Turk J Math
30 (2006) , 403 – 411.

Note on Generalized Jordan Derivations Associate with Hochschild 2-cocycles of Rings*

Atsushi Nakajima

Abstract

We introduce a new type of generalized derivations associate with Hochschild 2-cocycles and prove that every generalized Jordan derivation of this type is a generalized derivation under certain conditions. This result contains the results of I. N. Herstein [6, Theorem 3.1] and M. Ashraf and N-U. Rehman [1, Theorem].

Key Words: Derivation, Jordan derivation, generalized derivation, generalized Jordan derivation, Hochschild 2-cocycle .

1. Introduction

Let R be a ring and let x, y be arbitrary elements of R. M is called an R-bimodule if M is a left and a right R-module such that $x(my) = (xm)y$ for all $m \in M$. Let $f : R \to M$ be an additive map. f is called a generalized derivation if there exists a derivation $d : R \to M$ such that

$$f(xy) = f(x)y + xd(y). \tag{1}$$

and f is called a generalized Jordan derivation if there exists a Jordan derivation $J : R \to M$ such that

$$f(x^2) = f(x)x + xJ(x), \tag{2}$$

We denote (1) and (2) by $(f,\ d)$ and $(f,\ J)$, respectively. These types of generalized derivations were introduced by M. Brešar [2] and their properties have been discussed in

*Dedicated to Professor Şerif Yenigül on his 60th birthday many papers. In [7], another type of generalized derivations was defined by the author as follows. f is called a generalized derivation if there exists an element $\omega \in M$ such that

$$f(xy) = f(x)y + xf(y) + x\omega y, \tag{3}$$

and f is called a generalized Jordan derivation if

$$f(x^2) = f(x)x + xf(x) + x\omega x, \tag{4}$$

which denote by $(f,\ \omega)$. Some categorical properties of these generalized derivations were given in [7]. In the case R has an identity element 1, if $(f,\ d)$ is a generalized derivation of type (1), then $(f,\ -f(1))$ is a generalized derivation of type (3); and conversely, if $(f,\ \omega)$ is a generalized derivation of type (3), then $f + \omega_\ell : R \to M$ is a derivation and $(f,\ f + \omega_\ell)$ is a generalized derivation of type (1), where $\omega_\ell : R \ni x \mapsto \omega x \in M$.

A Jordan derivation of 2-torsion free prime rings is a derivation. This result was first proved by Herstein [6, Theorem 3.1] and was extended to 2-torsion free semiprime rings by Brešar [4, Theorem 1]. In [8], the author proved that a generalized Jordan derivation $(f,\ \omega)$ of type (4) is also a generalized derivation of type (3), and in [1] they proved that a generalized Jordan derivation $(f,\ d)$ of type (2) is a generalized derivation of type (1) under a certain commutator condition.

In this note, we introduce a new type of generalized derivations and show that our generalized Jordan derivation is a generalized derivation under certain conditions. This result contains the results of I. N. Herstein [6, Theorem 3.1], M. Ashraf and N-U. Rehman [1, Theorem].

Throughout the following, we assume that R is a ring, M is an R-bimodule and x, y, z are arbitrary elements of R, unless otherwise stated.

2. Definitions and Lemmas

Let $\alpha : R \times R \to M$ be a biadditive map, that is, an additive map on each components. α is called a *Hochschild* 2-*cocycle* if

$$x\alpha(y,\ z) - \alpha(xy,\ z) + \alpha(x,\ yz) - \alpha(x,\ y)z = 0. \tag{5}$$

2-cocycle α is called *symmetric* (resp. *skew symmetric*) if $\alpha(x,\ y) = \alpha(y,\ x)$ (resp. $\alpha(x,\ y) = -\alpha(y,\ x)$). An additive map $f : R \to M$ is called a *generalized derivation* if there exists a 2-cocycle α such that

$$f(xy) = f(x)y + xf(y) + \alpha(x,\ y), \tag{6}$$

and f is called a *generalized Jordan derivation* if

$$f(x^2) = f(x)x + xf(x) + \alpha(x,\ x). \tag{7}$$

We denote it by $(f,\ \alpha)$. If $\alpha = 0$, then they are the usual derivations and Jordan derivations. We give some examples of our generalized derivations.

Examples (1) If $(f,\ d)$ and $(f,\ \omega)$ are generalized derivations of types (1) and (3), respectively, then the maps

$$\alpha_1 : R \times R \ni (x,\ y) \mapsto x(d - f)(y) \in M \quad \text{and} \quad \alpha_2 : R \times R \ni (x,\ y) \mapsto x\omega y \in M$$

are biadditive and satisfy the 2-cocycle condition (5). Since $(f,\ d) = (f,\ \alpha_1)$ and $(f,\ \omega) = (f,\ \alpha_2)$, the usual generalized derivations are generalized derivations in our sense.

