Turk J Math
32 (2008) , 41 – 49.

Linear Connections on Light-like Manifolds

T. Dereli, Ş. Koçak and M. Limoncu

Abstract

It is well-known that a torsion-free linear connection on a light-like manifold (M, g) compatible with the degenerate metric g exists if and only if $Rad(TM)$ is a Killing distribution. In case of existence, there is an infinitude of connections with none distinguished. We propose a method to single out connections with the help of a special set of 1-forms by the condition that the 1-forms become parallel with respect to this connection. Such sets of 1-forms could be regarded as an additional structure imposed upon the light-like manifold. We consider also connections with torsion and with non-metricity on light-like manifolds.

1. Introduction

In the following we will adopt the terminology of the book Duggal-Bejancu [1]. A light-like manifold (M, g) is a smooth manifold M with a smooth symmetric tensor field of type (0,2) with constant index and nullity-degree (co-rank) on M. There are half a dozen other names for such manifolds, "degenerate manifolds" being maybe one of the most popular terms. There are some scattered (and only partly related and sometimes duplicated) works of mathematical and physical origin in the literature about connections on light-like manifolds (see [1] and references therein, and also [2], [3], [4],[5]).

It is an important result (it could be called the fundamental theorem of connections on light-like manifolds) that a torsion-free linear connection ∇ on M compatible with g ($\nabla g = 0$) exists if and only if $Rad(TM)$ is a Killing distribution ([DB]). Here, $Rad(TM)$ denotes the radical of g, that is the sub-bundle of TM with $(Rad(TM))_x = Rad(T_xM) =$

AMS Mathematics Subject Classification: Primary 55S10, 55S05

$\{\xi \in T_xM |\, g(\xi, v) = 0\,\forall\, v \in T_xM\}$ for $x \in M$. $Rad(TM)$ is a distribution of rank equal to the constant nullity-degree (co-rank) of g_x. A distribution D on M is called a Killing distribution if $L_X g = 0$ for each vector field $X \in \Gamma(D)$ (Γ section space), L being the Lie derivative.

To motivate our approach we first consider a light-like manifold (M, g) with nullity degree 1. In this case, $Rad(TM)$ is a 1-dimensional distribution (a line bundle) on M. We assume this line bundle to be trivial and choose a nowhere vanishing vector field $\xi \in \Gamma(Rad(TM))$, i.e. a trivialization of this line bundle. (Such a vector field is regarded by physicists to be a "time-vector field".) Additionally, we consider a 1-form field $\tau \in \Gamma(T^*M)$ such that $\tau(\xi) = 1$, i.e. $\tau_x(\xi_x) = 1$ for all $x \in M$. It can easily be seen that $\bar{g} := g + \tau \otimes \tau$ is a non-degenerate symmetric (0,2)-tensor field so that $(M, \bar{g})$ becomes a semi-Riemannian manifold.

Proposition 1.1 *i) Let ∇ be a torsion-free linear connection on M with $\nabla_X g = 0$ and $\nabla_X \tau = 0$ for all $X \in \Gamma(TM)$. Then, $L_\xi g = 0$, $d\tau = 0$ and ∇ is the Levi-Civita connection on $(M, \bar{g})$. ii) Assume $L_\xi g = 0$ and $d\tau = 0$. If ∇ is the Levi-Civita connection on $(M, \bar{g})$, then $\nabla_X g = 0$ and $\nabla_X \tau = 0$ for all $X \in \Gamma(TM)$.*

We give below a proof of this proposition independently of the above referred fundamental theorem. The following corollary illustrates our viewpoint: Selection of a unique connection from the infinitude of connections compatible with g with the help of an appropriate 1-form.

Corollary 1.2 *Let $L_\xi g = 0$ and let there exist a closed 1-form field τ with $\tau(\xi) = 1$. Then there exists a unique torsion-free linear connection ∇ on M with $\nabla_X g = 0$ and $\nabla_X \tau = 0$ for all $X \in \Gamma(TM)$.*

Proof. (of corollary) By part ii) of the proposition, at least one connection of the desired properties exists: The Levi-Civita of $\bar{g} = g + \tau \otimes \tau$. On the other hand, if any torsion-free ∇ satisfies $\nabla_X g = 0$ and $\nabla_X \tau = 0$, then $\nabla_X \bar{g} = \nabla_X g + \nabla_X \tau \otimes \tau + \tau \otimes \nabla_X \tau = 0$, so that ∇ is necessarily the Levi-Civita of $\bar{g}$. □

Proof. (of the proposition) i) It is obvious that Γ is the Levi-Civita connection on $(M, \bar{g})$ (as seen in the proof of the corollary).

