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Quasi-Permutation Representations of Groups of Order 64

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Dedicated to the memory of Brian Hartley

Abstract

In [1], we gave algorithms to calculate $c(G)$, $q(G)$ and $p(G)$ for a finite group G. In this paper, we will calculate $c(G)$, $q(G)$, $p(G)$ for non-abelian groups G, where $|G| = 64$.

Key Words: Quasi-permutation representations, 2-groups, Character theory.

1. Introduction

By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ denote the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [1]. It is easy to see that, for any finite group G

$$c(G) \leq q(G) \leq p(G).$$

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Now we would like to state a problem from Prof. Brian Hartley (1992-94).

Problem : Let G be a finite p-group. Find G such that

$$c(G) \neq q(G) \neq p(G).$$

In fact it is easy to prove that, when p is an odd prime, then

$$c(G) = q(G).$$

So in this case a good question to be asked is:

For a p-group G with p an odd prime, when is $q(G) \neq p(G)$?

Now let $p = 2$. In [2] we showed that, when G is a generalized quaternion group then

$$2c(G) = q(G) = p(G).$$

So in this case a good question to be asked is:

For a 2-group G, when is $c(G) < q(G) < p(G)$?

When G is a finite abelian group, then $c(G)$, $q(G)$ and $p(G)$ are given in [3]. Also $c(G)$, $q(G)$ and $p(G)$ are given in [4], for non-abelin groups of order ≤ 32. So in this paper, we will calculate $c(G)$, $q(G)$, $p(G)$ for non-abelian groups G, where $|G| = 64$. We will show that for 2-groups of order less than or equal to 64 at least two of $c(G)$, $q(G)$ and $p(G)$ always coincide. In fact, we verify by direct calculation that $q(G) = p(G)$ for all non-abelian groups of order 64. However, the question of whether or not there is a 2-group G with strict inequalities $c(G) < q(G) < p(G)$ is still open.

2. Groups of order 64

Since in this section we will use the classification of finite groups of order 64 from GAP [8], so we will use the numbering of our groups of order 64 as they appear in the library of small groups in GAP.

For the calculation of $q(G)$ we need the Schur index of irreducible characters. This will be calculated by using the following results.

Lemma 2.1 *Let G be a 2-group and $\chi \in \mathrm{Irr}(G)$. Then $m_{\mathbb{Q}}(\chi) = m_{\mathbb{R}}(\chi)$.*

Proof. See [[7], Satz 1]. □

Lemma 2.2 *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Let*

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Then

$$\nu(\chi) = \begin{cases} 1 & \textit{if } \chi = \bar{\chi} \textit{ and } m_{\mathbb{R}}(\chi) = 1 \\ -1 & \textit{if } \chi = \bar{\chi} \textit{ and } m_{\mathbb{R}}(\chi) = 2 \\ 0 \textit{ if } \chi \neq \bar{\chi} \end{cases} .$$

Proof. See [[5], page 191, Lemma 33.4]. □

Note : By Lemmas 2.1 and 2.2 one can calculate the Schur index of any irreducible character of a 2-group by calculating $\nu(\chi)$. Note that calculating $\nu(\chi)$ it is not so easy.

Lemma 2.3 *Let G be a 2-group with an irreducible character of degree 2. Then* $\det\chi$ *is the principle character if and only if the Schur index $m_{\mathbb{Q}}(\chi) = 2$.*

