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On Graded Weakly Prime Ideals

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Abstract

Let G be an arbitrary group with identity e, and let R be a G-graded commutative ring. Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied in [1]. Here we study the graded weakly prime ideals of a G-graded commutative ring. A number of results concerning graded weakly prime ideals are given. For example, we give some characterizations of graded weakly prime ideals and their homogeneous components.

Key Words: Graded rings, Graded weakly prime ideals.

1. Introduction

Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D. D. Anderson and E. Smith in [1]. Also, weakly primary ideals in a commutative ring with non-zero identity have been introduced and studied in [2]. Here we study the graded weakly prime ideals of a G-graded commutative ring. The purpose of this paper is to explore some basic facts of these class of ideals. Various properties of graded weakly prime ideals are considered. First, we show that if P is a graded weakly prime ideal, then for each $g \in G$, either P_g is a prime subgroup of R_g or $P_g^2 = 0$. Also, we show that if P and Q are graded weakly prime ideals such that P_g and Q_h are not prime for all $g, h \in G$ respectively, then $\mathrm{Grad}(P) = \mathrm{Grad}(Q) = \mathrm{Grad}(0)$ and $P + Q$ is a graded weakly prime ideal of $G(R)$. Next, we give some characterizations of graded weakly prime ideals and their homogeneous components (see sec. 2).

Before we state some results let us introduce some notation and terminology. Let G be an arbitrary group with identity e. By a G-graded commutative ring we mean a commutative ring R with non-zero identity together with a direct sum decomposition (as an additive group) $R = \oplus_{g\in G} R_g$ with the property that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$; here $R_g R_h$ denotes the additive subgroup of R consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in R_h$. We denote this by $G(R)$, and we consider supp $G(R) = \{g \in G : R_g \neq 0\}$. The summands R_g are called homogeneous components and elements of these summands are called homogeneous elements. If $a \in R$, then a can be written uniquely as $\sum_{g\in G} a_g$ where a_g is the component of a in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. Moreover, if $R = \oplus_{g\in G} R_g$ is a graded ring, then R_e is a subring of R, $1_R \in R_e$ and R_g is an R_e-module for all $g \in G$.

Let I be an ideal of R. For $g \in G$, let $I_g = I \cap R_g$. Then I is a graded ideal of $G(R)$ if $I = \oplus_{g\in G} I_g$. In this case, I_g is called the g-component of I for $g \in G$. Moreover, R/I becomes a G-graded ring with g-component $(R/I)_g = (R_g + I)/I \cong R_g/I_g$ for $g \in G$. Clearly, 0 is a graded ideal of $G(R)$. If I and J are graded ideals of $G(R)$, the ideal $\{a \in R : aJ \subseteq I\}$, denoted by $(I :_R J)$, is a graded ideal (see [4]). An ideal I of $G(R)$ is said to be graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I, denoted by Grad(I), is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of $G(R)$, then $r \in \text{Grad}(I)$ if and only if $r^n \in I$ for some positive integer n.

2. Graded Weakly Prime Ideals

Our starting point is the following definitions.

Definition 2.1 *Let P be a graded ideal of $G(R)$ and $g \in G$.*

(i) We say that P_g is a prime subgroup of R_g if $P_g \neq R_g$; and whenever $a, b \in R_g$ with $ab \in P_g$, then either $a \in P_g$ or $b \in P_g$.

(ii) We say that P is a graded weakly prime ideal of $G(R)$ if $P \neq R$; and whenever $a, b \in h(R)$ with $0 \neq ab \in P$, then either $a \in P$ or $b \in P$.

Clearly, a graded prime ideal of $G(R)$ is a graded weakly prime. However, since 0 is always a graded weakly prime ideal (by definition), a graded weakly prime ideal need not be graded prime.

