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Basic Properties and Multipliers Space on $L^1(G) \cap L(p,q)(G)$ Spaces

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Abstract

Let G be locally compact Abelian group with Haar measure. First is discussed some properties of $L^1(G) \cap L(p,q)(G)$ spaces. Then is mentioned the multipliers space on $L^1(G) \cap L(p,q)(G)$ spaces.

1. Introduction and Preliminaries

Let G be a locally compact abelian group with Haar measure μ. The spaces $B^p(G) = L^1(G) \cap L^p(G)$, $1 \leq p < \infty$ have been studied in [11], [13] and the others. The space $B^p(G)$ is a Banach algebra with respect to the norm $\|\cdot\|_{B^p}$ defined by $\|f\|_{B^p} = \|f\|_1 + \|f\|_p$ and the usual convolution product. The purpose of this paper is to extend some of the results on $B^p(G)$ to spaces

$$B(p,q)(G) = L^1(G) \cap L(p,q)(G),$$

and to discuss the properties of multipliers spaces of $B(p,q)(G)$, where $L(p,q)(G)$ is the usual Lorentz spaces. Many authors are discussed the space of multipliers of Segal algebras, multipliers from $L^1(G)$ into Segal algebras and multipliers from $L^1(G)$ into Banach spaces of functions in literature. Some of them are multipliers from $L^1(G)$ into Lorentz spaces in [3], multipliers of Banach spaces of functions in [5] and multipliers on $L^p(G, A)$ in [8]. The techniques mentioned in this papers will be used frequently. For convenience of the reader, we now review briefly what we need from the theory of $L(p,q)(G)$ spaces.

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Let (G, Σ, μ) be a measure space and let f be a measurable function on G. For each $y > 0$ let

$$\lambda_f(y) = \mu\{x \in G : \ |f(x)| > y\}.$$

The function λ_f is called the distribution function of f. The rearrangement of f is defined by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} = \sup\{y > 0 : \lambda_f(y) > t\}, \ t > 0,$$

where $\inf \phi = \infty$. Also, the average function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \ , \ t > 0.$$

Note that $\lambda_f(\cdot), f^*(\cdot)$ and $f^{**}(\cdot)$ are non-increasing and right continuous on $(0, \infty)$ ([2]). For $p, q \in (0, \infty)$ we define

$$\begin{aligned} \|f\|_{p,q}^* &= \|f\|_{p,q,\mu}^* = \left(\frac{q}{p}\int_0^\infty [f^*(t)]^q\, t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}} \\ \|f\|_{p,q} &= \|f\|_{p,q,\mu} = \left(\frac{q}{p}\int_0^\infty [f^{**}(t)]^q\, t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}}. \end{aligned}$$

Also, if $0 < p, q = \infty$ we define

$$\|f\|_{p,\infty}^* = \sup_{t>0} t^{\frac{1}{p}} f^*(t) \quad \text{and} \quad \|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^{**}(t).$$

For $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz spaces are denoted by $L(p,q)(G,\mu)$ (or in short, $L(p,q)(G)$) is defined to be the vector space of all (equivalence classes of) measurable functions f on G such that $\|f\|_{p,q}^* < \infty$. We know that $\|f\|_{p,p}^* = \|f\|_p$ and so $L^p(\mu) = L(p,p)(G)$ where $L^p(\mu)$ is the usual Lebesgue space. Also, $L(p,q_1)(G) \subset L(p,q_2)(G)$ for $q_1 \leq q_2$. In particular,

$$L(p,q_1)(G) \subset L(p,p)(G) = L^p(G) \subset L(p,q_2)(G) \subset L(p,\infty)(G)$$

for $0 < q_1 \leq p \leq q_2 \leq \infty$ ([2, 6]). It is also known that if $1 < p < \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q}^* \leq \|f\|_{p,q} \leq \frac{p}{p-1} \|f\|_{p,q}^*$$

for each $f \in L(p,q)(G)$ and $\left(L(p,q)(G), \|\cdot\|_{p,q}\right)$ is a Banach space ([6, 7]).

In [14], it was found that $B(p,q)(G)$ is a normed space with the norm $\|\cdot\|_B$ defined by $\|\cdot\|_B = \|\cdot\|_1 + \|\cdot\|_{p,q}$ and is a Segal Algebra; namely, it satisfies the properties:

1. $(B(p,q), \|\cdot\|_B)$ is a Homogeneous Banach space
2. $B(p,q)(G)$ is a Banach algebra with its norm $\|\cdot\|_B \geq \|\cdot\|_1$
3. $B(p,q)(G)$ is a dense subspace of $L^1(G)$ according to $\|\cdot\|_1$ norm.