First we show $d\tau = 0$:

$$(\nabla_X\tau)(Y) = \nabla_X(\tau(Y)) - \tau(\nabla_X Y) = 0$$
$$(\nabla_Y\tau)(X) = \nabla_Y(\tau(X)) - \tau(\nabla_Y X) = 0$$

for $X, Y \in \Gamma(TM)$ by assumption. Subtracting these equalities we get

$$X(\tau(Y)) - Y(\tau(X)) = \tau(\nabla_X Y - \nabla_Y X) = \tau([X,Y]),$$

which means $(d\tau)(X,Y) = 0$. □

Next, we will show $L_\xi g = 0$, but we first remark that $\nabla_X\xi = 0$ for $X \in \Gamma(TM)$: $\bar{g} = g + \tau\otimes\tau$ gives $\bar{g}(\xi, Y) = \tau(Y)$ for $Y \in \Gamma(TM)$. Then $\nabla_X(\bar{g}(\xi,Y)) = \nabla_X(\tau(Y))$,

$$(\nabla_X\bar{g})(\xi,Y) + \bar{g}(\nabla_X\xi, Y) + \bar{g}(\xi, \nabla_X Y) = (\nabla_X\tau)(Y) + \tau(\nabla_X Y).$$

$\nabla_X\bar{g} = 0$, $\nabla_X\tau = 0$ and $\bar{g}(\xi,\nabla_X Y) = \tau(\nabla_X Y)$ imply $\bar{g}(\nabla_X\xi, Y) = 0$, which gives $\nabla_X\xi = 0$ by non-degeneracy of $\bar{g}$.

Now,

$$(\nabla_X g)(Y,Z) = \nabla_X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$$

by assumption.

$$X(g(Y,Z)) - g([X,Y] + \nabla_Y X, Z) - g(Y, [X,Z] + \nabla_Z X) = 0$$
$$X(g(Y,Z)) - g(L_X Y, Z) - g(Y, L_X Z) - g(\nabla_Y X, Z) - g(Y, \nabla_Z X) = 0.$$

If we now take $X = \xi$, the last two terms vanish and we get

$$\xi(g(Y,Z)) - g(L_\xi Y, Z) - g(Y, L_\xi Z) = 0,$$

which means $(L_\xi g)(Y,Z) = 0$.

ii) It will be enough to show $\nabla_X\tau = 0$ as $\nabla_X g = 0$ is then a consequence of $\nabla_X\bar{g} = 0$.

We have

$$(\nabla_X\tau)(Y) = \nabla_X(\tau(Y)) - \tau(\nabla_X Y)$$

for $Y \in \Gamma(TM)$ and

$$(\nabla_X\tau)(Y) = X(\bar{g}(\xi,Y)) - \bar{g}(\xi, \nabla_X Y)),$$

since $\bar{g}(\xi, Y) = \tau(Y)$ for any Y. From the Koszul formula

$$\begin{aligned} 2\bar{g}(\nabla_X Y, Z) &= X\bar{g}(Y, Z) + Y\bar{g}(Z, X) - Z\bar{g}(X, Y) \\ &\quad -\bar{g}(X, [Y, Z]) + \bar{g}(Y, [Z, X]) + \bar{g}(Z, [X, Y]), \end{aligned}$$

we can write

$$\begin{aligned} (\nabla_X \tau)(Y) &= \frac{1}{2}(X\bar{g}(\xi, Y) - Y\bar{g}(\xi, X) - \bar{g}(\xi, [X, Y])) \\ &\quad +\frac{1}{2}(\xi\bar{g}(X, Y) - \bar{g}(X, [\xi, Y]) - \bar{g}(Y, [\xi, X])). \end{aligned}$$

The first term vanishes because of $d\tau = 0$:

$$\begin{aligned} (d\tau)(X, Y) &= X\tau(Y) - Y\tau(X) - \tau([X, Y]) = 0 \\ &= X\bar{g}(\xi, Y) - Y\bar{g}(\xi, X) - \bar{g}(\xi, [X, Y]) = 0. \end{aligned}$$

For the second term, it will be enough to show $L_\xi \bar{g} = 0$:

$$(L_\xi \bar{g})(X, Y) = \xi\bar{g}(X, Y) - \bar{g}(L_\xi X, Y) - \bar{g}(X, L_\xi Y).$$

To show $L_\xi \bar{g} = 0$, we first see $L_\xi \tau = 0$:

$$\begin{aligned} (L_\xi \tau)(X) &= L_\xi(\tau(X)) - \tau(L_\xi X) \\ &= \xi(\tau(X)) - \tau([\xi, X]) \\ &= X(\tau(\xi)) \\ &= 0, \end{aligned}$$

by $d\tau = 0$ and $\tau(\xi) = 1$. Now, $L_\xi \bar{g} = L_\xi g + (L_\xi \tau) \otimes \tau + \tau \otimes (L_\xi \tau) = 0$.

