(x, h(y)) + \|y - h(y)\|, & \text{if } x \in Z, y \in \lambda_\alpha \setminus h(y); \\ \delta(h(x), h(y)) + \|x - h(x)\| + \|y - h(y)\|, & \text{if } x \in \lambda_\alpha \setminus h(x), y \in \lambda_\beta \setminus h(y). \end{cases}$$

$\square$

Notice that if  $X = \mathbb{R}^2$  and  $Z$  is the  $x$ -axis, then this metric is the “river metric”, and if  $X = \mathbb{R}^2$  and  $Z = (0, 0)$  then it is the “radial metric” described in [5].

**Definition 2.1** Given a metric space  $(X, d)$ , the *hyperconvex hull* of  $X$  is another metric space  $(Y, \rho)$  such that  $X$  is contained isometrically in  $Y$ , where  $Y$  is a hyperconvex metric space and  $Y$  is minimal.

**Definition 2.2** A function  $f \in C(X)$  is called a *minimal extremal function* if

$$f(x) + f(y) \geq d(x, y)$$

and is pointwise minimal. That is, if  $g$  is another function with the same property such that  $g(x) \leq f(x)$  for all  $x \in X$ , then  $g = f$ .

The similarity between a minimally overlapping collection and a minimal extremal function is explained in the following remark.

**Remark 2.1** Suppose we have a minimally overlapping collection  $\{x_\alpha\} \in X$  and  $\{r_\alpha\} \in \mathbb{R}^+$ . We can think of this collection as a function

$$\tilde{f} : \{x_\alpha\}_{\alpha \in I} \rightarrow \mathbb{R}^+$$

defined by  $x_\alpha \mapsto r_\alpha$ . Because of a pairwise overlap we have  $\tilde{f}(x) + \tilde{f}(y) \geq d(x, y)$ . Moreover, we can extend  $\tilde{f}$  to  $f$  where

$$f : X \rightarrow \mathbb{R}^+$$

and

$$f(x) + f(y) \geq d(x, y).$$

To do this, we define  $f : X \rightarrow \mathbb{R}^+$  by

$$x \mapsto \inf_{x_\alpha} [d(x, x_\alpha) + r_\alpha].$$

It is easy to show that  $f$  is extremal [11].

There is an obvious family of minimal extremal functions on  $X$ , namely, select  $x \in X$  and define a function  $h_x$  by

$$h_x(z) = d(x, z)$$

for which obviously  $h_x(x) = 0$ . We will call these *distance cones*. One natural question is whether or not there are other minimal extremal functions besides distance cones? The answer to this question is in the connection between hyperconvexity and minimal extremal functions. It was shown by Isbell [11] that there are other extremal minimal functions precisely when the space is not hyperconvex. The following theorem (proof can be found in [11]) introduces the basic properties of the hyperconvex hull. In the following we will denote the hyperconvex hull by  $h(X)$ .

**Theorem 2.3** For a metric space  $(X, d)$ , consider the set

$$h(X) = \{f : X \rightarrow \mathbb{R} : f(x) + f(y) \geq d(x, y) \text{ and } f \text{ is minimal}\}$$



and the metric

$$\rho(f, g) = \sup_{x \in X} d(f(x), g(x))$$

on  $h(X)$ . Then:

- (1) A metric space  $(X, d)$  is hyperconvex if and only if every minimal extremal function is a distance cone.
- (2)  $(h(X), \rho)$  is well defined and hyperconvex.
- (3)  $X$  is isometrically embedded in  $h(X)$ , via the map  $d : X \rightarrow h(X)$  defined by  $d_x(y) = d(x, y)$ .
- (4) If  $X \subset A \subset h(X)$ , then  $h(A)$  is isometric to  $h(X)$ .
- (5) If  $f \in h(X)$  and the distance cone  $h_v \in h(X)$ , then  $\rho(h_v, f) = f(v)$ .
- (6) If we have  $f \in h(X)$ , then  $f(x) = \sup_{w \in X} \{d(x, w) - f(w)\}$ .
- (7) If  $f \in h(X)$ , then  $f$  is continuous. That is, we have  $X \hookrightarrow h(X) \hookrightarrow B(X)$  where the first mapping is the mapping  $d$  defined in (3) the second map is the natural embedding of  $h(X)$  into  $B(X)$ .