Proof. See [[6], Theorem 3]. □

Theorem 2.4 *Let G be a group of order 64. Then the following table holds.*

G	$c(G)$	$q(G) = p(G)$									
$(64,3)$	16	16	$(64,38)$	20	20	$(64,74)$	12	16	$(64,109)$	12	12
$(64,4)$	16	16	$(64,39)$	20	20	$(64,75)$	12	12	$(64,110)$	18	18
$(64,5)$	16	16	$(64,40)$	32	32	$(64,76)$	12	16	$(64,111)$	16	16
$(64,6)$	16	16	$(64,41)$	16	16	$(64,77)$	12	12,	$(64,112)$	16	16
$(64,7)$	16	16	$(64,42)$	16	16	$(64,78)$	16	16	$(64,113)$	16	16
$(64,8)$	16	16	$(64,43)$	16	32	$(64,79)$	12	16	$(64,114)$	24	24
$(64,9)$	16	16	$(64,44)$	20	20	$(64,80)$	12	16	$(64,115)$	12	12
$(64,10)$	16	16	$(64,45)$	16	16	$(64,81)$	16	20	$(64,116)$	12	12
$(64,11)$	16	16	$(64,46)$	16	16	$(64,82)$	24	24	$(64,117)$	12	12
$(64,12)$	16	16	$(64,47)$	20	20	$(64,84)$	14	14	$(64,118)$	12	12
$(64,13)$	16	16	$(64,48)$	20	20	$(64,85)$	12	12	$(64,119)$	12	12
$(64,14)$	16	16	$(64,49)$	32	32	$(64,86)$	20	20	$(64,120)$	12	20
$(64,15)$	16	16	$(64,51)$	32	32	$(64,87)$	14	14	$(64,121)$	12	12
$(64,16)$	16	16	$(64,52)$	32	32	$(64,88)$	12	12	$(64,122)$	12	20
$(64,17)$	16	16	$(64,53)$	32	32	$(64,89)$	20	20	$(64,123)$	12	12
$(64,18)$	16	16	$(64,54)$	32	64	$(64,90)$	10	10	$(64,124)$	16	16
$(64,19)$	16	16	$(64,56)$	14	14	$(64,91)$	16	16	$(64,125)$	16	16
$(64,20)$	12	12	$(64,57)$	16	16	$(64,92)$	10	10	$(64,126)$	12	16
$(64,21)$	16	16	$(64,58)$	12	12	$(64,93)$	10	18	$(64,127)$	12	16
$(64,22)$	20	20	$(64,59)$	12	12	$(64,94)$	16	16	$(64,128)$	12	12
$(64,23)$	12	12	$(64,60)$	12	12	$(64,95)$	14	14	$(64,129)$	12	12
$(64,24)$	12	12	$(64,61)$	12	12	$(64,96)$	14	14	$(64,130)$	12	12
$(64,25)$	16	16	$(64,62)$	16	16	$(64,97)$	20	20	$(64,131)$	12	12
$(64,27)$	20	20	$(64,63)$	16	16	$(64,98)$	12	12	$(64,132)$	12	20
$(64,28)$	16	16	$(64,64)$	20	20	$(64,99)$	12	12	$(64,133)$	12	20
$(64,29)$	20	20	$(64,65)$	12	12	$(64,100)$	12	20	$(64,134)$	8	8
$(64,30)$	16	16	$(64,66)$	12	12	$(64,101)$	10	10	$(64,135)$	16	16
$(64,31)$	32	32	$(64,67)$	12	12	$(64,102)$	16	16	$(64,136)$	16	16
$(64,32)$	8	8	$(64,68)$	16	16	$(64,103)$	14	14	$(64,137)$	8	16
$(64,33)$	16	16	$(64,69)$	12	12	$(64,104)$	12	12	$(64,138)$	8	8
$(64,34)$	8	8	$(64,70)$	12	12	$(64,105)$	20	20	$(64,139)$	16	16
$(64,35)$	16	16	$(64,71)$	12	12	$(64,106)$	14	14	$(64,140)$	12	12
$(64,36)$	16	16	$(64,72)$	12	16	$(64,107)$	14	14	$(64,141)$	12	12