This proposition can be extended in several directions. The nullity degree (co-rank) of g can be higher (multi-time models in physics), the connection can have torsion and/or non-metricity Q. We give below a theorem comprising all these cases.

Recall that a connection ∇ is said to have non-metricity Q, if $(\nabla_Z g)(X, Y) = Q(Z, X, Y)$ for $X, Y, Z \in \Gamma(TM)$. It follows from the Koszul-method that, for a given non-degenerate metric on M, an anti-symmetric (1,2)-tensor $T(X, Y)$ and a (0,3)-tensor $Q(Z, X, Y)$ symmetric in X and Y, there exists a unique linear connection on TM having torsion T and non-metricity Q.

Theorem 1.3 *Let (M, g) be a light-like manifold with constant index and nullity degree r ($Rad(TM)$ is an r-distribution). Assume $Rad(TM)$ to be a trivial bundle with everywhere independent vector fields $\xi_i \in \Gamma(Rad(TM))$, $i = 1, ..., r$. Let τ_i $(i = 1, ..., r)$ be a set of 1-form fields satisfying $\tau_i(\xi_j) = \delta_{ij}$, $i, j = 1, ..., r$. $\bar{g} := g + \sum_{k=1}^{r} \tau_k \otimes \tau_k$ is then a non-degenerate metric on M.*

i) Let ∇ be a linear connection on M with torsion T ($T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ for $X, Y \in \Gamma(TM)$) and with non-metricity (0,3)-tensor Q, i.e. $(\nabla_Z g)(X, Y) = Q(Z, X, Y)$. Assume additionally that the 1-forms τ_i are parallel with respect to ∇: $\nabla_X \tau_i = 0$ for $X \in \Gamma(TM)$ and $i = 1, ..., r$.

Then the following relationships hold:

$$(d\tau_i)(X, Y) = \tau_i(T(X, Y)) \tag{1.1}$$

$$(L_{\xi_i} g)(X, Y) = g(X, T(\xi_i, Y)) + g(Y, T(\xi_i, X)) + Q(\xi_i, X, Y) - Q(X, Y, \xi_i) - Q(Y, X, \xi_i) \tag{1.2}$$

for $X, Y \in \Gamma(TM)$, $i = 1, ..., r$.

ii) Let T be an anti-symmetric (1,2)-tensor field and Q a (0,3)-tensor field (symmetric in the last two variables) satisfying (1.1) and (1.2).

If ∇ is the connection on $(M, \bar{g})$ with torsion T and non-metricity Q (i.e. $(\nabla_Z \bar{g})(X, Y) = Q(Z, X, Y)$), then ∇ has also non-metricity Q with respect to g (i.e. $(\nabla_Z g)(X, Y) = Q(Z, X, Y)$) and the 1-forms τ_i are parallel with respect to ∇ for $i = 1, ..., r$.

Corollary 1.4 *In the setting of the above theorem, given any torsion tensor T and non-metricity tensor Q on M satisfying (1.1) and (1.2), there exists a unique linear connection on M having torsion T, non-metricity Q for g and making τ_i $(i = 1, ..., r)$ parallel.*

It might be useful to re-word the corollary for the metric-compatible case.

Corollary 1.5 *In the setting of the above theorem, given any torsion tensor T satisfying*

$$(d\tau_i)(X, Y) = \tau_i(T(X, Y))$$

and

$$(L_{\xi_i} g)(X, Y) = g(X, T(\xi_i, Y)) + g(Y, T(\xi_i, X))$$

for $i = 1, ..., r$, there exists a unique linear connection on M compatible with g, having torsion T and making τ_i $(i = 1, ..., r)$ parallel.

As a last corollary, we note also the torsion-free case.