### 3. Metric Trees

In the following we denote the distance between two points  $x, y \in X$  by  $xy := d(x, y)$ .

**Definition 3.1** A *metric tree*  $X$  is a metric space  $(X, d)$  satisfying the following two axioms:

- (i) For every  $x, y \in X, x \neq y$ , there is a uniquely determined isometry

$$\varphi_{xy} : [0, d(x, y)] \rightarrow X$$

such that  $\varphi_{xy}(0) = x, \varphi_{xy}(d(x, y)) = y$ ; and

- (ii) For every one-to-one continuous mapping  $f : [0, 1] \rightarrow X$  and every  $t \in [0, 1]$ , we have

$$d(f(0), f(t)) + d(f(t), f(1)) = d(f(0), f(1)).$$

It is known [6] that any metric tree  $X$  has the four-point property, but only a connected, complete metric space with the four-point property is a metric tree. Since a metric tree is a space in which there is only one path between two points  $x$  and  $y$ , this would imply that if  $z$  is a point between  $x$  and  $y$  (that is, if  $xz + zy = xy$ ), then we know

that  $z$  is actually on the path between  $x$  and  $y$ . This motivates the next concept of a metric interval.

A *metric interval*  $\langle x, y \rangle$  is defined as

$$\langle x, y \rangle := \{z \in X : xz + zy = xy\}.$$

Consider the function

$$h_x : \langle x, y \rangle \rightarrow [0, xy]$$

defined by  $h_x(z) = xz$ . That is  $h_x$  is the restriction of the distance cone to the metric interval. It was proved in [9] that  $(X, d)$ , satisfying only *the first property of a metric tree*, is equivalent to  $h_x$  being a *bijective isometry*, which says that a metric interval is the same as an interval in  $\mathbb{R}$ .

We need the following three lemmas in order to prove Theorem 3.2 below. Ideas behind these lemmas can be found in [9]. Nevertheless, we reconstruct and expand these ideas using Isbell's [11] notation. Below in lemmas 3.1 and 3.2, we give a more detailed version of the proof given in [9]. In [9], to prove the fact that the four-point property is inherited by the hyperconvex hull, the concept of "thready spaces" was used which will be omitted in our discussion.

**Lemma 3.1** (Dress [9])

(a) In a metric tree  $(X, d)$ , for any points  $x, y$  and  $z$  the intersection

$$\langle x, y \rangle \cap \langle x, z \rangle$$

is a metric segment ending at some point  $u$ .

(b) In a metric tree  $(X, d)$ , we have

$$\langle x, y \rangle \cap \langle y, z \rangle \cap \langle z, x \rangle \neq \emptyset$$

for all  $x, y, z \in X$ .

Part (a) of the above lemma tells that if a portion of the metric space looks like a line segment, and this segment splits into two, the pieces can never connect again, so it must look rather like a tree. Part (b) expresses that metric trees are median.

**Lemma 3.2** In a metric tree  $(X, d)$  for  $x, y \in X$  we have

$$\langle x, y \rangle_X = \langle x, y \rangle_{h(X)},$$

where  $\langle x, y \rangle_X = \{z \in X : xz + zy = xy\}$ . Similarly  $\langle x, y \rangle_{h(X)} = \{z \in h(X) : xz + zy = xy\}$ .

**Proof:**  $X$ , and therefore  $h(X)$ , are trees and it is clear that  $\langle x, y \rangle_X \subset \langle x, y \rangle_{h(X)}$ . To show the other inclusion, consider the map  $h_x : \langle x, y \rangle_{h(X)} \rightarrow [0, xy]$  defined by  $z \mapsto zx$ . This is a bijective isometry since  $h(X)$  is a tree. On the other hand we also know that for all  $r \in [0, xy]$ , there exists  $x_r \in \langle x, y \rangle_X$  with  $xx_r = r$  because  $X$  is a tree. Therefore, if we take  $z \in \langle x, y \rangle_{h(X)}$ , we have  $zx = wx$  for some  $w \in \langle x, y \rangle_X$ . Therefore  $h_x(z) = h_x(w)$ . Since  $h_x$  is injective, we have  $z = w$ .  $\square$