Before beginning the next part of the paper, let's give some basic propositions about $B(p,q)(G)$ which are easy to prove using the properties of $L(p,q)(G)$ mentioned in [1],[2],[3],[6],[7] and [14]. On the other hand, we will say a few words about proofs of some propositions.

Proposition 1 $(B(p,q), \|\cdot\|_B)$ *is strongly character invariant and the map* $f \to M_t f$ *and the function* $t \to M_t f$ *are continuous where* $M_t f(x) = \langle x, t \rangle f(x)$ *for all* $f \in B(p,q)$, $x \in G$ *and* $t \in \widehat{G}$ *([3],[12]).*

Proof. $L^1(G)$ and $L(p,q)(G)$ are strongly character invariant and the functions $t \to M_t f$ and $f \to M_t f$ are continuous in both spaces. □

Proposition 2 *For* $0 < q_1 \leq p \leq q_2 \leq \infty$, *we have the following inclusions as similar to Lorentz spaces [2],[6] and [12]*

$$B(p,q_1)(G) \subset B(p,p)(G) = B^p(G) \subset B(p,q_2)(G) \subset B(p,\infty)(G).$$

Proposition 3 $(B(p,q), \|\cdot\|_B)$ *has a minimal approximate identity in* $L^1(G)$ *for* $1 < p < \infty$ *and* $1 \leq q < \infty$ *([3]).*

Proposition 4 $(B(p,q), \|\cdot\|_B)$ *is an essential Banach* $L^1(G)$*-module.*

Proof. Let $f \in L^1(G)$ and $g \in B(p,q)$. Since $L(p,q)$ is an essential Banach $L^1(G)$-module for $1 < p < \infty,\ 0 \le q < \infty$, ([1]) we have

$$\|f * g\|_B = \|f * g\|_1 + \|f * g\|_{pq} \le \|f\|_1 \|g\|_B .$$

Also, by using the approximate identity of $L^1(G)$, say (e_α); we have $\|e_\alpha * g - g\|_B \to 0$. Therefore we get that $(B(p,q), \|\cdot\|_B)$ is an essential Banach $L^1(G)$-module. □

2. Multipliers space on $B(p,q)(G)$

Let us denote the space of all bounded linear operators on $B(p,q)$ as M_{pq}, which is a Banach algebra under the usual operator norm. Besides this, let $Hom_{L^1(G)}(B(p,q)(G), B(p,q)(G))$ be the space of all module homomorphisms of $L^1(G)$-module $B(p,q)(G)$, that is, an operator $T \in M_{pq}$ satisfies $T(f * g) = f * T(g)$ for all $f \in L^1(G)$ and $g \in B(p,q)(G)$. The module homomorphisms space, called the *multipliers space*

$$Hom_{L^1(G)}(B(p,q)(G), B(p,q)(G)) = Hom_{L^1(G)}(B(p,q)(G))$$

is a Banach $L^1(G)$-module by $(f \circ T)(g) = f * T(g) = T(f * g)$ for all $g \in B(p,q)(G)$.

Now, let us fix $f \in L^1(G)$ and define $W_f : B(p,q) \to B(p,q)$ as $W_f(g) = f * g$ for all $f \in L^1(G)$ and $g \in B(p,q)$. It is easy to see that W_f is linear and bounded.

Proposition 5 *The set*

$$\Lambda = \overline{span\{W_f \ \mid \ f \in L^1(G)\}} = \overline{\{W_f \ \mid \ f \in L^1(G)\}}$$

is a complete subalgebra of M_{pq} and it possesses a minimal approximate identity.

