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A Generalization of Ankeny and Rivlin's Result on the Maximum Modulus of Polynomials not Vanishing in the Interior of the Unit Circle

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Abstract

For an arbitrary entire function $f(z)$, let

$$M(f,r) = \max_{|z|=r} |f(z)|.$$

For a polynomial $p(z)$ of degree n, it is known that

$$M(p,R) \leq R^n M(p,1), \quad R > 1.$$

By considering the polynomial $p(z)$ with no zeros in $|z| < 1$, Ankeny and Rivlin obtained the refinement

$$M(p,R) \leq \{(R^n+1)/2\} M(p,1), \quad R > 1.$$

By considering the polynomial $p(z)$ with no zeros in $|z| < k, (k \geq 1)$ and simultaneously thinking of s^{th} derivative $(0 \leq s < n)$ of the polynomial, we have obtained the generalization

$$M(p^{(s)},R) \leq \begin{cases} (1/2)\{\frac{d^s}{dR^s}(R^n+k^n)\}(2/(1+k))^n M(p,1), & R \geq k, \\ (1/(R^s+k^s))[\{\frac{d^s}{dx^s}(1+x^n)\}_{x=1}]((R+k)/(1+k))^n M(p,1), & 1 \leq R \leq k, \end{cases}$$

of Ankeny and Rivlin's result.

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1. Introduction and statement of results

For an arbitrary entire function $f(z)$, let

$$M(f,r) = \max_{|z|=r} |f(z)|.$$

As a consequence of maximum modulus principle, we have the following result.

Theorem A *If $p(z)$ is a polynomial of degree n, then*

$$M(p,R) \leq R^n M(p,1), \quad R > 1,$$

with equality only for $p(z) = \lambda z^n$.

Ankeny and Rivlin [1] considered polynomials not vanishing in the interior of the unit circle and obtained the following refinement of Theorem A.

Theorem B *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then*

$$M(p,R) \leq \{(1+R^n)/2\} M(p,1), \quad R > 1,$$

with equality only for $p(z) = \lambda + \mu z^n$, with $|\lambda| = |\mu|$.

In this paper, we have obtained a generalization of Theorem B, by considering polynomials with no zeros in $|z| < k, k \geq 1$ and simultaneously thinking of the s^{th} derivative, $(0 \leq s < n)$, of the polynomial, instead of the polynomial itself. More precisely we have proved the following theorem.

Theorem *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < k, (k \geq 1)$ then for $0 \leq s < n$*

$$M(p^{(s)},R) \leq \begin{cases} (1/2)\{\frac{d^s}{dR^s}(R^n + k^n)\}(2/(1+k))^n M(p,1), \ R \geq k, & (1.1) \\ (1/(R^s+k^s))[\{\frac{d^s}{dx^s}(1+x^n)\}_{x=1}]((R+k)/(1+k))^n M(p,1), 1 \leq R \leq k. & (1.2) \end{cases}$$

Equality holds in (1.1) (with $k = 1$ & $s = 0$) for $p(z) = z^n + 1$ and equality holds in (1.2) (with $s = 1$) for $p(z) = (z+k)^n$.

2. Lemmas

For the proof of the theorem we require following lemmas.

Lemma 1 *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \le 1$. If $p(z)$ is a polynomial of degree at most n such that*

$$|p(z)| \le |P(z)|, \quad |z| = 1, \tag{2.1}$$

then for $0 \le s < n$,

$$|p^{(s)}(z)| \le |P^{(s)}(z)|, \quad |z| \ge 1. \tag{2.2}$$

Proof of Lemma 1 By using (2.1), we can say that an application of maximum modulus principle to the function $p(z)/P(z)$ will yield

$$|p(z)| \le |P(z)|, \quad |z| \ge 1. \tag{2.3}$$

Therefore the polynomial

$$p(z) - \lambda P(z)$$

will not vanish in $|z| > 1$ for every λ with $|\lambda| > 1$. Gauss-Lucas theorem will then imply that polynomial

$$p^{(s)}(z) - \lambda P^{(s)}(z), \quad 1 \le s < n$$

will not vanish in $|z| > 1$ for every λ with $|\lambda| > 1$ and therefore

$$|p^{(s)}(z)| \le |P^{(s)}(z)|, \quad |z| > 1,$$

leading to

$$|p^{(s)}(z)| \le |P^{(s)}(z)|, \quad |z| \ge 1, \& 1 \le s < n,$$

which, on being combined with(2.3), completes the proof of Lemma 1. □

Lemma 2 *If $p(z)$ is a polynomial of degree at most n then for $0 \le s < n$,*

$$|p^{(s)}(z)| + |q^{(s)}(z)| \le \{|\frac{d^s}{dz^s}(1)| + |\frac{d^s}{dz^s}(z^n)|\} M(p, 1), \quad |z| \ge 1, \tag{2.4}$$

where

$$q(z) = z^n \overline{p(1/\overline{z})}. \tag{2.5}$$

Proof of Lemma 2 We consider the polynomial

$$t(z) = p(z) - \lambda M(p, 1), \quad |\lambda| > 1,$$

of degree at most n. Then the polynomial

$$T(z) = z^n \overline{t(1/\overline{z})} = q(z) - \overline{\lambda} M(p, 1) z^n, \quad \text{(by (2.5))},$$

