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Weighted Norm Inequalities for a Class of Rough Maximal Operators

H. M. Al-Qassem

Abstract

We consider maximal singular integral operators arising from rough kernels satisfying an H^1-type condition on the unit $(n-1)$-sphere and prove weighted L^p estimates for certain radial weights. We also prove weighted L^p estimates with A_p-weights where in this case the H^1-type condition is replaced by an L^q-type condition with $q > 1$. Some applications of these results are also obtained regarding singular integrals and Marcinkiewicz integrals. Our results are essential extensions and improvements of some known results.

Key words and phrases: L^p boundedness, Hardy space, maximal operators, Fourier transform, rough kernel, A_p weight.

1. Introduction and Results

Throughout this paper, let $\mathbf{R}^n$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Let $x' = x/|x|$ for $x \in \mathbf{R}^n \backslash \{0\}$, p' denote the conjugate index of p (that is, $1/p + 1/p' = 1$) and $\{\Omega_j\}$ be an arbitrary but fixed countable subset of $L^1(\mathbf{S}^{n-1})$ with

$$\int_{\mathbf{S}^{n-1}} \Omega_j(y) d\sigma(y) = 0. \tag{1.1}$$

Let Ω be an arbitrary but fixed function defined on $\mathbf{S}^{n-1}$ with $\Omega \in L^1(\mathbf{S}^{n-1})$ and satisfies the cancellation condition (1.1) with Ω_j replaced by Ω. Let $\mathcal{R}(\mathbf{R}_+)$ denote the set of all

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functions b on $\mathbf{R}_+$ satisfying the condition

$$\|b\|_{L^2(\mathbf{R}_+,dr/r)} = \left(\int_{\mathbf{R}_+} |b(r)|^2 \frac{dr}{r} \right)^{1/2} \leq 1$$

and let $M(\{\Omega_j\})$ be the class of all kernels of the form

$$K(ty) = t^{-n} \sum_j b_j(t)\Omega_j(y) \text{ (defined for } t > 0 \text{ and } y \in \mathbf{S}^{n-1}),$$

where

$$\int_0^\infty \sum_j |b_j(t)|^2 \frac{dt}{t} \leq 1.$$

In 1992, L. K. Chen and X. Wang [5] studied the L^p boundedness of the maximal operator $\sup_{K \in M(\{\Omega_j\})} |T_K f|$, where the operator T_K is defined by

$$T_K f(x) = \int_0^\infty \int_{\mathbf{S}^{n-1}} f(x - ty)K(ty)t^{n-1} d\sigma(y)dt.$$

In [5], L. K. Chen and X. Wang proved the following:

Theorem A *Let $2n/(2n-1) < p < \infty$ and let $\{\Omega_j\}$ be a countable subset of $L^2(S^{n-1})$ with $\sum_j \|\Omega_j\|^2_{L^2(\mathbf{S}^{n-1})} < \infty$. Then $\sup_{K \in M(\{\Omega_j\})} |T_K f|$ is bounded on $L^p(\mathbf{R}^n)$. Moreover, the range of p is the best possible.*

We notice that if we take in the definition of $M(\{\Omega_j\})$ our countable set $\{\Omega_j\}$ to be the the singleton Ω, where Ω is a fixed function defined on $\mathbf{S}^{n-1}$ with $\Omega \in L^2(\mathbf{S}^{n-1})$ and satisfies (1.1) and if we take the countable set $\{b_j\}$ to be the singleton b and letting b vary with b belongs to the class $\mathcal{R}(\mathbf{R}_+)$, the maximal function $\sup_{K \in M(\{\Omega_j\})} |T_K f|$ will reduce to the maximal operator $\mathcal{M}_\Omega(f)$ given by

$$\mathcal{M}_\Omega f(x) = \sup_{b \in \mathcal{R}(\mathbf{R}_+)} \left| \int_{\mathbf{R}^n} f(x-y) \frac{\Omega(y)}{|y|^n} b(|y|)dy \right|.$$

Thus obviously, the maximal operator $\sup_{K \in M(\{\Omega_j\})} |T_K f|$ is a natural extension of the maximal operator $\mathcal{M}_\Omega$. Therefore, as an immediately corollary of Theorem A we get the following:

Theorem B *Let Ω be a function defined on $\mathbf{S}^{n-1}$ with $\Omega \in L^2(\mathbf{S}^{n-1})$ and satisfies (1.1). Then $\mathcal{M}_\Omega(f)$ is bounded on $L^p(\mathbf{R}^n)$ for $2n/(2n-1) < p < \infty$.*

We remark that the maximal operator $\mathcal{M}_\Omega$ was formally introduced by L. K. Chen and H. Lin in [4] who proved Theorem B under the stronger condition $\Omega \in C(\mathbf{S}^{n-1})$ (see also [5]). The study of the maximal operator $\mathcal{M}_\Omega$ has attracted the attention of many authors in recent years. For example, see [1], [2], [9], [18] and [24].