(2) If $f : R \to M$ is a left mutiplier, that is, f is additive and $f(xy) = f(x)y$, then by $f(xy) = f(x)y + xf(y) + x(-f)(y)$, we have a 2-cocycle $\alpha_3 : R \times R \ni (x,\ y) \mapsto x(-f)(y) \in M$ and $f = (f,\ \alpha_3)$. Thus a left multiplier is also a generalized derivation.

(3) Let f be a $(\sigma,\ \tau)$-derivation, that is, σ and τ are ring homomorphisms of R and $f(xy) = f(x)\sigma(y) + \tau(x)f(y)$. Then the map

$$\alpha_4 : R \times R \ni (x,\ y) \mapsto f(x)(\sigma(y) - y) + (\tau(x) - x)f(y) \in M$$

is biadditive and satisfies the 2-cocycle condition. Since $f(xy) = f(x)y + xf(y) + \alpha_4(x,\ y)$, $(\sigma,\ \tau)$-derivation f is also a generalized derivation $(f,\ \alpha_4)$.

(4) In general, we have the following. Let $f : R \to M$ be an additive map and let $\alpha : R \times R \to M$ be a biadditive map. If $f(xy) = f(x)y + xf(y) + \alpha(x,\ y)$ holds, then by the associativity $f((xy)z) = f(x(yz))$, α satisfies the 2-cocycle condition. Thus $(f,\ \alpha)$ is a generalized derivation in our sense.

Now the following lemma is elementary and can be found everywhere such as in [3, Proposition 2] or [1, Lemma 2.1].

Lemma 1 *Let $(f,\ d) : R \to M$ be a generalized Jordan derivation and M a 2-torsion free module, where $d : R \to M$ is a derivation. Then the following relations hold:*

(1) $f(xy+yx)=f(x)y+xd(y)+f(y)x+yd(x)$;
(2) $f(xyx)=f(x)yx+xd(y)x+xyd(x)$;
(3) $f(xyz+zyx)=f(x)yz+xd(y)z+xyd(z)+f(z)yx+zd(y)x+zyd(x)$.

In our case, the above relations are generalized as follows.

Lemma 2 *Let* $(f,\ \alpha): R\rightarrow M$ *be a generalized Jordan derivation associate with Hochschild 2-cocycle* α *and* M *a 2-torsion free module. Then the following relations hold:*
(1) $f(xy+yx)=f(x)y+xf(y)+\alpha(x,\ y)+f(y)x+yf(x)+\alpha(y,\ x)$,
(2) $f(xyx)=f(x)yx+xf(y)x+xyf(x)+x\alpha(y,\ x)+\alpha(x,\ yx)$,
(3) $f(xyz+zyx)=f(x)yz+xf(y)z+xyf(z)+x\alpha(y,\ z)+\alpha(x,\ yz)$
$+\ f(z)yx+zf(y)x+zyf(x)+z\alpha(y,\ x)+\alpha(z,\ yx)$.

Proof. (1) Since $f(x^2)=f(x)x+xf(x)+\alpha(x,\ x)$, (1) is easily obtained by $f(xy+yx)=f((x+y)^2)-f(x^2)-f(y^2)$.

(2) Replacing y by $xy+yx$ in (1) and using the 2-cocycle condition (5), we have

$$\begin{aligned}
2f(xyx) &= f(x(xy+yx)+(xy+yx)x)-f(x^2y+yx^2)\\
&= 2\{f(x)yx+xf(y)x+xyf(x)\}\\
&\quad +x\{\alpha(x,\ y)+\alpha(y,\ x)\}+\alpha(x,\ xy)+\alpha(x,\ yx)\\
&\quad +\{\alpha(x,\ y)+\alpha(y,\ x)\}x+\alpha(xy,\ x)+\alpha(yx,\ x)\\
&\quad -\{\alpha(x,\ x)y+\alpha(x^2,\ y)+y\alpha(x,\ x)+\alpha(y,\ x^2)\}\\
&= 2\{f(x)yx+xf(y)x+xyf(x)\}\\
&\quad +\{x\alpha(x,\ y)-\alpha(x^2,\ y)+\alpha(x,\ xy)-\alpha(x,\ x)y\}\\
&\quad -\{y\alpha(x,\ x)-\alpha(yx,\ x)+\alpha(y,\ x^2)-\alpha(y,\ x)x\}\\
&\quad +x\alpha(y,\ x)+\alpha(x,\ yx)+\alpha(x,\ y)x+\alpha(xy,\ x)\\
&= 2\{f(x)yx+xf(y)x+xyf(x)\}\\
&\quad +x\alpha(y,\ x)+\alpha(x,\ yx)+\alpha(x,\ y)x+\alpha(xy,\ x).
\end{aligned}$$

Since $x\alpha(y,\ x)+\alpha(x,\ yx)=\alpha(xy,\ x)+\alpha(x,\ y)x$ and M is 2-torsion free, we have the relation (2).