Proposition 2.2 *Let $P = \oplus_{g\in G} P_g$ be a graded weakly prime ideal of $G(R)$. Then for each $g \in G$, either P_g is a prime subgroup of R_g or $P_g^2 = 0$.*

Proof. It is enough to show that if $P_g^2 \neq 0$ for some $g \in G$, then P_g is a prime subgroup of R_g. Let $pq \in P_g \subseteq P$ where $p, q \in R_g$. If $pq \neq 0$, then P weakly prime gives either $p \in P_g$ or $q \in P_g$. So suppose that $pq = 0$. If $pP_g \neq 0$, then there is an element c of P_g such that $pc \neq 0$, so $0 \neq pc = p(c+q) \in P$; hence either $p \in P$ or $(c+q) \in P$. As $c \in P$ we have either $p \in P_g$ or $q \in P_g$. So we can assume that $pP_g = 0$. Similarly, we can assume that $qP_g = 0$. Since $P_g^2 \neq 0$, there exist $c, d \in P_g$ such that $cd \neq 0$. Then $(p+c)(q+d) = cd \in P$, so either $p + c \in P$ or $q + d \in P$, and hence either $p \in P_g$ or $q \in P_g$. Thus P_g is prime. □

Proposition 2.3 *Let P be a graded weakly prime ideal of $G(R)$ and $g \in G$. Then for $a \in R_g - P_g$, either $(P_g :_{R_e} a) = P_e$ or $(P_g :_{R_e} a) = (0 :_{R_e} a)$.*

Proof. It is well known that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them; so for $a \in R_g - P_g$, it is enough to show that $(P_g :_{R_e} a) = P_e \cup (0 :_{R_e} a) = H$.

If $b \in P_e$, then $ab \in R_g \cap P = P_g$, so $b \in (P_g :_{R_e} a)$. Clearly, $(0 :_{R_e} a) \subseteq (P_g :_{R_e} a)$; hence $H \subseteq (P_g :_{R_e} a)$. For the other containment, assume that $c \in (P_g :_{R_e} a)$. If $0 \neq ac \in P_g \subseteq P$, then P graded weakly prime gives $c \in P$; hence $c \in P_e \subseteq H$. If $ac = 0$, then $c \in (0 :_{R_e} a) \subseteq H$, as needed. □

Theorem 2.4 *Let $P = \oplus_{g\in G} P_g$ be a graded weakly prime ideal of $G(R)$ such that P_g is not a prime subgroup of R_g for every $g \in G$. Then* Grad$(P) =$ Grad(0).

Proof. Since Grad$(0) \subseteq$ Grad(P) is trivial, we will prove the reverse inclusion. Let $p \in P$. By Proposition 2.2, $P_g^2 = 0 \in (0)$ for every $g \in G$, so $p \in$ Grad(0); hence $P \subseteq$ Grad(0). It follows that Grad$(P) \subseteq$ Grad(0) by [4, Proposition 1.2], as required. □

Proposition 2.5 *Let $I \subseteq P$ be graded ideals of $G(R)$ with $P \neq R$. Then the following hold:*

(i) If P is graded weakly prime, then P/I is graded weakly prime.

(ii) If I and P/I are graded weakly prime, then P is graded weakly prime.

Proof. (i) Let $0 \neq (a+I)(b+I) = ab + I \in P/I$ where $a, b \in h(R)$, so $ab \in P$. If $ab = 0 \in I$, then $(a+I)(b+I) = 0$, a contradiction. If $ab \neq 0$, P graded weakly prime gives either $a \in P$ or $b \in P$; hence either $a + I \in P/I$ or $b + I \in P/I$, as required.

(ii) Let $0 \neq ab \in P$ where $a, b \in h(R)$, so $(a+I)(b+I) \in P/I$. If $ab \in I$, then I graded weakly prime gives either $a \in I \subseteq P$ or $b \in I \subseteq P$. So we may assume that $ab \notin I$. Then either $a+I \in P/I$ or $b+I \in P/I$ since P/I is graded weakly prime. It follows that either $a \in P$ or $b \in P$, as needed. □

Theorem 2.6 *Let P and Q be graded weakly prime ideals of $G(R)$ such that P_g and Q_h are not prime subgroups of R_g and R_h respectively for all $g, h \in G$. Then $P + Q$ is a graded weakly prime ideal of $G(R)$.*

Proof. By Theorem 2.4, we have $\mathrm{Grad}(P) + \mathrm{Grad}(Q) = \mathrm{Grad}(0) \neq R$, so $P + Q$ is a proper ideal of R. Since $(P+Q)/Q \cong Q/(P \cap Q)$, we get $(P+Q)/Q$ is graded weakly prime by Propositin 2.5 (i). Now the assertion follows from Proposition 2.5 (ii). □

Let M be an R-module. A proper submodule N of M is prime if for any $r \in R$ and $m \in M$ such that $rm \in N$, either $rM \subseteq N$ or $m \in N$. It is easy to show that if N is a prime submodule of M then the annihilator of the module M/N is a prime ideal of R. A proper submodule N of a module M over a commutative ring R is said to be weakly prime submodule if whenever $0 \neq rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$. The following lemma is well-known, but we write it here for the sake of references.

