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Cover for Modules and Injective Modules

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Abstract

Let R be a commutative ring with identity and M be an R-module with $Spec(M) \neq \phi$. A cover of the R-submodule K of M is a subset C of $Spec(M)$ satisfying that for any $x \in K, x \neq 0$, there is $N \in C$ such that $ann(x) \subset (N : M)$. If we denote by $J = \bigcap_{N \in C} (N : M)$ and assume that M is finitely generated, then $JM = M$ implies that $M = 0, M$ is called C-injective provided each R-homomorphism $\phi : (N : M) \to M$ with $N \in C$ can be lifted to an R-homomorphism $\lambda : R \to M$. If R is a commutative Noetherian ring and $C' = Spec(R)$, where $C' = \{(N : M) | N \in C\}$, then every C-injective R-module is injective.

Key Words: Commutative ring, D-prime module cover, prime submodule, injective module, quasi-injective and injective hull.

Definition. Let M be an R-module. A proper submodule P of M is a prime submodule, if $rm \in P$, for $r \in R$ and $m \in M$ implies that either $m \in P$ or $rM \subset P$. The set of all prime submodules of M is called the spectrum of M and denoted by $Spec(M)$.

Definition. Let M be an R-module. A subset C of $Spec(M)$ is a cover of M, if for every $0 \neq x \in M$ there exists $P \in C$ such that $ann(x) \subset (P : M)$. If C is a finite set, then C is called a finite cover.

Definition. An R-module M is called D-prime provided that $M \neq 0$ and $ann(N) = ann(M)$, for all non-zero submodule N of M.

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1. Cover for Modules and Localization

Lemma 1. Let M be a non-zero R-module and C a cover of M and $J = \bigcap_{P \in C} (P : M)$ if $JM = M$, then $M = 0$.

Proof. Suppose that $M \neq 0$ and $JM = M$, then there exists $r \in R$ such that $r - 1 \in J$ and $rM = 0$, so $rm = 0$ for all $m \in M$ and $r \in ann(m)$. Hence $r \in J$, that is a contradiction. □

Lemma 2. Let R be a Noetherian ring, M is a finitely generated R-module, C a cover of $M, I \subset \bigcap_{P \in C} (P : M)$. Then $\bigcap_{n=1}^{\infty} I^n M = 0$.

Proof. Let $\bigcap_{n=1}^{\infty} I^n M = K$. Then by Krull's Theorem $IK = K$ and by Lemma 1, $K = 0$. □

Lemma 3. Let C be a finite subset of $Spec(M)$ such that $(P : M)$ is maximal for every $P \in C$, and $J = \bigcap_{P \in C} (P : M)$. If $\bigcap_{n=1}^{\infty} J^n M = 0$, then C is a finite cover of M.

Proof. If C is not a cover of M, then there is an element $0 \neq x \in M$ such that $ann(x) \not\subset (P : M)$ for all $P \in C$. Hence $ann(x) + (P : M) = R$. Let $1 = r + s$ with $s \in (P : M)$ and $r \in ann(x)$. Then for every $n \in N, 1^n = (r + s)^n = r' + s', r' \in ann(x), s' \in (P : M)^n$, so $x = r'x + s'x = s'x$. Hence $Rx = (P : M)^n x$, for every $P \in C$, and so $J^n x = Rx$. Hence $\bigcap_{n=1}^{\infty} J^n M \neq 0$, which is a contradiction. □

Theorem 4. Let R be a Noetherian ring and M a faithful finitely generated R-module. Then M has a finite cover C and $\bigcap_{n=1}^{\infty} J^n M = 0$, where $J = \bigcap_{P \in C} (P : M)$. In particular, if $M = R$, then $\bigcap_{n=1}^{\infty} J^n = 0$.

Proof. See [1. Theorem 6]. □

Theorem 5. Let M be a finitely generated R-module and C is a subset of $Spec(M)$. If for every prime ideal P of R and $N \in C, N_p \neq M_P$, then C is a cover for M over

R if and only if C_P is a cover for M_P over R_P, for every prime ideal P of R, where $C_P = \{N_P | N \in C\}$.