Proof. By the definition of Λ, it is easy to see that Λ is a complete subalgebra of M_{pq} under the operator norm with usual composition. For each $f \in L^1(G)$ and $h \in B(p,q)$, if we define $W_f(h) = f * h$, then we have

$$\|W_f\| = \sup_{\|h\|_B \le 1} \|W_f(h)\|_B = \sup_{\|h\|_B \le 1} \|f * h\|_B \le \|f\|_1 , \tag{1}$$

and for all $f, g \in L^1(G)$, $h \in B(p,q)$, one can write

$$\begin{aligned} (W_f - W_g)(h) &= f * h - g * h = (f - g) * h = W_{f-g}(h) \qquad (2) \\ (W_f \circ W_g)(h) &= W_f(g * h) = f * g * h = W_{f*g}(h) . \end{aligned}$$

Let $f \in L^1(G)$. Using (1),(2) and the minimal approximate identity of $L^1(G)$ say (e_α), we get

$$\begin{aligned}\overline{\lim_\alpha} \|W_{e_\alpha} \circ W_f - W_f\| &= \overline{\lim_\alpha} \|W_{e_\alpha * f} - W_f\| \\ &= \overline{\lim_\alpha} \|W_{e_\alpha * f - f}\| \\ &\leq \overline{\lim_\alpha} \|e_\alpha * f - f\|_1 = 0.\end{aligned}$$

Consequently, we have $\overline{\lim_\alpha} \|W_{e_\alpha} \circ T - T\| = 0$ for all $T \in \Lambda$. □

Proposition 6 *The space Λ is a complete subalgebra of $Hom_{L^1(G)}(B(p,q)(G))$.*

Proof. Let $f \in L^1(G)$, then $W_f \in M_{pq}$. Since $B(p,q)$ is an essential Banach $L^1(G)$-module, we have

$$W_f(g * h) = f * g * h = g * W_f(h)$$

for all $g \in L^1(G)$ and $h \in B(p,q)$. Thus W_f belongs to $Hom_{L^1(G)}(B(p,q)(G))$. Since $Hom_{L^1(G)}(B(p,q)(G))$ is a Banach space under the usual operator norm, Λ is a complete subalgebra of $Hom_{L^1(G)}(B(p,q)(G))$. □

Proposition 7 *The space Λ is an essential Banach $L^1(G)$-module.*

Proof. Let $g \in L^1(G)$ and $W_f \in \Lambda$. Define $g \circ W_f : B(p,q) \to B(p,q)$ by letting $(g \circ W_f)(h) = W_f(h * g) = W_f(g * h)$ for each $h \in B(p,q)$. In this case, we find

$$\begin{aligned}\|g \circ W_f\| &= \sup_{\|h\|_B \leq 1} \|(g \circ W_f)(h)\|_B = \sup_{\|h\|_B \leq 1} \|W_f(g * h)\|_B \\ &\leq \|W_f\| \sup_{\|h\|_B \leq 1} \|g * h\|_B \leq \|W_f\| \|g\|_1 .\end{aligned}$$

As a result, Λ is a Banach $L^1(G)$-module. On the other hand, since $L^1(G)$ has a bounded approximate identity (e_α), $(e_\alpha \geq 0)$ which is in $C_c(G)$, the set of all continuous functions with a compact support, such that it is also an approximate identity in $B(p,q)$

by proposition 3. Then, for any $W_f \in \Lambda$, we have

$$\begin{aligned} \|e_\alpha \circ W_f - W_f\| &= \sup_{\|u\|_B \le 1} \|(e_\alpha \circ W_f - W_f)(u)\|_B \\ &= \sup_{\|u\|_B \le 1} \|f * u * e_\alpha - f * u\|_B \\ &\le \sup_{\|u\|_B \le 1} \|f * e_\alpha - f\|_1 \|u\|_B \\ &= \|f * e_\alpha - f\|_1 \to 0 \end{aligned}$$

by proposition 4. Therefore Λ is an essential Banach $L^1(G)$ -module. Also for any $f \in L^1(G)$ and $W_{e_\alpha} \in \Lambda$, we have

$$\begin{aligned} \lim_\alpha \|f - f \circ W_{e_\alpha}\| &= \lim_\alpha \left(\sup_{\|u\|_B \le 1} \|(f - f \circ W_{e_\alpha})(u)\|_B \right) \\ &= \lim_\alpha \left(\sup_{\|u\|_B \le 1} \|f * u - e_\alpha * (f * u)\|_B \right) \\ &\le \lim_\alpha \left(\sup_{\|u\|_B \le 1} \|f - e_\alpha * f\|_1 \|u\|_B \right) \\ &\le \lim_\alpha \|f - e_\alpha * f\|_1 = 0. \end{aligned}$$

So $f \in \overline{L^1(G) \circ \Lambda}$, namely $f \in \Lambda$. That is to say $L^1(G) \subset \Lambda$. □

Proposition 8 *Let $T \in Hom_{L^1(G)}(B(p,q)(G))$. Therefore $T \circ W \in \Lambda$ for each $W \in \Lambda$.*