This paper aims at extending the result in Theorem A in several directions: (1) by allowing the countable set $\{\Omega_j\}$ to be a subset of $L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq \infty$ instead of being $\{\Omega_j\}$ a subset of $L^2(\mathbf{S}^{n-1})$, (2) by allowing the countable set $\{\Omega_j\}$ to be a subset of the Hardy space $H^1(\mathbf{S}^{n-1})$ which contains $L^2(\mathbf{S}^{n-1})$ as a proper subset, (3) by investigating the weighted L^p boundedness of these operators instead of investigating their $L^p(\mathbf{R}^n)$ boundedness, and (4) by considering maximal operators along some types of submanifolds.

Before stating our results, we first introduce some notations and give some definitions.

Definition *For $q \geq 1$, let $\mathcal{L}_q(\{\Omega_j\})$ be the class of all kernels of the form*

$$K(ty) = t^{-n} \sum_j b_j(t)\Omega_j(y) \ \text{(defined for } t > 0 \text{ and } y \in \mathbf{S}^{n-1}),$$

where

$$\int_0^\infty \sum_j |b_j(t)|^2 \frac{dt}{t} \leq 1$$

and $\{\Omega_j\}$ is a fixed countable subset of $L^q(\mathbf{S}^{n-1})$ with Ω_j satisfying (1.1) and

$$\sum_j \|\Omega_j\|^2_{L^q(\mathbf{S}^{n-1})} < \infty.$$

Definition *Let $\mathcal{H}^1(\{\Omega_j\})$ denote the class of kernels of the form*

$$K(ty) = t^{-n} \sum_j b_j(t)\Omega_j(y) \ \text{(defined for } t > 0 \text{ and } y \in \mathbf{S}^{n-1}),$$

where

$$\int_0^\infty \sum_j |b_j(t)|^2 \frac{dt}{t} \leq 1$$

and $\{\Omega_j\}$ *is a fixed countable fixed subset of* $H^1(\mathbf{S}^{n-1})$ *with* Ω_j *satisfying (1.1) and*

$$\sum_j \|\Omega_j\|^2_{H^1(\mathbf{S}^{n-1})} < \infty.$$

Here $H^1(\mathbf{S}^{n-1})$ denotes the Hardy space on $\mathbf{S}^{n-1}$ in the sense of Coifman and Weiss [6] and its definition will be reviewed in Section 2. It is well-known that

$$C(\mathbf{S}^{n-1}) \subset L^q(\mathbf{S}^{n-1})(q>1) \subset L(\log L)(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}) \subset L^1(\mathbf{S}^{n-1}). \tag{1.2}$$

The inclusions in (1.2) are proper. In light of (1.2), it is easy to verify that the following inclusions hold:

$$\mathcal{L}_q(\{\Omega_j\}) \text{ (for } q > 1) \subsetneq \mathcal{H}^1(\{\Omega_j\}) \subsetneq \mathcal{L}_1(\{\Omega_j\}); \tag{1.3}$$

$$\mathcal{L}_q(\{\Omega_j\}) \text{ (for } q \geq 2) \subset \mathcal{L}_2(\{\Omega_j\}) \subset \mathcal{L}_q(\{\Omega_j\}) \text{ (for } 1 < q \leq 2). \tag{1.4}$$

Thus we have

$$\sup_{K\in\mathcal{L}_q(\{\Omega_j\})} |T_K f| \quad (\text{for } q > 1) \leq \sup_{K\in\mathcal{H}^1(\{\Omega_j\})} |T_K f|;$$

$$\sup_{K\in\mathcal{L}_q(\{\Omega_j\})} |T_K f| \quad (\text{for } q \geq 2) \leq \sup_{K\in M} |T_K f| \leq \sup_{K\in\mathcal{L}_q(\{\Omega_j\})} |T_K f| \ (\text{for } 1 < q \leq 2).$$

We shall need the following definitions which are closely related to those appearing in [15]:

Definition *We say that a function* Φ *satisfies "hypothesis I" if*

(a) Φ *is an increasing* C^1 *function on* $[0,\infty)$ *with* $\Phi(0)=0$,

(b) $\Phi'(t)$ *is increasing on* $(0,\infty)$ *or* Φ *satisfies* $\Phi(2t) \geq \eta\Phi(t)$ *for some fixed* $\eta > 1$ *and* $\Phi'(t)$ *is decreasing on* $(0,\infty)$.