$(64, 142)$	12	12	$(64, 172)$	16	24	$(64, 204)$	10	14	$(64, 234)$	16	16
$(64, 143)$	12	20	$(64, 173)$	12	12	$(64, 205)$	14	14	$(64, 235)$	12	16
$(64, 144)$	12	12	$(64, 174)$	12	12	$(64, 206)$	12	12	$(64, 236)$	16	16
$(64, 145)$	12	20	$(64, 175)$	12	20	$(64, 207)$	14	14	$(64, 237)$	16	16
$(64, 146)$	12	12	$(64, 176)$	20	20	$(64, 208)$	14	18	$(64, 238)$	12	16
$(64, 147)$	12	12	$(64, 177)$	12	12	$(64, 209)$	18	18	$(64, 239)$	8	16
$(64, 148)$	12	20	$(64, 178)$	12	20	$(64, 210)$	16	16	$(64, 240)$	16	16
$(64, 149)$	12	12	$(64, 179)$	12	16	$(64, 211)$	10	10	$(64, 241)$	16	16
$(64, 150)$	12	12	$(64, 180)$	20	24	$(64, 212)$	10	14	$(64, 242)$	16	16
$(64, 151)$	12	20	$(64, 181)$	12	16	$(64, 213)$	12	12	$(64, 243)$	16	16
$(64, 152)$	16	16	$(64, 182)$	12	16	$(64, 214)$	12	16	$(64, 244)$	16	24
$(64, 153)$	16	16	$(64, 184)$	18	18	$(64, 215)$	12	12	$(64, 245)$	16	32
$(64, 154)$	16	32	$(64, 185)$	32	32	$(64, 216)$	12	12	$(64, 247)$	12	12
$(64, 155)$	12	16	$(64, 186)$	18	18	$(64, 217)$	12	20	$(64, 248)$	18	18
$(64, 156)$	12	16	$(64, 187)$	18	18	$(64, 218)$	12	12	$(64, 249)$	16	16
$(64, 157)$	12	16	$(64, 188)$	18	34	$(64, 219)$	16	16	$(64, 250)$	12	12
$(64, 158)$	12	16	$(64, 189)$	32	32	$(64, 220)$	16	16	$(64, 251)$	12	12
$(64, 159)$	12	16	$(64, 190)$	16	16	$(64, 221)$	16	16	$(64, 252)$	12	20
$(64, 160)$	12	24	$(64, 191)$	16	32	$(64, 222)$	16	24	$(64, 253)$	18	18
$(64, 161)$	16	16	$(64, 193)$	12	12	$(64, 223)$	16	16	$(64, 254)$	10	10
$(64, 162)$	16	16	$(64, 194)$	12	12	$(64, 224)$	12	16	$(64, 255)$	10	18
$(64, 163)$	16	16	$(64, 195)$	14	14	$(64, 225)$	12	16	$(64, 256)$	16	16
$(64, 164)$	16	16	$(64, 196)$	10	10	$(64, 226)$	8	8	$(64, 257)$	16	16
$(64, 165)$	16	16	$(64, 197)$	10	14	$(64, 227)$	12	12	$(64, 258)$	16	16
$(64, 166)$	16	24	$(64, 198)$	12	12	$(64, 228)$	12	12	$(64, 259)$	16	32
$(64, 167)$	16	16	$(64, 199)$	12	12	$(64, 229)$	12	12	$(64, 261)$	10	10
$(64, 168)$	16	16	$(64, 200)$	12	20	$(64, 230)$	8	12	$(64, 262)$	10	14
$(64, 169)$	24	24	$(64, 201)$	12	12	$(64, 231)$	12	12	$(64, 263)$	12	12
$(64, 170)$	16	16	$(64, 202)$	10	10	$(64, 232)$	16	16	$(64, 264)$	10	10
$(64, 171)$	16	16	$(64, 203)$	10	10	$(64, 233)$	16	16	$(64, 265)$	10	10
									$(64, 266)$	16	16

Proof. We used the GAP for the character tables and the subgroups and the core of subgroups. Also we used Lemmas 2.2 and 2.3 and $\nu(\chi)$ for Schur indices. Finally we used [[1], Corollaries 2.4 and 3.11] for groups with cyclic center and [[1], Theorems 2.2 and 3.6] for groups with non-cyclic center in order to calculate $c(G)$, $q(G)$ and $p(G)$. □

Corollary 2.5 *Let G be a finite group of order 64. Then $q(G) = p(G)$.*

Proof. For abelian G this is proved in [3], and for non-abelian G this is established in Theorem 2.4 □

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