Corollary 1.6 *In the setting of the above theorem, assume the 1-forms τ_i to be closed and $L_{\xi_i} g = 0$ for $i = 1, ..., r$. Then there exists a unique torsion-free and g-compatible connection ∇ with $\nabla\tau_i = 0$ $(i = 1, ..., r)$.*

Proof. (of the theorem) i) From the equalities

$$(\nabla_X \tau_i)(Y) = X\tau_i(Y) - \tau_i(\nabla_X Y) = 0$$
$$(\nabla_Y \tau_i)(X) = Y\tau_i(X) - \tau_i(\nabla_Y X) = 0,$$

we get

$$X\tau_i(Y) - Y\tau_i(X) - \tau_i([X, Y] + T(X, Y)) = 0,$$

which means

$$(d\tau_i)(X, Y) = X\tau_i(Y) - Y\tau_i(X) - \tau_i([X, Y]) = \tau_i(T(X, Y)),$$

so that the first relationship (1.1) holds. □

Now, let us start with

$$(\nabla_Z g)(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = Q(Z, X, Y), \tag{1.3}$$

and insert ξ_i for Z (using torsion):

$$\xi_i g(X, Y) - g(\nabla_X \xi_i + [\xi_i, X] + T(\xi_i, X), Y)$$
$$-g(X, \nabla_Y \xi_i + [\xi_i, Y] + T(\xi_i, Y)) = Q(\xi_i, X, Y),$$

$$\xi_i g(X, Y) - g([\xi_i, X], Y) - g(X, [\xi_i, Y]) = Q(\xi_i, X, Y) + g(\nabla_X \xi_i, Y) + g(X, \nabla_Y \xi_i)$$
$$+g(T(\xi_i, X), Y) + g(X, T(\xi_i, Y)).$$

As $(L_{\xi_i} g)(X, Y)$ equals the left-hand side, it is enough to see

$$g(\nabla_X \xi_i, Y) = -Q(X, Y, \xi_i)$$

for any $X, Y \in \Gamma(TM)$ for the second relationship to hold. Now, inserting ξ_i for Y in (1.3), we get

$$Zg(X, \xi_i) - g(\nabla_Z X, \xi_i) - g(X, \nabla_Z \xi_i) = Q(Z, X, \xi_i),$$

which reduces to $-g(X, \nabla_Z \xi_i) = Q(Z, X, \xi_i)$ for any $X, Z \in \Gamma(TM)$, since $\xi_i \in \Gamma(Rad(TM))$.

ii)We first note $\bar{g}(\xi_i, Y) = \tau_i(Y)$ for $Y \in \Gamma(TM)$ by definition of $\bar{g}$.

The following equations can be verified by direct computation using (1.1) and (1.2):

$$\begin{aligned} X\tau_i(Y) - \tau_i(\nabla_X Y) = \frac{1}{2}(& X\bar{g}(\xi_i, Y) - Y\bar{g}(\xi_i, X) + \xi_i \bar{g}(X, Y) \\ & + \bar{g}(X, [Y, \xi_i]) - \bar{g}(Y, [\xi_i, X]) - \bar{g}(\xi_i, [X, Y]) \\ & + Q(X, Y, \xi_i) + Q(Y, X, \xi_i) - Q(\xi_i, X, Y) \\ & + \bar{g}(X, T(Y, \xi_i)) + \bar{g}(Y, T(X, \xi_i)) - \bar{g}(\xi_i, T(X, Y))) \end{aligned}$$

$$\begin{aligned} = \frac{1}{2}(& X\tau_i(Y) - Y\tau_i(X) - \tau_i([X, Y]) \\ & + \xi_i \bar{g}(X, Y) - \bar{g}([\xi_i, X], Y) - \bar{g}(X, [\xi_i, Y]) \\ & + Q(X, Y, \xi_i) + Q(Y, X, \xi_i) - Q(\xi_i, X, Y) \\ & + \bar{g}(X, T(Y, \xi_i)) + \bar{g}(Y, T(X, \xi_i)) - \tau_i(T(X, Y))) \end{aligned}$$

$$\begin{aligned} X\tau_i(Y) - \tau_i(\nabla_X Y) = \frac{1}{2}(& (d\tau_i)(X, Y) - \tau_i(T(X, Y)) + (L_{\xi_i}\bar{g})(X, Y) \\ & + Q(X, Y, \xi_i) + Q(Y, X, \xi_i) - Q(\xi_i, X, Y) \\ & + \bar{g}(X, T(Y, \xi_i)) + \bar{g}(Y, T(X, \xi_i))). \end{aligned}$$