(3) Replace x by $x+z$ in (2), then (3) is easily seen. □

The following lemma is useful in the calculations of 2-torsion free semiprime rings which can be found in [5, Lemmas 1.1 and 1.2].

Lemma 3 *(1) Let R be a 2-torsion free semiprime ring and $a,\ b \in R$. If $axb + bxa = 0$ for all $x \in R$, then $axb = bxa = 0$ for all $x \in R$. Especially, $ab = ba = 0$.*

(2) Let $G_1,\ G_2, \cdots, G_n$ be additive groups and R a semiprime ring. Suppose that mappings $S : G_1 \times G_2 \times \cdots \times G_n \to R$ and $T : G_1 \times G_2 \times \cdots \times G_n \to R$ are additive in each argument. If $S(a_1, a_2, \cdots, a_n)xT(a_1, a_2, \cdots, a_n) = 0$ for all $x \in R$, $a_i \in G_i$, $i = 1, 2, \cdots, n$, then $S(a_1, a_2, \cdots, a_n)xT(b_1, b_2, \cdots, b_n) = 0$ for all $x \in R$, $a_i,\ b_i \in G_i$, $i = 1, 2, \cdots, n$.

Now for all $x,\ y \in R$, we set

$$F(x,\ y) = f(xy) - f(x)y - xf(y), \tag{8}$$

and

$$\delta(x,\ y) = F(x,\ y) - \alpha(x,\ y). \tag{9}$$

Then $F(x,\ y)$ and $\delta(x,\ y)$ are biadditive and by Lemma 2 (1), we have

$$\delta(x,\ y) + \delta(y,\ x) = 0. \tag{10}$$

Lemma 4 *Let $(f,\ \alpha) : R \to M$ be a generalized Jordan derivation and M a 2-torsion free module. Then the following relations hold:*

(1) $\delta(x,\ y)z[x,\ y] + [x,\ y]z\delta(x,\ y) = 0$ where $[x,\ y] = xy - yx$,

(2) $\delta(x,\ y)[x,\ y] = 0$.

Proof. (1) By (2) and (3) of Lemma 2, we have

$$\begin{aligned}
0 &= f((xy)z(yx) + (yx)z(xy)) - f(x(yzy)x + y(xzx)y) \\
&= F(x,\ y)zyx + F(y,\ x)zxy + xyzF(y,\ x) + yxzF(x,\ y) \\
&\quad + xy\{\alpha(z,\ yx) - \alpha(z,\ y)x\} + \{\alpha(xy,\ zyx) - \alpha(x,\ yzyx)\} \\
&\quad + yx\{\alpha(z,\ xy) - \alpha(z,\ x)y\} + \{\alpha(yx,\ zxy) - \alpha(y,\ xzxy)\} \\
&\quad - \{x\alpha(y,\ zy)x + x\alpha(yzy,\ x) + y\alpha(x,\ zx)y + y\alpha(xzx,\ y)\}(*).
\end{aligned}$$

Since α is a 2-cocycle, we have the following relations:

(1) $xy\{\alpha(z,\ yx) - \alpha(z,\ y)x\} = xy\{\alpha(zy,\ x) - z\alpha(y,\ x)\}$,
(2) $\alpha(xy,\ zyx) - \alpha(x,\ yzyx) = x\alpha(y,\ zyx) - \alpha(x,\ y)zyx$,
(3) $yx(\alpha(z,\ xy) - \alpha(z,\ x)y\} = yx\{\alpha(zx,\ y) - z\alpha(x,\ y))$,
(4) $\alpha(yx,\ zxy) - \alpha(y,\ xzxy) = y\alpha(x,\ zxy) - \alpha(y,\ x)zxy$. □