Proof. Since $B(p,q)(G)$ is a Segal algebra, it is easy to see that

$$\Lambda = \overline{span\{W_f \mid f \in L^1(G)\}} = \overline{span\{W_g \mid g \in B(p,q)(G)\}}.$$

Let us take any $W_g \in \Lambda$. Then for all $h \in B(p,q)(G)$, we get

$$(T \circ W_g)(h) = T(g * h) = T(g) * h = W_{T(g)}(h)$$

and $T \circ W_g \in \Lambda$, since $T(g) \in B(p,q)(G)$. Now take any $W \in \Lambda$. By the definition of Λ, for all $\varepsilon > 0$ we can find $g \in B(p,q)(G)$ such that $\|W - W_g\| < \frac{\varepsilon}{\|T\|}$. Since $T \circ W_g \in \Lambda$

and T is bounded on $B(p,q)(G)$, we have

$$\begin{aligned}\|T\circ W - T\circ W_g\| &= \sup_{\|h\|_B\le 1}\|(T\circ W)(h)-(T\circ W_g)(h)\|_B\\ &= \sup_{\|h\|_B\le 1}\|T(W(h))-T(g*h)\|_B\\ &\le \|T\|\sup_{\|h\|_B\le 1}\|W(h)-g*h\|_B\\ &= \|T\|\sup_{\|h\|_B\le 1}\|W(h)-W_g(h)\|_B\\ &= \|T\|\,\|W-W_g\|<\varepsilon.\end{aligned}$$

Therefore we say that $T\circ W\in\overline{span\{W_g \mid g\in B(p,q)(G)\}}=\Lambda$. □

Theorem 9 *Let G be a locally compact abelian group. Then $M(\Lambda)$, the space of multipliers on Banach algebra Λ, is isometrically isomorphic to the space $Hom_{L^1(G)}(B(p,q)(G))$.*

Proof. Define a mapping $\Psi : Hom_{L^1(G)}(B(p,q)(G))\to M(\Lambda)$ by letting $\Psi(T)=\rho_T$ for each $T\in Hom_{L^1(G)}(B(p,q)(G))$, where $\rho_T(S)=T\circ S$ for all $S\in\Lambda$. Note that Ψ is well-defined by Proposition 8; and moreover, if $\rho_T(S\circ K)=T\circ S\circ K=\rho_T(S)\circ K$ for all $S,K\in\Lambda$, then we see that $\Psi(T)=\rho_T\in M(\Lambda)$. It is obvious that the mapping Ψ is linear and injective. Also, for $T\in Hom_{L^1(G)}(B(p,q)(G))$ and any $S\in\Lambda$, we have

$$\begin{aligned}\|T\circ S\| &= \sup_{\|g\|_B\le 1}\|(T\circ S)(g)\|_B=\sup_{\|g\|_B\le 1}\|T(S(g))\|_B\\ &\le \|T\|\sup_{\|g\|_B\le 1}\|S(g)\|_B=\|T\|\,\|S\|,\end{aligned}$$

and so we can obtain the relation

$$\|\rho_T\|=\sup_{S\in\Lambda}\frac{\|\rho_T(S)\|}{\|S\|}=\sup_{S\in\Lambda}\frac{\|T\circ S\|}{\|S\|}\le\|T\|.$$

On the other hand, since $\{W_{e_\alpha}\}$ is a minimal approximate identity for the space Λ, we get

$$\|\rho_T\|=\sup_{S\in\Lambda}\frac{\|T\circ S\|}{\|S\|}\ge\sup_{\alpha}\frac{\|T\circ W_{e_\alpha}\|}{\|W_{e_\alpha}\|}\ge\sup_{\alpha}\|T\circ W_{e_\alpha}\|\ge\|T\|$$

and $\|\rho_T\| = \|T\|$.