Thus we get

$$\begin{aligned} (\nabla_X \tau_i)(Y) = \frac{1}{2}(& (L_{\xi_i}\bar{g})(X, Y) + Q(X, Y, \xi_i) + Q(Y, X, \xi_i) \\ & - Q(\xi_i, X, Y) + \bar{g}(X, T(Y, \xi_i)) + \bar{g}(Y, T(X, \xi_i))). \end{aligned} \tag{1.4}$$

Let us now compute $L_{\xi_i}\bar{g}$ using $\bar{g} = g + \sum_{k=1}^{r} \tau_k \otimes \tau_k$:

$$(L_{\xi_i}\bar{g})(X, Y) = (L_{\xi_i} g)(X, Y) + \sum_{k=1}^{r}((L_{\xi_i}\tau_k)(X)\tau_k(Y) + \tau_k(X)(L_{\xi_i}\tau_k)(Y))$$

$$\begin{aligned}(L_{\xi_i}\tau_k)(X) &= \xi_i\tau_k(X) - \tau_k([\xi_i, X]) \\ &= \xi_i\tau_k(X) - X\tau_k(\xi_i) - \tau_k([\xi_i, X]) \\ &= (d\tau_k)(\xi_i, X) \\ &= \tau_k(T(\xi_i, X)),\end{aligned}$$

since $\tau_k(\xi_i) = \delta_{ki}$ and the last equality by assumption (1.1). Thus we obtain

$$\begin{aligned}(L_{\xi_i}\bar{g})(X,Y) &= (L_{\xi_i}g)(X,Y) + \sum_{k=1}^{r}(\tau_k(T(\xi_i,X))\tau_k(Y) + \tau_k(X)\tau_k(T(\xi_i,Y))) \\ &= (L_{\xi_i}g)(X,Y) - \sum_{k=1}^{r}\tau_k(X)\tau_k(T(Y,\xi_i)) - \sum_{k=1}^{r}\tau_k(Y)\tau_k(T(X,\xi_i)) \\ &= (L_{\xi_i}g)(X,Y) - \sum_{k=1}^{r}(\tau_k\otimes\tau_k)(X,T(Y,\xi_i)) - \sum_{k=1}^{r}(\tau_k\otimes\tau_k)(Y,T(X,\xi_i))\end{aligned}$$

$$\begin{aligned}&= -Q(X,Y,\xi_i) - Q(Y,X,\xi_i) + Q(\xi_i,X,Y) \\ &\quad -g(X,T(Y,\xi_i)) - g(Y,T(X,\xi_i)) \\ &\quad -\sum_{k=1}^{r}(\tau_k\otimes\tau_k)(X,T(Y,\xi_i)) - \sum_{k=1}^{r}(\tau_k\otimes\tau_k)(Y,T(X,\xi_i)) \\ &= -Q(X,Y,\xi_i) - Q(Y,X,\xi_i) + Q(\xi_i,X,Y) \\ &\quad -\bar{g}(X,T(Y,\xi_i)) - \bar{g}(Y,T(X,\xi_i)).\end{aligned}$$

Inserting this into the equation (1.4) we get

$$(\nabla_X\tau_i)(Y) = 0.$$

The other assertion, $(\nabla_Z g)(X,Y) = Q(Z,X,Y)$ is now a consequence of

$$\nabla_Z\bar{g} = \nabla_Z g + \nabla_Z(\sum_{k=1}^{r}\tau_k\otimes\tau_k) = \nabla_Z g.$$

References

[1] Duggal, K. L., Bejancu, A.: **Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications**, New York, Kluwer Academic Publishers (1996).

[2] Dombrowski, H. D., Horneffer, K.: *Die Differentialgeometrie des Galileischen Relativätsprinzips*, Math. Zeitschr., 86, 291 (1964).

[3] Crampin, M.: *On differentiable manifolds with degenerate metrics*, Proc. Camb. Phil. Soc., 64, 307 (1968).

[4] Künzle, H. P.: *Galilei and Lorentz structures on space-time: Comparison of the corresponding geometry and physics*, Ann. Inst. Henri Poincarè, XVII, 337 (1972).

[5] Kozlov, S. E.: *Levi-Civita Connections On Degenerate Pseudo-Riemannian Manifolds*, J. Math. Scie., 104, No. 4, (2001).

T. DERELİ
Department of Physics, Koç University,
34450 Sarıyer, İstanbul-TURKEY
e-mail: tdereli@ku.edu.tr
Ş. KOÇAK, M. LİMONCU
Department of Mathematics, Anadolu University,
26470 Eskişehir-TURKEY
e-mail: skocak@anadolu.edu.tr
e-mail: mlimoncu@anadolu.edu.tr

Received 26.10.2006