Substituting from (1) to (4) in the above relation (∗) and using the 2-cocycle condition (5) we get

$$\begin{aligned} 0 &= \delta(x,\ y)zyx + \delta(y,\ x)zxy + xyz\delta(y,\ x) + yxz\delta(x,\ y) \\ &+ x\{y\alpha(zy,\ x) - \alpha(yzy,\ x) + \alpha(y,\ zyx) - \alpha(y,\ zy)x\} \\ &+ y\{x\alpha(zx,\ y) - \alpha(xzx,\ y) + \alpha(x,\ zxy) - \alpha(x,\ zx)y\} \\ &= \delta(x,\ y)zyx + \delta(y,\ x)zxy + xyz\delta(y,\ x) + yxz\delta(x,\ y). \end{aligned}$$

Since $\delta(x,\ y) = -\delta(y,\ x)$ by (10), we have

$$\delta(x,\ y)z[x,\ y] + [x,\ y]z\delta(x,\ y) = 0.$$

(2) Similarly by Lemma 2 (2) and (3), we have

$$\begin{aligned} 0 &= f((xy)^2 + xy^2x) - f(xy(xy) + (xy)yx) \\ &= F(x,\ y)[x,\ y] + \{\alpha(xy,\ xy) - x\alpha(y,\ xy) - \alpha(x,\ yxy)\} \\ &+ x\{\alpha(y^2,\ x) + \alpha(y,\ y)x - y\alpha(y,\ x)\} + \alpha(x,\ y^2x) - \alpha(xy,\ yx)(**). \end{aligned}$$

Substituting

$$\alpha(xy,\ xy) = x\alpha(y,\ xy) + \alpha(x,\ yxy) - \alpha(x,\ y)xy \quad \text{and}$$
$$\alpha(y^2,\ x) = y\alpha(y,\ x) + \alpha(y,\ yx) - \alpha(y,\ y)x$$

in the relation (∗∗), we get

$$F(x,\ y)[x,\ y] - \alpha(x,\ y)xy + x\alpha(y,\ yx) + \alpha(x,\ y^2x) - \alpha(xy,\ yx) = 0.$$

Since $x\alpha(y,\ yx) + \alpha(x,\ y^2x) - \alpha(xy,\ yx) = \alpha(x,\ y)yx$, we have

$$\delta(x,\ y)[x,\ y] = 0.$$

Lemma 5 *Let R be a 2-torsion free ring and G_1, G_2 additive groups. Let $S,\ T$: $G_1 \times G_2 \to R$ be biadditive maps. Assume that $S(x_1,\ x_2)T(x_1,\ x_2) = 0$ for all $x_i \in G_i$, $i = 1, 2$. If there exists a non-zero divisor $T(a_1,\ a_2)$ for some $a_i \in G_i$, $i = 1, 2$, then $S(x_1,\ x_2) = 0$ for all $x_i \in G_i$, $i = 1, 2$.*

Proof. We may assume that $T(a_1,\ a_2)$ is a non-zero divisor for some $a_i \in G_i$, $i = 1, 2$ and so $S(a_1,\ a_2) = 0$. Then by $S(x_1 + a_1,\ x_2)T(x_1 + a_1,\ x_2) = 0$, we have

$$S(x_1,\ x_2)T(a_1,\ x_2) + S(a_1,\ x_2)T(x_1,\ x_2) = 0 \tag{11}$$

for all $x_i \in G_i$, $i = 1, 2$. Replacing x_2 by $x_2 + a_2$ in (11), and using (11) again, we get

$$S(x_1,\ x_2)T(a_1,\ a_2) + S(x_1,\ a_2)T(a_1,\ x_2) + S(x_1,\ a_2)T(a_1,\ a_2) + S(a_1,\ x_2)T(x_1,\ a_2) = 0$$

for all $x_i \in G_i$, $i = 1, 2$. Take $x_1 = a_1$ in the above relation, we have $2S(a_1,\ x_2)T(a_1,\ a_2) = 0$. Since $T(a_1,\ a_2)$ is non-zero divisor and R is 2-torsion free, we see $S(a_1,\ x_2) = 0$ for all $x_2 \in G_2$. Then by (11), we get $S(x_1,\ x_2)T(a_1,\ x_2) = 0$ and so $S(x_1,\ a_2) = 0$. Thus we have $S(x_1,\ x_2) = 0$ for all $x_i \in G_i$, $i = 1, 2$. □

3. Generalized Jordan Derivations

In this section, we show that a generalized Jordan derivation $(f,\ \alpha)$ is a generalized derivation under certain conditions.

Theorem 6 *Let R be a 2-torsion free ring and let $(f,\ \alpha) : R \to R$ be a generalized Jordan derivation associate with Hochschild 2-cocycle α. If R satisfies one of the following conditions, then $(f,\ \alpha)$ is a generalized derivation.*

(1) R is a non-commutative prime ring.
(2) There exist $a, b \in R$ such that $[a,\ b]$ is a non-zero divisor.
(3) R is commutative and α is symmetric.