Definition *We say that* Φ *satisfies "hypothesis D" if*

(a') Φ *is a decreasing* C^1 *function on* $[0,\infty)$ *with* $\Phi(0)=0$,

(b') $\Phi'(t)$ *is decreasing on* $(0,\infty)$ *or* $\Phi(t) \geq \eta\Phi(2t)$ *for some fixed* $\eta > 1$ and $\Phi'(t)$ *is increasing on* $(0,\infty)$.

Model functions for the Φ satisfy hypothesis I are $\Phi(t) = t^d$ with $d > 0$, and their linear combinations with positive coefficients. Model functions for the Φ satisfy hypothesis D are $\Phi(t) = t^r$ with $r < 0$, and their linear combinations with positive coefficients.

Throughout this paper, for a nonnegative locally integrable function ω we shall write $\|f\|_{p,\omega}$ (or $\|f\|_{L^p(\omega)}$) for $\left(\int_{\mathbf{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p}$. When $\omega \equiv 1$, we shall simply write $\|f\|_p$ (or $\|f\|_{L^p}$) for $\|f\|_{p,\omega}$.

We now state our main results.

Theorem 1.1 *Let $1 < q \leq \infty$. Then $\sup_{K \in \mathcal{L}_q(\{\Omega_j\})} |T_K f|$ is bounded on $L^p(\omega)$ if p and ω satisfy one of the following conditions:*

(a) $\delta \leq p < \infty$ and $\omega \in A_{p/\delta}$;

(b) $2n\delta/(2n + n\delta - 2) < p < 2$, $\omega(x) = |x|^{\alpha}$, $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np - 2n - p)$, where $\delta = \max\{2, q'\}$.

Here $A_p = A_p(\mathbf{R}^n)$ represents the collection of Muckenhoupt's A_p-weights whose definition will be recalled in Section 2.

Corollary 1 *Assume that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$ and satisfies the vanishing condition (1.1). Then $\mathcal{M}_{\Omega}$ is bounded on $L^p(\omega)$ if p and ω satisfy the same conditions as in Theorem 1.1.*

For radial weights we are able to prove the following sharper and more general result:

Theorem 1.2 *Assume that Φ satisfies either hypothesis I or D. Then*

(a) $\sup_{K \in \mathcal{H}^1(\{\Omega_j\})} |T_{K,\Phi} f|$ is bounded on $L^p(\omega)$ if $\omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+)$ and $2 \leq p < \infty$;

(b) $\sup_{K \in \mathcal{L}_q(\{\Omega_j\})} |T_{K,\Phi} f|$ (for $q > 1$) is bounded on $L^p\left(|x|^{\alpha}\right)$ if $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np - 2n - p)$ and $2n\delta/(2n + n\delta - 2) < p < 2$, where

$$T_{K,\Phi} f(x) = \int_0^{\infty} \int_{\mathbf{S}^{n-1}} f(x - \Phi(t)y) K(ty) t^{n-1} d\sigma(y) dt,$$

and $\tilde{A}^I_p(\mathbf{R}_+)$ is a special class of radial weights introduced by Duoandikoetxea [10]. The definition of $\tilde{A}^I_p(\mathbf{R}_+)$ will be reviewed in Section 2.

Corollary 2 *Let Ω be a homogeneous function of degree zero satisfies the vanishing condition (1.1). Assume that Φ satisfies either hypothesis I or D. Then the maximal operator $\mathcal{M}_{\Omega,\Phi}$ defined by*

$$\mathcal{M}_{\Omega,\Phi} f(x) = \sup_{h \in \mathcal{R}(\mathbf{R}_+)} \left| \int_{\mathbf{R}^n} f(x - \Phi(|y|)y') \frac{\Omega(y)}{|y|^n} h(|y|) dy \right|$$

is bounded on $L^p(\omega)$ if Ω, p and ω satisfy the same conditions as in Theorem 1.2.

Remarks (1) By the relationships (1.3)–(1.4) one sees that, even in the special cases $\Phi(t) \equiv t$ and $\omega = 1$, our results represent substantial extensions of Theorem A.

(2) Corollary 1 was proved by Y. Ding and H. Qingzheng in [9] under the condition $\Omega \in L^2(\mathbf{S}^{n-1})$. Later on, Corollaries 1 and 2 were proved by Al-Qassem in [1].

(3) The main tools used in this paper come from [1], [2], [18], [11] and [15], among others.

Throughout the paper the letter C will denote a positive constant whose value may change at each occurrence.