Lemma 2.7 *Let R be a commutative ring, M an R-module, and N a proper submodule of M. Then the following assertions are equivalent.*

(i) N is a prime submodule of M.

(ii) $IB \subseteq N$, with I an ideal of R, and B a submodule of M, implies that $I \subseteq (N : M)$ or $B \subseteq N$.

Lemma 2.8 *Let $P = \oplus_{g \in G} P_g$ be a graded weakly prime ideal of $G(R)$. Then P_g is a weakly prime submodule of the R_e-module R_g for every $g \in G$.*

Proof. Suppose that P is a graded weakly prime ideal of $G(R)$. For $g \in G$, assume that $0 \neq ab \in P_g \subseteq P$ where $a \in R_g$ and $b \in R_e$, so P graded weakly prime gives either

$a \in P$ or $b \in P$. If $a \in P$, then $a \in P_g$. If $b \in P$, then $b \in (P_g :_{R_e} R_g)$. So P_g is weakly prime. □

Proposition 2.9 *Let $P = \oplus_{g \in G} P_g$ be a graded weakly prime ideal of $G(R)$. Then for each $g \in G$, either P_g is a prime submodule of the R_e-module R_g or $(P_g :_{R_e} R_g)P_g = 0$.*

Proof. By Lemma 2.8, P_g is a weakly prime submodule of R_g for every $g \in G$. It is enough to show that if $(P_g :_{R_e} R_g)P_g \neq 0$ for some $g \in G$, then P_g is prime. Let $pq \in P_g$, where $p \in R_g$ and $q \in R_e$. If $pq \neq 0$, then either $p \in P_g$ or $q \in (P_g :_{R_e} R_g)$ since P_g is weakly prime. So suppose that $pq = 0$. If $qP_g \neq 0$, then there is an element p' of P_g such that $qp' \neq 0$, so $0 \neq qp' = q(p' + p) \in P_g$, and hence P_g weakly prime gives either $q \in (P_g :_{R_e} R_g)$ or $(p' + p) \in P_g$. As $p' \in P_g$ we have either $q \in (P_g :_{R_e} R_g)$ or $p \in P_g$. So we can assume that $pP_g = 0$. Suppose that $p(P_g :_{R_e} R_g) \neq 0$, say $pc \neq 0$ where $c \in (P_g :_{R_e} R_g)$. Then $0 \neq pc = p(c + q) \in P_g$ and P_g weakly prime gives either $p \in P_g$ or $q \in (P_g :_{R_e} R_g)$ since $c \in (P_g :_{R_e} R_g)$. So we can assume that $p(P_g :_{R_e} R_g) = 0$.

Since $(P_g :_{R_e} R_g)P_g \neq 0$, there exist $c \in (P_g :_{R_e} R_g)$ and $d \in P_g$ such that $cd \neq 0$. Then $(q + c)(p + d) = cd \in P_g$, so either $q + c \in (P_g :_{R_e} R_g)$ or $p + d \in P_g$, and hence either $q \in (P_g :_{R_e} R_g)$ or $p \in P_g$. Thus P_g is prime. □

We next give three other characterizations of homogeneous components of graded ideals.

Theorem 2.10 *Let P be a proper graded ideal of $G(R)$ and $g \in G$. Then the following assertions are equivalent.*

(i) If whenever $0 \neq IB \subseteq P_g$ with I an ideal of R_e and B a submodule of R_g implies that $I \subseteq (P_g :_{R_e} R_g)$ or $B \subseteq P_g$.

(ii) P_g is a weakly prime submodule of R_g.

(iii) For $a \in R_g - P_g$, $(P_g :_{R_e} a) = (P_g :_{R_e} R_g) \cup (0 :_{R_e} a)$.

(iv) For $a \in R_g - P_g$, $(P_g :_{R_e} a) = (P_g :_{R_e} R_g)$ or $(P_g :_{R_e} a) = (0 :_{R_e} a)$.