**Lemma 3.3** If  $(X, d)$  has the four-point property, then  $h(X)$  has the four-point property.

**Proof:** First we show that if the metric space  $(X, d)$  has the four-point property, and if  $f \in h(X)$ , then  $X \cup \{f\}$  has the four-point property. Suppose  $f, x, y, v \in X \cup \{f\}$ . Then

$$\begin{aligned} xy + \rho(h_v, f) &= xy + f(v) = \sup_{w \in X} \{xy + vw - f(w)\} \\ &\leq \max \left\{ \sup_{w \in X} \{xv + yw - f(w)\}, \sup_{w \in X} \{xw + yv - f(w)\} \right\} \\ &= \max\{xv + f(y), yv + f(x)\} = \max\{xv + \rho(h_y, f), yv + \rho(h_x, f)\}. \end{aligned}$$

This proves that  $X \cup \{f\}$  has the four-point property. To prove that  $h(X)$  has the four-point property, use item (4) of Theorem 2.3 and  $X \subset X \cup \{f\} \subset h(X)$ , which yields  $h(X \cup \{f\}) = h(X)$ . Using the argument above, by taking  $f_2 \in h(X \cup \{f_1\})$ , we see that  $X \cup \{f_1, f_2\}$  has the four-point property. Continuing in this manner and adding one point at a time concludes the proof.  $\square$

**Theorem 3.1** (Dress [9]) A metric space is a metric tree if and only if it is complete, connected and satisfies the four-point property.

**Theorem 3.2** Every complete metric tree is hyperconvex.

**Proof:** Suppose  $(X, d)$  is a metric tree. Then by the above theorem it has the four-point property, which in turn implies that  $h(X)$  has the four-point property. Since the hyperconvex hull is connected,  $h(X)$  is a metric tree as well. We would like to prove that any minimal extremal function  $f \in h(X)$  is a distance cone. (i.e.,  $f$  has a zero). This is sufficient because, as described in Remark 2.1, any pairwise overlapping collection can be extended to a minimal extremal function, and this function having a zero means that the point  $x$  where  $f(x) = 0$  will be within the radius of each closed ball in the original collection. In the following we identify a point  $x \in X$  with its isometric image  $h_x \in h(X)$ . Start by fixing an  $x \in X$  and use the minimality of  $f$  to obtain that, for each  $\epsilon > 0$ , there is a point  $y$ , depending on  $\epsilon$ , with  $f(x) + f(y) \geq d(x, y) + \epsilon$ . Equivalently, for all  $n \in \mathbb{N}$ , set  $\epsilon = 1/n$  and find  $x_n$  with  $f(x) + f(x_n) \geq d(x, x_n) + 1/n$ . Now, using Lemma 3.1 part (b.) there is an element  $g_n \in h(X)$  with

$$g_n \in \langle x, x_n \rangle_{h(X)} \cap \langle x_n, f \rangle_{h(X)} \cap \langle f, x \rangle_{h(X)}.$$

This means  $xg_n + g_nf = xf$  and  $x_n g_n + g_n f = x_n f$ , giving us

$$2g_n f + xg_n + g_n x_n = f(x) + f(x_n).$$

This equality is further reduced to

$$2g_n f + x x_n = f(x) + f(x_n)$$

using the fact that  $xg_n + g_n x_n = x x_n$ . Rewriting, we have

$$g_n f = 1/2(f(x) + f(x_n) - x x_n) \leq 1/2n.$$

We now use Lemma 3.2 to write  $g_n \in \langle x, y \rangle_{h(X)} = \langle x, y \rangle_X$ . However, all elements of  $\langle x, y \rangle_X$  are distance cones, therefore  $g_n = h_{y_n}$  for some point  $y_n \in X$ , and  $f g_n = f(y_n)$ . Since  $f g_n \leq 1/2n$ , we have a sequence of points  $\{y_n\}$  with  $f(y_n) \leq 1/2n$ .  $\{y_n\}$  is a Cauchy sequence. Completeness gives us a limit point  $y^*$  in  $X$  and the continuity of  $f$  implies  $f(y^*) = 0$ .  $\square$

**Remark 3.1** There are two equivalent definitions of a metric tree. One definition is due to A. Dress (named as T-theory). This definition yields several “properties” of metric intervals. The other definition was given by J. Tits [18] (named as  $\mathbb{R}$ -trees), which lists “properties” of metric intervals as part of the definition. W. A. Kirk [14], using J. Tits’

definition, proved that a metric space is a complete R-tree if and only if it is hyperconvex and has unique metric segments. Here we use A. Dress' definition to show all complete metric trees are hyperconvex. Moreover, Kirk's method of proof is quite different than ours. Our aim is to use the elegant and geometrical nature of **the four-point property** for metric trees when making the connection between hyperconvexity and metric trees.