2. Definitions and Lemmas

Let us begin by recalling the definition of the Hardy space H^1 on the unit sphere $\mathbf{S}^{n-1}$.

Definition 2.1 *The Hardy space $H^1(\mathbf{S}^{n-1})$ is the linear space of distributions $f \in \mathcal{S}'(\mathbf{S}^{n-1})$ with norm $\|f\|_{H^1(\mathbf{S}^{n-1})} = \|P^+ f\|_{L^1(\mathbf{S}^{n-1})} < \infty$, where $P^+ f(x')$ denotes the radial maximal function of f. The space $H^1(\mathbf{S}^{n-1})$ was studied in [6] (see also [7]). A function $a : \mathbf{S}^{n-1} \to \mathbf{C}$ is called an H^1 atom if it satisfies the following:*

(i) $\operatorname{supp}(a) \subset \mathbf{S}^{n-1} \cap \mathbf{B}(x_0, \rho)$ *for some $x_0 \in \mathbf{S}^{n-1}$ and $\rho > 0$, where $\mathbf{B}(x_0, \rho)$ is the ball with center x_0 and radius ρ;*

(ii) $\|a\|_\infty \le \rho^{-n+1}$;

(iii) $\int_{\mathbf{S}^{n-1}} a(y) d\sigma(y) = 0$.

¿From [6] or [7], we find that any $\Omega \in H^1(\mathbf{S}^{n-1})$ with the mean zero property (1.1) has an atomic decomposition $\Omega = \sum_{j=1}^{\infty} c_j a_j$, where $\{c_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$, $\{a_j\}$ is a sequence of H^1 atoms on $\mathbf{S}^{n-1}$ and

$$\sum_{j=1}^{\infty} |c_j| \le C \|\Omega\|_{H^1(\mathbf{S}^{n-1})}$$

with C independent of Ω.

Definition 2.2 *A locally integrable nonnegative function ω is said to belong to $A_p(\mathbf{R}^n)$*

$(1 \le p < \infty)$ if there is a positive constant C such that

$$\sup_{Q\subset \mathbf{R}^n} \left(|Q|^{-1} \int_Q \omega(x)dx\right) \left(|Q|^{-1} \int_Q \omega(x)^{-1/(p-1)}dx\right)^{p-1} \le C, \text{ for } 1 < p < \infty$$

and $\omega \in A_1(\mathbf{R}^n)$ if $M^*\omega(x) \le C\omega(x)$ a.e. $x \in \mathbf{R}^n$, where Q denotes a cube in $\mathbf{R}^n$ with its sides parallel to the coordinate axes and $M^* f$ denotes the usual Hardy-Littlewood maximal function.

Now, we give the definition of certain radial weights ([15], [10]).

Definition 2.3 *Let $\omega(t) \ge 0$ and $\omega \in L^1_{loc}(\mathbf{R}_+)$. For $1 < p < \infty$, we say that $\omega \in A_p(\mathbf{R}_+)$ if there is a positive constant C such that for any interval $I \subset \mathbf{R}_+$,*

$$\left(|I|^{-1} \int_I \omega(t)dt\right) \left(|I|^{-1} \int_I \omega(t)^{-1/(p-1)}dt\right)^{p-1} \le C < \infty.$$

We say that $\omega \in A_1(\mathbf{R}_+)$ if there is a positive constant C such that

$$|I|^{-1} \int_I \omega(t)dt \le C \text{ } ess \inf_{t\in I} \omega(t) \text{ for any interval } I \subset \mathbf{R}_+.$$

It is easy to verify that $\omega \in A_1(\mathbf{R}_+)$ if and only if there is a positive constant C such that

$$M^*\omega(t) \le C\omega(t) \text{ for } a.e.\ t \in \mathbf{R}_+$$

Definition 2.4 *Let $1 \le p < \infty$. If $\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p}$, where either $\nu_j \in A_1(\mathbf{R}_+)$ is decreasing or $\nu_j^2 \in A_1(\mathbf{R}_+)$, $j = 1, 2$, then we say that $\omega \in \tilde{A}_p(\mathbf{R}_+)$.*

Let $A_p^I(\mathbf{R}^n)$ be the weight class defined by using all n-dimensional intervals with sides parallel to coordinate axes (see [17]). It is well-known that $|x|^\gamma \in \tilde{A}_p^I$ for $-1 < \gamma < p - 1$ (see [17]). Let $\tilde{A}_p^I(\mathbf{R}_+)$ be the class of all weights $\omega(t)$ so that $\omega(t) \in \tilde{A}_p(\mathbf{R}_+)$ and $\omega(|x|) \in A_p^I(\mathbf{R}^n)$. If $\omega \in \tilde{A}_p(\mathbf{R}_+)$, it follows from [10] that M^*f is bounded on $L^p(\mathbf{R}^n, \omega(|x|)dx)$. Therefore, if $\omega(t) \in \tilde{A}_p(\mathbf{R}_+)$, then $\omega(|x|) \in A_p(\mathbf{R}^n)$.