Finally we show that the mapping $\Psi : Hom_{L^1(G)}(B(p,q)(G)) \to M(\Lambda)$ is onto. Let ρ be an element of $M(\Lambda)$ and (e_α) an approximate identity for $L_1(G)$. Since $\Lambda \subset Hom_{L^1(G)}(B(p,q)(G))$ and $\rho e_\alpha \in \Lambda$, for any $f \in L^1(G)$ and $g \in B(p,q)$, we have

$$\rho e_\alpha (f * g) = (f \circ (\rho e_\alpha))(g). \tag{3}$$

Also $M(\Lambda) \subset Hom_{L^1(G)}(\Lambda)$ implies that

$$\rho (f * e_\alpha)(g) = (f \circ (\rho e_\alpha))(g). \tag{4}$$

Therefore by (3) and (4), we get

$$\rho e_\alpha (f * g) = (f \circ (\rho e_\alpha))(g) = \rho (f * e_\alpha)(g).$$

So for each $f \in L^1(G)$ and $g \in B(p,q)$, we obtain

$$\begin{aligned} \lim_\alpha \|\rho (f * e_\alpha)(g) - \rho f(g)\|_B &= \lim_\alpha \|(\rho (f * e_\alpha) - \rho f)(g)\|_B \\ &= \lim_\alpha \|\rho (f * e_\alpha - f)(g)\|_B \\ &\le \lim_\alpha \|\rho (f * e_\alpha - f)\| \, \|g\|_B \\ &\le \|\rho\| \lim_\alpha \|f * e_\alpha - f\|_1 \, \|g\|_B = 0 \end{aligned}$$

Thus we get

$$\lim_\alpha (\rho e_\alpha)(f * g) = \lim_\alpha (f \circ (\rho e_\alpha))(g) = \lim_\alpha \rho (f * e_\alpha)(g) = \rho f(g).$$

Since the space $B(p,q)$ is an essential Banach $L^1(G)$-module by proposition 4, the limit of $(\rho e_\alpha)(f * g) = (f \circ (\rho e_\alpha))(g)$ exists and equal to $f * T(g) \in B(p,q)$ while T is an operator in $Hom_{L^1(G)}(B(p,q))$. Therefore, since the limits $\lim_\alpha (\rho e_\alpha)(f * g) = \lim_\alpha (f \circ (\rho e_\alpha))(g) = \rho f(g)$ exist, we can write $f \circ T = \rho f$ for all $f \in L^1(G)$. Then $e_\alpha \circ T \circ W = (\rho e_\alpha) \circ W = \rho (e_\alpha \circ W)$ can be written for all $W \in \Lambda$. By proposition 7, for all $W \in \Lambda$, we get $T \circ W = \rho(W)$ or $\rho_T(W) = \rho(W)$. Therefore $\rho_T = \rho$. □

References

[1] Blozinski, A.P.: *On a convolution theorem for $L(p,q)$ spaces*, Trans. Amer. Math. Soc. **164**, 255-264, (1972).

[2] Blozinski, A.P.: *Convolution of $L(p,q)$ functions*, Proc. of the Amer. Math. Soc. **32-1**, 237-240, (1972).

[3] Chen, Y.K. and Lai, H.C.: *Multipliers of Lorentz spaces*, Hokkaido Math. J.**4**, 247-260, (1975).

[4] Duyar, C. and Gürkanlı, A.T.: *Multipliers and Relative completion in weighted Lorentz spaces*,Acta Math. Sci.**23**, 467-476, (2003).

[5] Feichtinger, H.: *Multipliers of Banach spaces of functions on groups*, Math. Z. **157**, 47-58, (1976).

[6] Hunt, R.A.: *On $L(p,q)$ spaces*, L'enseignement Mathematique,**TXII-4**, 249-276, (1966).

[7] O'Neil, R.: *Convolution operators and $L(p,q)$ spaces,* DMJ **30**, 129-142, (1963).

[8] Öztop, S.: *A note on multipliers of $L^p(G,A)$*, J. Aust. Math. Soc., **74**, 25-34, (2003).

[9] Saeki, S. and Thome, E.L.: *Lorentz spaces as L^1-modules and multipliers*, Hokkaido Math. J. **23**, 55-92, (1994).

[10] Wang, A.C.: *Homogeneous Banach Algebras*, M.Dekker INC.Newyork 1980.

[11] Warner, C.R.: *Closed ideals in the group algebra $L^1(G) \cap L^2(G)$*, Trans. Amer. Math. Soc.,**121**,408-423, (1966).

[12] Yap, L.Y.H.: *Some remarks on convolution operators and $L(p,q)$ spaces,* DMJ, **36**, 647-658, (1969).

[13] Yap, L.Y.H.: *Ideals in subalgebras of the group algebras,* Studia Math. **35**, 165-175, (1972).

[14] Yap, L.Y.H.: On Two classes of subalgebras of $L^1(G)$, Proc. Japan Acad., 48, 315-319, (1972).

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