of degree n, possesses the characteristic

$$|t(z)| \le |T(z)|, \quad |z| = 1$$

and has all its zeros in $|z| \le 1$. Therefore on applying Lemma 1 to polynomials $t(z)$ and $T(z)$ we get for $0 \le s < n$ and $|\lambda| > 1$

$$|p^{(s)}(z) - \lambda M(p, 1) \frac{d^s}{dz^s}(1)| \le |q^{(s)}(z) - \overline{\lambda} M(p, 1) \frac{d^s}{dz^s}(z^n)|, \quad |z| \ge 1,$$

which, by choosing $\arg \lambda$ suitably, can be rewritten as

$$|p^{(s)}(z)| - |\lambda| M(p, 1) |\frac{d^s}{dz^s}(1)| \le ||\lambda| M(p, 1) |\frac{d^s}{dz^s}(z^n)| - |q^{(s)}(z)||, \quad |z| \ge 1. \tag{2.6}$$

We can apply Lemma 1 to polynomials $q(z)$ and

$$z^n M(p, 1)$$

also, and obtain for $0 \le s < n$

$$|q^{(s)}(z)| \le M(p, 1) |\frac{d^s}{dz^s}(z^n)|, \quad |z| \ge 1,$$

which helps us to rewrite(2.6) as

$$|p^{(s)}(z)| - |\lambda| M(p, 1) |\frac{d^s}{dz^s}(1)| \le |\lambda| M(p, 1) |\frac{d^s}{dz^s}(z^n)| - |q^{(s)}(z)|, \quad |z| \ge 1, |\lambda| > 1, 0 \le s < n.$$

Now on letting

$$|\lambda| \to 1,$$

(2.4) follows. □

Lemma 3 *If $P(z)$ is a polynomial of degree n, having no zeros in $|z| < k, (k \geq 1)$, with*

$$M(P, 1) = 1$$

then for $1 \leq R \leq k^2$

$$M(P, R) \leq ((R + k)/(1 + k))^n.$$

This lemma is due to Aziz and Mohammad [2].

Lemma 4 *Let $P(z)$ be a polynomial of degree n, having no zeros in $|z| < k, (k \geq 1)$. Then*

$$|P(z)| \leq 1 \textit{ for } |z| \leq 1$$

implies

$$|P^{(s)}(z)| \leq n(n-1)\dots(n-s+1)/(1+k^s) \textit{ for } |z| \leq 1 \textit{ and } s \geq 1.$$

This lemma is due to Govil and Rahman [3].
From Lemma 4 we easily get

Lemma 5 *If $P(z)$ is a polynomial of degree n, having no zeros in $|z| < k, (k \geq 1)$ then for $0 \leq s < n$*

$$M(P^{(s)}, 1) \leq (1/(1+k^s))M(P, 1)[\{\frac{d^s}{dx^s}(1+x^n)\}_{x=1}].$$

3. Proof of Theorem 1

We consider the polynomial

$$P(z) = p(kz). \tag{3.1}$$

Then the polynomial

$$Q(z) = z^n \overline{P(1/\overline{z})}$$

possesses the characteristic

$$|P(z)| \leq |Q(z)|, \quad |z| = 1$$

and has all its zeros in $|z| \le 1$. Therefore on applying Lemma 1 to the polynomials $P(z)$ and $Q(z)$ we get for $0 \le s < n$ and $t \ge 1$

$$|P^{(s)}(te^{i\theta})| \le |Q^{(s)}(te^{i\theta})|, \quad 0 \le \theta \le 2\pi. \tag{3.2}$$

Further, by Lemma 2 we have for $t \ge 1$ and $0 \le s < n$

$$|P^{(s)}(te^{i\theta})| + |Q^{(s)}(te^{i\theta})| \le \{\frac{d^s}{dt^s}(t^n + 1)\}M(P, 1), \quad 0 \le \theta \le 2\pi,$$

which, by (3.2), implies that

$$|P^{(s)}(te^{i\theta})| \le (1/2)\{\frac{d^s}{dt^s}(1 + t^n)\}M(p, 1),$$

i.e.

$$\begin{aligned} |p^{(s)}(kte^{i\theta})| &\le (1/(2k^s))\{\frac{d^s}{dt^s}(1 + t^n)\}M(p, k),, \text{ (by (3.1))}, \\ &\le (1/(2k^s))(2k/(1 + k))^n M(p, 1)\{\frac{d^s}{dt^s}(1 + t^n)\}, \text{(by Lemma 3)}, \end{aligned}$$

thereby leading to inequality (1.1).

Now by applying Lemma 5 to the polynomial $p(Rz), (1 \le R \le k)$, having no zeros in $|z| < k/R$, we have for $0 \le s < n$

$$M(p^{(s)}, R) \le (1/(R^s + k^s))M(p, R)[\{\frac{d^s}{dx^s}(1 + x^n)\}_{x=1}],$$

and the inequality (1.2) follows by using Lemma 3. This completes the proof of Theorem 1. □

References

[1] Ankeny, N. C. and Rivlin, T. J.: On a theorem of S. Bernstein, *Pac. J. Maths.* 5, 849–852 1955.

[2] Aziz, A. and Mohammad, Q. G.: Growth of polynomials with zeros outside a circle, *Proc. Amer. Math. Soc.* 81, 549–553 1981.

[3] Govil, N. K. and Rahman, Q. I.: Functions of exponential type not vanishing in a half plane and related polynomials, *Trans. Amer. Math. Soc.* 137, 501–517 1969.

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