By following the same argument as in the proof of the elementary properties of A_p weight class (see for example [16]) we get the following:

Lemma 2.5 *If $1 \le p < \infty$, then the weight class $\tilde{A}_p^I(\mathbf{R}_+)$ has the following properties:*

(i) $\tilde{A}_{p_1}^I \subset \tilde{A}_{p_1}^I$, *if* $1 \le p_1 < p_2 < \infty$;

(ii) For any $\omega \in \tilde{A}_p^I$, *there exists an* $\varepsilon > 0$ *such that* $\omega^{1+\varepsilon} \in \tilde{A}_p^I$;

(iii) For any $\omega \in \tilde{A}_p^I$ *and* $p > 1$, *there exists an* $\varepsilon > 0$ *such that* $p - \varepsilon > 1$ *and* $\omega \in \tilde{A}_{p-\varepsilon}^I$.

For a fixed $\rho > 0$, we let $B_\rho(\xi) = (\rho^2\xi_1, \rho\xi_2, \dots, \rho\xi_n)$. Also, for $k \in \mathbf{Z}$, set $\theta_k = \Phi(2^k)$ if Φ satisfies hypothesis I and $\theta_k = (\Phi(2^k))^{-1}$ if Φ satisfies hypothesis D. Then by the conditions of Φ, it is easy to see that $\{\theta_k\}$ is a lacunary sequence of positive numbers with $\inf_{k\in\mathbf{Z}} \frac{\theta_{k+1}}{\theta_k} \ge \lambda > 1$, where $\lambda = \min\{2, \eta\}$.

By following the same argument as in the proof of Lemma 2.1 in [15], we get the following:

Lemma 2.6 *Suppose that* $a(\cdot)$ *is an* H^1 atom *on* $\mathbf{S}^{n-1}$ *with* supp*(a)*$\subseteq \mathbf{B}(\mathbf{e}, \rho) \cap \mathbf{S}^{n-1}$, *where* $\mathbf{e} = (1, \dots, 0) \in \mathbf{S}^{n-1}$. *Let*

$$F_{a,\Phi,k}(\xi) = \left(\int_{2^k}^{2^{k+1}} \left| \int_{\mathbf{S}^{n-1}} a(y') e^{-i\Phi(t)<\xi,y'>} d\sigma(y') \right|^2 \frac{dt}{t} \right)^{1/2}.$$

Then there exist positive constants β, C *independent of* k, ξ *and* ρ *such that if* Φ *satisfies hypothesis I,*

$$|F_{a,\Phi,k}(\xi)| \le C \min\left\{ \theta_k^{-\beta} \left|B_\rho(\xi)\right|^{-\beta}, \theta_{k+1}^{\beta} \left|B_\rho(\xi)\right|^{\beta} \right\};$$

and if Φ *satisfies hypothesis D,*

$$|F_{a,\Phi,k}(\xi)| \le C \min\left\{ \theta_{k+1}^{-\beta} \left|B_\rho(\xi)\right|^{\beta}, \theta_k^{\beta} \left|B_\rho(\xi)\right|^{-\beta} \right\}.$$

For an $\Omega \in L^1(\mathbf{S}^{n-1})$ and a C^1 function Φ defined on $\mathbf{R}_+$, we define the maximal operator

$$M^*_{\Phi,\Omega} f(x) = \sup_{k\in\mathbf{Z}} \left| \int_{2^k \le |y| < 2^{k+1}} f(x - \Phi(|y|)y') \frac{|\Omega(y')|}{|y|^n} dy \right|.$$

If $\Phi(t) \equiv t$, we denote $M^*_{\Phi,\Omega}$ by M^*_Ω.

By the same argument as in ([22], p.57) , we get the following lemma.

Lemma 2.7 *Let φ be a nonnegative, decreasing function on $[0,\infty)$ with*

$$\int_{[0,\infty)} \varphi(t)dt = 1.$$

Then

$$\left| \int_{[0,\infty)} f(x-ty')\varphi(t)dt \right| \le M_{y'}f(x),$$

where

$$M_{y'}f(x) = \sup_{R\in\mathbf{R}} \frac{1}{R}\int_0^R |f(x-sy')|\, ds$$

is the Hardy-Littlewood maximal function of f in the direction of y'.