Proof. $(i) \Longrightarrow (ii)$ Let $0 \neq ab \in P_g$ where $a \in R_g$ and $b \in R_e$. Take $I = R_e b$ and $B = R_e a$. Then $0 \neq IB \subseteq P_g$, so either $I \subseteq (P_g :_{R_e} R_g)$ or $B \subseteq P_g$; hence either $a \in P_g$ or $b \in (P_g :_{R_e} R_g)$. Thus P_g is weakly prime.

$(ii) \Rightarrow (i)$ Suppose first that P_g is a weakly prime submodule of R_g. If P_g is prime, then the result follows by Lemma 2.7. So we can assume that P_g is weakly prime that is not prime. Let $0 \neq IB \subseteq P_g$ with $x \in B - P_g$. We show that $I \subseteq (P_g :_{R_e} R_g)$. Let $r \in I$. If $rx \neq 0$, then P_g weakly prime gives $r \in (P_g :_{R_e} R_g)$. So assume that $rx = 0$. If $rB \neq 0$, then $rd \neq 0$ for some $0 \neq d \in B \subseteq R_g$. If $d \in P_g$, then $r(d+x) \in P_g$ gives either $r \in (P_g :_{R_e} R_g)$ or $d + x \in P_g$, so $r \in (P_g :_{R_e} R_g)$ since $d \in P_g$. If $d \notin P_g$, then $rd \in P_g$ gives $r \in (P_g :_{R_e} R_g)$. So we can assume that $rB = 0$. Suppose that $Ix \neq 0$, say $ax \neq 0$ where $a \in I$. Then P_g weakly prime gives $a \in (P_g :_{R_e} R_g)$. It follows from the equality $(r + a)x = ax$ that $r \in (P_g :_{R_e} R_g)$, so $I \subseteq (P_g :_{R_e} R_g)$. Therefore we can assume that $Ix = 0$.

Since $IB \neq 0$, there exist $s \in I$ and $b \in B$ such that $sb \neq 0$. As $0 \neq s(b+x) = sb \in P_g$ we divided the proof into the following cases:

Case 1 $s \notin (P_g :_{R_e} R_g)$ and $b + x \notin P_g$.

Since $s(b + x) = sb \in P_g$, P_g weakly prime gives either $b + x \in P_g$ or $s \in (P_g :_{R_e} R_g)$, a contradiction.

Case 2 $s \notin (P_g :_{R_e} R_g)$ and $b + x \in P_g$.

As $0 \neq sb \in P_g$ we have $b \in P_g$, so $x \in P_g$, a contradiction.

Case 3 $s \in (P_g :_{R_e} R_g)$ and $b + x \in P_g$.

Since $b + x \in P_g$, we obtain $b \notin P_g$ (otherwise $x \in P_g$). As $0 \neq b(r + s) \in P_g$, we get $r \in (P_g :_{R_e} R_g)$. Thus $I \subseteq (P_g :_{R_e} R_g)$.

Case 4 $s \in (P_g :_{R_e} R_g)$ and $b + x \notin P_g$.

Since $0 \neq (r+s)(b+x) = sb \in P_g$ it follows that $r+s \in (P_g :_{R_e} R_g)$, so $r \in (P_g :_{R_e} R_g)$. Hence $I \subseteq (P_g :_{R_e} R_g)$.

$(ii) \Rightarrow (iii)$ Clearly, if $a \in R_g - P_g$, then $H = (P_g :_{R_e} R_g) \cup (0 :_{R_e} a) \subseteq (P_g :_{R_e} a)$. Let $b \in (P_g :_{R_e} a)$ where $a \in R_g - P_g$. Then $ab \in P_g$. If $ab \neq 0$, then $b \in (P_g :_{R_e} R_g)$ since P_g is weakly prime, so $b \in H$. If $ab = 0$, then $b \in (0 :_{R_e} a)$, so $b \in H$, and hence we have equality.

$(iii) \Rightarrow (iv)$ Is obvious.

$(iv) \Rightarrow (ii)$ Suppose that $0 \neq ab \in P_g$ with $b \in R_e$ and $a \in R_g - P_g$. Then $b \in (P_g :_{R_e} a)$ and $b \notin (0 :_{R_e} a)$. It follows from (iv) that $b \in (P_g :_{R_e} a) = (P_g :_{R_e} R_g)$, as required. □

Lemma 2.11 *Let P be a graded ideal of $G(R)$. Then the following assertions are equivalent.*

(i) P is a graded prime ideal of $G(R)$.