Proof. (1) By Lemma 4 (1) and Lemma 3 (1), we have $\delta(x,\ y)z[x,\ y] = 0$. Since R is non-commutative, there exist $a, b \in R$ such that $[a,\ b] \neq 0$. Then by Lemma 3 (2), we have $\delta(x,\ y)z[a,\ b] = 0$ and by the primeness of R, $\delta(x,\ y) = 0$ for all $x, y \in R$. Thus $(f,\ \alpha)$ is a generalized derivation.

(2) Suppose that $[a,\ b]$ is a nonzero divisor and so $\delta(a,\ b) = 0$ for some $a,\ b \in R$. Since $\delta(x,\ y)$ and $[x,\ y]$ are biadditive maps, then by Lemma 5, we have $\delta(x,\ y) = 0$.

(3) Since R is commutative and α is symmetric, then by Lemma 2 (1), $(f,\ \alpha)$ is a generalized derivation.

Let $\xi : R \to M$ be a left Jordan multiplier, that is, ξ is additive and $\xi(x^2) = \xi(x)x$, then by the similar calculations as in the proof of Lemma 2 and Lemma 4 (2), we have the following relations:

$$\xi(xy + yx) = \xi(x)y + \xi(y)x, \qquad 2\xi(xyx) = 2\xi(x)yx,$$

$$2\xi(xyz + zyx) = 2(\xi(x)yz + \xi(z)yx), \qquad 2(\xi(xy) - \xi(x)y)[x,\ y] = 0.$$

If R is 2-torsion free and has a non-zero divisor $[a,\ b]$ for some $a,\ b \in R$, then by the above relations, we have $\xi(ab) = \xi(a)b$ and thus by Lemma 5, $\xi(xy) = \xi(x)y$ for all x, $y \in R$. Therefore a left Jordan multiplier is a left multiplier. If $(f,\ J)$ is a generalized Jordan derivation of type (2), then

$$f(x^2) = f(x)x + xf(x) + x(J - f)(x)$$

and $\xi = J - f$ is a left Jordan multiplier. In this case, it is easy to see that $\alpha : R \times R \ni (x,\ y) \mapsto x\xi(y) \in M$ is 2-cocycle. Thus by Theorem 6 (2), we have the following which is a slight generalization of [1, Theorem]. □

Corollary 7 *Let R be a 2-torsion free ring with non-zero divisor $[a,\ b]$ for some $a,\ b \in R$. Then a generalized Jordan derivation $(f,\ J)$ is a generalized derivation.*

Finally, we note the following. Let $(f,\ \alpha)$ be a generalized derivation. Then by (6), α is symmetric if and only if $f([x,\ y]) = [f(x),\ y] + [x,\ f(y)]$, that is, f is a Lie derivation. And α is skew symmetric if and only if $f\{x,\ y\} = \{f(x),\ y\} + \{x,\ f(y)\}$, where $\{x,\ y\} = xy + yx$. Thus in case of R is 2-torsion free, this means that α is skew symmetric if and only if f is a Jordan derivation. Therefore the notion of our generalized derivations has many common properties of the notion of several types of derivations defined untill now.

References

[1] Ashraf M. and UR-Rehman N.: On Jordan generalized derivations in rings, Math. J. Okayama Univ. **42**(2000), 7-9.

[2] Brešar M.: On the distance of the compositions of two derivations to the generalized derivations, Glasgow Math. J. **33**(1991), 80-93.

[3] Brešar M. and Vukman J.: Jordan derivations on prime rings, Bull. Austral. Math. Soc. **37**(1988), 321-322.

[4] Brešar M.: Jordan derivations on semiprime rings, Proc. A.M.S. **104**(1988), 1003-1006.

[5] Brešar M.: Jordan mappings of semiprime rings, J. Algebra **127**(1989), 218-228.

[6] Herstein, I.N.: Jordan derivations of prime rings, Proc. A.M.S. **8**(1957), 1104-1110.

[7] Nakajima, A.: On categorical properties of generalized derivations, Scientiae Mathematicae **2**(1999), 345-352.

[8] Nakajima, A.: Generalized Jordan derivations, International Symposium on Ring Theory, 235-243, Birkhäuser, Boston, 2001.

Atsushi NAKAJIMA
Received 13.04.2005
Department of Environmental and Mathematical Sciences,
Faculty of Environmental Science and Technology,
Okayama University,
Tsushima, Okayama 700-8530, JAPAN
e-mail: nakajima@ems.okayama-u.ac.jp