By Lemma 2.7 and following a similar argument as in [1] and [15] we get:

Lemma 2.8 *Let $\Omega \in L^1(\mathbf{S}^{n-1})$ and $\omega \in \tilde{A}_p(\mathbf{R}_+)$, $1 < p < \infty$. Assume Φ satisfies either hypothesis I or D. Then*

$$\left\| M^*_{\Phi,\Omega}(f) \right\|_{L^p(\omega)} \le C_p \left\|\Omega\right\|_{L^1(\mathbf{S}^{n-1})} \left\|f\right\|_{L^p(\omega)}, \tag{2.1}$$

where C_p is a constant independent of Ω and $f \in L^p(\omega)$.

By the proof of the Theorem 5 in ([10], p. 873), we have the following:

Lemma 2.9 *Let $\Omega \in L^d(\mathbf{S}^{n-1})$ for some $d > 1$. If p, d and ω satisfy one of the following conditions:*
(a) $d' \le p < \infty$, $p \ne 1$ and $\omega \in A_{p/d'}$;
(b) $1 < p \le d$, $p \ne \infty$, $\omega^{1-p'} \in A_{p'/d'}$,
then there is a $C > 0$, independent of f and Ω such that

$$\left\|M^*_\Omega f\right\|_{L^p(\omega)} \le C \left\|\Omega\right\|_{L^d(\mathbf{S}^{n-1})} \left\|f\right\|_{L^p(\omega)}.$$

Let $\mathcal{M}_{Sph}$ be the spherical maximal operator defined by

$$\mathcal{M}_{Sph}f(x) = \sup_{r>0} \int_{\mathbf{S}^{n-1}} |f(x - r\theta)|\, d\sigma(\theta).$$

We shall need the following result concerning the weighted L^p boundedness of $\mathcal{M}_{Sph}$ with power weights.

Lemma 2.10 ([13]) *Suppose that* $n \geq 2$, $p > n/(n-1)$ *and* $1-n < \alpha < (n-1)(p-1)-1$. *Then* $\mathcal{M}_{Sph}(f)$ *is bounded on* $L^p(\mathbf{R}^n, |x|^{\alpha})$.

Lemma 2.11 *Suppose that* $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $q > 1$. *Then for some positive constant* C, *we have*

$$\left| \int_{\mathbf{S}^{n-1}} \Omega(\xi) f(\xi) d\sigma(\xi) \right|^2 \leq C \left\| \Omega \right\|_q^{\min\{2,q\}} \int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{\max\{0,2-q\}} |f(\xi)|^2 \, d\sigma(\xi) \tag{2.2}$$

for arbitrary functions f.

Proof. When $q \geq 2$ (so that $q' \leq 2$), from Hölder's inequality we have

$$\begin{aligned} \left| \int_{\mathbf{S}^{n-1}} \Omega(\xi) f(\xi) d\sigma(\xi) \right|^2 &\leq \|\Omega\|_q^2 \left(\int_{\mathbf{S}^{n-1}} |f(\xi)|^{q'} \, d\sigma(\xi) \right)^{2/q'} \\ &\leq \|\Omega\|_q^2 \int_{\mathbf{S}^{n-1}} |f(\xi)|^2 \, d\sigma(\xi), \end{aligned}$$

which is the statement of the lemma for the case $q \geq 2$.

When $1 < q < 2$ (so that $q' > 2$), the conclusion of the lemma follows from Schwarz's inequality and the fact that $\Omega \in L^q(\mathbf{S}^{n-1})$. This finishes the proof of the lemma. □

Lemma 2.12 *Suppose that* $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $q > 1$. *Let* $\delta = \max\{2, q'\}$. *Then for some positive constant* C, *we have*

$$\int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{\max\{0,2-q\}} |\omega(x - t\xi)| \, d\sigma(\xi) \leq C \left\| \Omega \right\|_q^{\max\{0,2-q\}} \left(\mathcal{M}_{Sph} \left(|\omega|^{\delta/2} \right)(x) \right)^{2/\delta} \tag{2.3}$$

for all positive real numbers t, $x \in \mathbf{R}^n$ *and arbitrary functions* ω.

Proof. As in the proof of Lemma 2.11, we shall consider the cases $q \geq 2$ and $1 < q < 2$ separately. We notice that if $q \geq 2$, the inequality (2.7) is obvious. However, if $1 < q < 2$, (2.7) follows easily from Hölder's inequality and noticing that $(\frac{q}{2-q})' = q'/2$. The lemma is proved. □

3. Proof of Main Results

We shall present the proof of Theorem 1.2 only for the case Φ satisfies hypothesis I, since the proof of these theorems for the case Φ satisfies hypothesis D is essentially the same. We shall start first by proving Theorem 1.2.