(ii) For each $g, h \in G$, the inclusion $AB \subseteq P$ with submodules A of R_g and B of R_h implies that $A \subseteq P$ or $B \subseteq P$.

Proof. $(i) \Longrightarrow (ii)$ Suppose that P is a graded prime ideal of $G(R)$. For $g, h \in G$, assume that A is an R_e-submodule of R_g and B is an R_e-submodule of R_h such that $AB \subseteq P$ with $x \in A - P$. We want to prove that $B \subseteq P$. Let $a \in B$. Then $ax \in P$, so $a \in P$ since P is graded prime.

$(ii) \Longrightarrow (i)$ Suppose that $cd \in P$ where $c, d \in h(R)$. There are elements $r, s \in G$ such that $c \in R_r$ and $d \in R_s$. Then $R_e c$ and $R_e d$ are submodules of R_r and R_s respectively with $(c)(d) \subseteq P$, so either $(c) \subseteq P$ or $(d) \subseteq P$ by (ii); hence either $c \in P$ or $d \in P$. So P is graded prime. □

We next give an other characterization of graded weakly prime ideals.

Theorem 2.12 *Let $P = \oplus_{g \in G} P_g$ be a graded ideal of $G(R)$ with $P \neq R$. Then the following assertions are equivalent.*

(i) P is a graded weakly prime ideal of $G(R)$.

(ii) For each $g, h \in G$, the inclusion $0 \neq AB \subseteq P$ with submodules A of R_g and B of R_h implies that $A \subseteq P$ or $B \subseteq P$.

Proof. $(i) \Rightarrow (ii)$ Suppose that P is a graded weakly prime ideal of $G(R)$. For $g, h \in G$, assume that $0 \neq AB \subseteq P$ where A is a submodule of R_g and B is a submodule of R_h with $x \in B - P$. We show that $A \subseteq P$. If P is graded prime, then the result follows by Lemma 2.11. So we can assume that P is graded weakly prime that is not graded prime. Let $y \in A$. If $xy \neq 0$, then P graded weakly prime gives $y \in P$. So assume that $xy = 0$. First suppose that $yB \neq 0$, say $yd \neq 0$ where $0 \neq d \in B \subseteq R_h$. If $d \in P$, then $0 \neq y(x + d) \in P$ gives either $y \in P$ or $x + d \in P$. Hence $y \in P$ since $x \notin P$. If $d \notin P$, then $y \in P$ since $0 \neq dy \in P$ and P is graded weakly prime. So we can assume that $yB = 0$. Suppose that $xA \neq 0$, say $xb \neq 0$ where $b \in A \subseteq R_g$. Then P graded weakly prime gives $b \in P$. Since $0 \neq x(y + b) \in P$, we obtain $y + b \in P$; hence $y \in P$. So we can assume that $xA = 0$.

Since $AB \neq 0$, there exist $c \in A$ and $d \in B$ with $0 \neq cd \in P$, so either $c \in P$ or $d \in P$. As $0 \neq (x + d)(y + c) = cd \in P$, we divided the proof the following cases:

Case 1 $c \in P$ and $y + c \notin P$.

Since $0 \neq (x + d)(y + c) = cd \in P$ it follows that $x + d \in P$. As $0 \neq d(y + c) \in P$, we obtain $d \in P$; hence $x \in P$, a contradiction.

Case 2 $c \notin P$ and $y + c \in P$.

As $0 \neq cd \in P$ and $c(x+d) \in P$, we get $d \in P$ and $x + d \in P$; hence $x \in P$, a contradiction.

Case 3 $c \notin P$ and $y + c \notin P$.

By assumption, $d \in P$ and $x + d \in P$, so $x \in P$, a contradiction.

Case 4 $c \in P$ and $y + c \in P$.

Clearly, $y \in P$. Thus $A \subseteq P$.

$(ii) \Longrightarrow (i)$ Let $0 \neq ab \in P$ where $a \in R_g$ and $b \in R_h$ for some $h, g \in G$. Take $A = R_e a \subseteq R_g$ and $B = R_e b \subseteq R_h$. Then $0 \neq AB \subseteq P$, so either $A \subseteq P$ or $B \subseteq P$; hence either $a \in P$ or $b \in P$. Thus P is graded weakly prime. $\square$

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