Proof. Let $\frac{m}{s} \in M_P$. Since $m \in M$ and C is a cover for M, there exists $N \in C$ such that $ann(m) \subset (N : M)$. Let $r/s \in ann(\frac{m}{s})$. Since $ann(m_P) \subset (N_P : M_P), r/s \in (N_P : M_P)$ and $ann(\frac{m}{s}) \subset (N_P : M_P)$ so C_P is a cover for M_P over R_P. Let $m \in M$, then $\frac{m}{1} \in M_P$ so there exists $N_P \in C_P$ such that $ann(\frac{m}{1}) \subset (N_P : M_P)$. Now let $r \in ann(m)$. Then $\frac{r}{1}.\frac{y}{1} \in N_P$, where $\frac{y}{1} \in M_P$, so $\frac{ry}{1} = \frac{n}{s}$ for some $n \in N$; and so there exists $s' \in R - P$ such that $rss'y = s'n \in N$. Hence $ss'(ry) \in N$, and since $ss' \notin (N : M)$, $ry \in N$, so $rM \subset N$, and $ann(x) \subset (N : M)$. □

Theorem 6. Let R be a reduced ring and C is a subset of $Spec(R)$. Then C is a cover for R as an R-module if and only if $C[|x|]$ is a cover for $R[|x|]$, where $C[|x|] = \{P[|x|] | P \in C\}$.

Proof. Let C be a cover for R and $g(x) \in ann(f(x))$ for $f(x), g(x) \in R[|x|]$. If $g(x) = \sum_{n=0}^{\infty} b_i x^i$ and $f(x) = \sum_{n=0}^{\infty} a_i x^i$, then for every $i, b_i f(x) = 0$, so for every $i, b_i \in ann(a_0) \subset P$, for some $P \in C$ and hence $g(x) \in P[|x|]$. Conversely if $C[|x|]$ is a cover for $R[|x|]$ and let $a \in R, r \in R$ such that $r \in ann(a) \subset P[|x|]$, for some $P[|x|] \in C[|x|]$. So $ra = 0$ and hence $r \in P[|x|] \cap R$, so $r \in P$. Then $ann(a) \subset P$. Hence C is a cover for R. □

Proposition 7. Let R be a ring and C is a subset of $Spec(R)$. Then C is a cover for R as an R-module if and only if $C[x] = \{P[x] | P \in C\}$ is a cover for $R[x]$ as an $R[x]$-module.

Proof. Let C be a cover for R and $f(x) \in R[x], g(x) \in ann(f(x))$, then $f(x)g(x) = 0$. If $g(x) = \sum_{i=0}^{k} b_i x^i$, and $f(x) = \sum_{i=1}^{m} a_i x^i$, then there is an element a such that $ag(x) = 0$ so $b_i \in ann(a) \subset P$ for some $P \in C$. So $g(x) \in P[x]$ and hence $C[x]$ is a cover for $R[x]$.

Conversely, let $C[x]$ be a cover for $R[x]$, and let $a \in R, r \in R$, and $r \in ann(a)$. As $ann(a) \subset P[x]$ so $r \in P[x] \cap R = P$. Thus C is a cover for R. □

2. C-injective Modules

Definition. Let R be a ring M, X are R-modules, C is a cover of M. We say that X is C-injective provided every R-homomorphism $\phi : (N : M) \to X$, where $N \in C$ can be lifted to an R-homomorphism $\lambda : R \to X$. In the next results we shall be interested in ring R with the following properties:

(P1) for every proper ideal I there exists a finite set of prime ideals $P_1, P_2, \cdots, P_n$ such that $P_1P_2 \cdots P_n \leq I \leq P_1 \cap P_2 \cap \cdots \cap P_n$.

(P2) The ascending chain condition on prime ideals.

Proposition 8. Let R be a Noetherian ring. Then R satisfies (P1) and (P2).

Proof. Since R is Noetherian then R satisfies (P2). Suppose R does not satisfy (P1).

Let $S = \{J|$ (P1) fails for $J\}$. Suppose I be a maximal element of S. Then I is not prime ideal, so there exists ideal I_1 and I_2 properly containing I such that $I_1I_2 \leq I$. By the choice of I, (P1) holds for each I_1 and I_2, and hence for I, which is a contradiction. □

Proposition 9. Let R be a ring which satisfies (P1) and (P2). Then every non-zero R-module contains a D-prime submodule.

Proof. Let M be a non-zero R-module. Let $I = ann(M)$. The there exists prime ideal $P_1, P_2, \cdots, P_n$ such that $P_1P_2 \cdots P_n \leq I \leq P_1 \cap P_2 \cap \cdots \cap P_n$. Thus $P_1P_2 \cdots P_nM = 0$ and it follows that there exists P_k such that $P_km = 0$, for some $m \in M$. Suppose $B = \{P : P$ is a prime ideal and $Px = 0$ for some $x \in M\}$. Let Q be a maximal element of B and let $y \in M$ such that $Qy = 0$. We show that $N = Ry$ is a D-prime submodule of M. let K be a non-zero submodule of N. Then $Q \leq ann(K)$, we show that $Q = ann(K)$. Let $Q \neq ann(K)$.