Proof of Theorem 1.2 for condition (a)

Assume $K \in \mathcal{H}^1(\{\Omega_j\})$. By definition of $T_{K,\Phi}$ we have

$$T_{K,\Phi}f(x) = \int_0^\infty \sum_j b_j(t) \int_{\mathbf{S}^{n-1}} \Omega_j(y) f(x - \Phi(t)y) d\sigma(y) \frac{dt}{t}.$$

By Shwarz inequality and the definition of the kernel K, we get

$$|T_{K,\Phi}f(x)| \leq \left(\int_0^\infty \sum_j \left| g_{\Omega_j}^{(t)} f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \equiv gf(x), \tag{3.1}$$

where

$$g_{\Omega_j}^{(t)} f(x) = \int_{\mathbf{S}^{n-1}} \Omega_j(y) f(x - \Phi(t)y) d\sigma(y).$$

Thus the proof of Theorem 1.2 for condition (a) is proved if we can show that

$$\|g(f)\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)} \tag{3.2}$$

for $\omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+)$, $2 \leq p < \infty$ and for a constant C independent of the kernel K. So let us turn to the proof of (3.2). Since $\Omega_j \in H^1(\mathbf{S}^{n-1})$ has the mean zero property (1.1), we can write $\Omega_j = \sum\limits_{s=1}^{\infty} C_{s,j} a_{s,j}$, where each $a_{s,j}$ is an H^1 atom and

$$\sum_{s=1}^{\infty} |C_{s,j}| \leq C \|\Omega_j\|_{H^1(\mathbf{S}^{n-1})}$$

with a constant C independent of Ω_j. Thus,

$$gf(x) = \left(\sum_j \int_0^\infty \left| \sum_{s=1}^\infty C_{s,j} g_{a_{s,j}}^{(t)} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{3.3}$$

where $g_{a_{s,j}}^{(t)} f(x)$ is defined in the same way as $g_{\Omega_j}^{(t)} f(x)$ except that we replace Ω_j by $a_{s,j}$. By applying Minkowski's inequality, we get

$$gf(x) \leq \sum_{s=1}^\infty \left(\sum_j |C_{s,j}|^2 \left| G_{a_{s,j}} f(x) \right|^2 \right)^{1/2}, \tag{3.4}$$

where

$$G_{a_{s,j}} f(x) = \left(\int_0^\infty \left| g_{a_{s,j}}^{(t)} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

The key step in the proof of (3.2), is to prove the following inequality:

$$\left\| G_{a_{s,j}}(f) \right\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)} \text{ for } \omega \in \tilde{A}_{p/2}^I(\mathbf{R}_+) \text{ and } 2 \leq p < \infty, \tag{3.5}$$

where C is independent of the atoms $a_{s,j}(\cdot)$ and f. Before presenting a proof of (3.5), let us prove (3.2) by employing (3.5). By (3.4) and (3.5) we have

$$\begin{aligned}
\|g(f)\|_{L^p(\omega)} &\leq \sum_{s=1}^\infty \left\| \left(\sum_j |C_{s,j}|^2 \left| G_{a_{s,j}} f \right|^2 \right)^{1/2} \right\|_{L^p(\omega)} \\
&\leq \sum_{s=1}^\infty \sum_j |C_{s,j}|^2 \left\| \left| G_{a_{s,j}} f \right|^2 \right\|_{L^{p/2}(\omega)}^{1/2} \\
&\leq \sum_{s=1}^\infty \sum_j |C_{s,j}|^2 \left\| \left| G_{a_{s,j}} f \right| \right\|_{L^p(\omega)} \\
&\leq C \sum_j \|\Omega_j\|_{H^1(\mathbf{S}^{n-1})}^2 \|f\|_{L^p(\omega)} \\
&\leq C \|f\|_{L^p(\omega)}.
\end{aligned}$$