#### 4. Extension Theorems and Metric Trees

The theory of Banach spaces could not have developed without the Hahn-Banach theorem. So it is natural to ask whether the same type of extension theorem is true in the context of metric spaces. This question have led Aronszajn and Panitchpakti [1] to the theory of hyperconvex spaces. They established the following theorem.

**Theorem 4.1** Let  $X$  be a metric space.  $X$  is hyperconvex if and only if every mapping  $T$  of a metric space  $Y$  into  $X$  with some subadditive modulus of continuity  $\delta(\epsilon)$  has, for any space  $Z$  containing  $Y$  metrically, an extension  $\tilde{T} : Z \rightarrow X$  with the same modulus  $\delta(\epsilon)$ .

It is worth noting that earlier L. Nachbin in [16] proved a generalization of the Hahn-Banach theorem, stating that if the target space of a bounded linear map is an arbitrary real normed space, instead of the real numbers, then the extension is possible exactly when this target space is hyperconvex (he did not use the term "hyperconvex"). Extension theory for general bounded linear operators has a lot of unanswered questions even for basic cases. However, if one restricts the discussion to the extension of compact operators, there are a lot of elegant results (see [19]). In the following, we discuss  $P_1$  spaces.

**Definition 4.1** A metric space  $(X, d)$  has the *binary ball intersection property* if given any collection of closed balls that intersect pairwise, their total intersection is non-empty.

It is clear that if a metric space is hyperconvex then it has the binary ball intersection property. For if the collection  $\{B(x_i, r_i)\}$  intersects pairwise and if  $x \in B(x_i, r_i) \cap B(x_j, r_j)$ , then by the triangle inequality  $d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) \leq r_i + r_j$  is satisfied. However the binary ball intersection property does not imply hyperconvexity. If a space has the binary ball intersection property with the additional assumption that it is totally convex [9], then it is hyperconvex.

**Definition 4.2** A Banach space  $X$  is called *1-injective*, or a  $P_1$ -space, if for every space  $Y$  containing  $X$  there is a projection  $P$  from  $Y$  onto  $X$  with  $\|P\| \leq 1$ .

A real Banach space  $X$  is  $P_1$  if and only if it has the binary intersection property for balls, hence if and only if it is an absolute 1-Lipschitz retract (see [19]). The work of Nachbin, Goodner, Kelly and Hasumi characterizes real and complex  $P_1$ -spaces as the  $C(K)$  spaces for extremally disconnected compact Hausdorff spaces  $K$ . For details see [6].

An example of a  $P_1$ -space is a real  $L_\infty(\mu)$  space with  $\mu$  finite. This space has the binary intersection property, and hence it is a  $P_1$ -space.

**Theorem 4.2** Suppose  $X$  is a real Banach space that satisfies the four-point property. Then  $X$  is a  $P_1$ -space.

**Proof.** The result is a consequence of Theorem 3.1 and 3.2, and the fact that:

$$\begin{aligned} X \text{ is hyperconvex} &\Rightarrow \\ &X \text{ has the binary intersection property} \\ &\Leftrightarrow X \text{ is a } P_1\text{-space} \\ &\Leftrightarrow X \text{ is an absolute 1-Lipschitz retract.} \end{aligned}$$

□

**Remark 4.1** Matoušek in [15] proves the following theorem. Let  $Y$  be a metric tree and  $X \subset Y$ , and let  $f$  be a mapping of  $X$  into a Banach space  $Z$  with Lipschitz constant  $L$ . Then,  $f$  can be extended onto  $Y$  with Lipschitz constant  $CL$ , where  $C$  is an absolute constant. He also uses the four-point property in his proof.

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Received 20.03.2007

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