Then there exists prime ideal $q_1, q_2, \cdots, q_m$ such that $q_1q_2 \cdots q_m \leq ann(K) \leq q_1 \cap q_2 \cap \cdots \cap q_m$. If follows that $q_1q_2 \cdots q_mK = 0$, and there exists $x \in K$ such that $q_ix = 0$ for some i. But $Q < ann(K) \leq q_i$ and this contradicts the choice of Q. Hence $Q = ann(K)$, and so N is D-prime submodule of M.

Theorem 10. Let M be an R-module and R satisfies (P1) and (P2), C a cover of M. Then M is C-injective if and only if M is an injective R-module.

Proof. Let M be a C-injective and I be an ideal of R and $\phi : I \to M$ an R-

homomorphism. By zorn lemma there exists an ideal J containing I maximal with respect to the property that ϕ can be lifted to a homomorphism $\lambda : J \to M$. We show that $J = R$. Suppose $J \neq R$. Thus $\frac{R}{J}$ is a non-zero R-module and so $\frac{R}{J}$ has a D-prime submodule. □

Let K be an ideal containing J such that $\frac{K}{J}$ is a D-prime submodule of $\frac{R}{J}$. Let $k \in K, k \notin J$. Then $\frac{(RK+J)}{J}$ is a D-prime module. Let $P = \{r \in R | rk \in J\}$. Then $\frac{R}{P} \simeq \frac{Rk+J}{J}$, and hence P is a prime ideal of R. As $P = (N : M)$, where $N \in C$, define $\gamma : P \to M$ $\gamma(x) = \lambda(kx)$. Then γ is a homomorphism, and because $P = (N : M)$ for $N \in C$, there exists $m \in M$ such that $\gamma(x) = mx$. Now define $\theta : kR + J \to M$ by $\theta(rk + j) = rm + \lambda(j)$, so θ is well-defined, θ is a homomorphism and θ extends λ and hence ϕ. This contradiction shows that $J = R$. it follows that M is injective.

3. Quasi-Injective Modules

Definition. An R-module M is said to be quasi-injective if every R-homomorphism $\phi : N \to M, N$ a submodule of M, is induced by an R-endomorphism of M.

Notation. Let C be a cover for R-module M, denote $C(M) = \{x \in M | (N : M) \subset ann(x)$, for some $N \in C\}$.

Lemma 11. $C(M)$ is a submodule of M.
Proof. It is obvious. □

Theorem 12. An R-module M is quasi-injective if and only if $M = E[C(M)]$, where $E[C(M)]$ is injective hull of $C(M)$.

Proof. If M is quasi-injective. Then $M \leq E[C(M)]$, we show that $E[C(M)] \leq M$. Let $y \in E[C(M)]$, then there exists $N \in C$ such that $(N : M) \subset ann(y)$; and since C is a cover for M there exists $x \in M$ such that $ann(x) \subset (N : M)$. We define $\alpha : Rx \to Ry$ by $\alpha(x) = y$. Let $E = E[M]$, so we have the mapping

$$\begin{array}{ccccccc} 0 \to & Rx & \to & M & \to E \\ & \alpha \downarrow & & \swarrow \lambda & \\ & E & & & \end{array}$$

Now $\phi = \lambda/M$ maps x onto y; and since M is quasi-injective, it is fully invariant in E, then $y \in M$ so that $E[C(M)] \leq M$, and equality holds. Conversely, suppose that

$M = E[C(M)]$, since $E[C(M)]$ is a injective R-module so is M, and since every injective R-module is quasi-injective. Hence M is quasi-injective R-modules. □

Corollary 13. Let C be a cover for an R-module M. Then the following are equivalent.

(1) M is quasi-injective R-module.

(2) M is a injective R-module.

(3) $M = E[C(M)]$.

References

[1] Amiri, N., Ershad, M. and Sharif, H.: Cover for Modules, To appear in International Journal Of Commutative Ring.

[2] Anderson, F. W. and Fuller, K. P.: Ring and Categories of Modules Springer-Verlag (1974).

[3] Gooderl, K. R.: Von Neumann Regular Ring, Pitman, 1979.

[4] Matsumura, H.: Commutative ring theory, Cambridge university Press, 1986.

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