Let us now turn to the proof of (3.5). The proof of (3.5) follows a similar argument employed in [1] except for minor modifications. For the reader's convenience and since we need to employ some parts of this proof, we shall present a sketch of the proof of this inequality and omit some details. For simplicity of the notation, we denote $a_{s,j}(\cdot)$ by $a(\cdot)$ and $G_{a_{s,j}}(f)$ by $G_a(f)$. In the following we assume that a is an H^1atom with $\mathrm{supp}(a) \subset \mathbf{S}^{n-1} \cap \mathbf{B}(x_0, \rho)$ for some $x_0 \in \mathbf{S}^{n-1}$ and $\rho > 0$. Since the weight function ω is radial, by using an appropriate rotation on $\mathbf{S}^{n-1}$, we may assume that $x_0 = (0, \ldots, 0, 1)$. Let $\{\Gamma_l\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the intervals $I_l = [\theta_{l+1}^{-1}, \theta_{l-1}^{-1}]$. More precisely, we require

$$\begin{aligned} \Gamma_l &\in C^{\infty},\ 0 \leq \Gamma_l \leq 1,\ \sum_l \Gamma_l(t) = 1, \\ \mathrm{supp}\ \Gamma_l &\subseteq I_l,\ \left| \frac{d^s \Gamma_l(t)}{dt^s} \right| \leq \frac{C}{t^s}. \end{aligned}$$

Define the multiplier operators S_l in $\mathbf{R}^n$ by

$$\widehat{(S_l f)}(\xi) = \Gamma_l(\left|B_{\rho}(\xi)\right|)\hat{f}(\xi).$$

Then for any $k \in \mathbf{Z}$ and $f \in \mathcal{S}(\mathbf{R}^n)$, we have

$$f(x) = \sum_{l \in \mathbf{Z}} S_{l+k} f(x).$$

Therefore, by Minkowski's inequality we have

$$\begin{aligned} G_a f(x) &= \left(\sum_{k \in \mathbf{Z}} \int_{2^k}^{2^{k+1}} \left| \sum_{l \in \mathbf{Z}} \int_{\mathbf{S}^{n-1}} a(y) \left(S_{l+k} f\right)(x - \Phi(t)y) d\sigma(y) \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq \sum_{l \in \mathbf{Z}} H_l f(x), \end{aligned} \tag{3.6}$$

where

$$H_l f(x) = \left(\sum_{k \in \mathbf{Z}} \int_{2^k}^{2^{k+1}} \left| \int_{\mathbf{S}^{n-1}} a(y) \left(S_{l+k} f\right)(x - \Phi(t)y) d\sigma(y) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

By applying Plancherel's theorem and Lemma 2.5, we get

$$\left\| H_l(f) \right\|_2 \leq C \lambda^{-\alpha|l|} \left\| f \right\|_2. \tag{3.7}$$

Again, by the same argument as in [1] we have

$$\|H_l(f)\|_{p,\omega} \le C\, \|f\|_{p,\omega} \ \text{ for } 2 \le p < \infty \text{ and } \omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+). \tag{3.8}$$

By interpolating between (3.7) and (3.8) with $\omega = 1$, we get

$$\|H_l(f)\|_p \le C_p \lambda^{-\vartheta|l|}\, \|f\|_p \tag{3.9}$$

for $2 \le p < \infty$ and for some $\vartheta > 0$.

Now, using Lemma 2.5, (3.8)–(3.9) and Stein-Weiss' interpolation theorem with change of measures [23], we claim that there exists $\mu = \mu(p) \in (0,1)$ such that

$$\|H_l(f)\|_{p,\omega} \le C_p \lambda^{-\tau|l|}\, \|f\|_{p,\omega} \ \text{ for } 2 \le p < \infty \text{ and } \omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+) \tag{3.10}$$

which in turn implies

$$\|G_a(f)\|_{p,\omega} \le C_p \sum_l \|H_l(f)\|_{p,\omega} \le C_p\, \|f\|_{p,\omega} \tag{3.11}$$

for $2 \le p < \infty$ and $\omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+)$.

To prove (3.10) we consider first the case $p > 2$. Choose a $p_1 > p$. By Lemma 2.5, there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon}$ so that $\omega^{1+\varepsilon} \subseteq \tilde{A}^I_{p/2}(\mathbf{R}_+) \subseteq \tilde{A}^I_{p_1/2}(\mathbf{R}_+)$. Thus, by (3.8), we have

$$\|H_l(f)\|_{p_1,\omega^{1+\varepsilon}} \le C\, \|f\|_{p_1,\omega^{1+\varepsilon}}\,. \tag{3.12}$$

Therefore, using interpolation with change of measures, we may interpolate between (3.8) and (3.12) to get (3.10) for $p > 2$. To handle the case $p = 2$, choose an $\varepsilon' > 0$ such that $\omega^{1+\varepsilon} \subseteq \tilde{A}^I_1(\mathbf{R}_+)$. By (3.8) we have

$$\|H_l(f)\|_{2,\omega^{1+\varepsilon'}} \le C\, \|f\|_{2,\omega^{1+\varepsilon'}}\,. \tag{3.13}$$

As above, interpolating between (3.8) and (3.13) yields (3.10) for the case $p = 2$. This finishes the proof Theorem 1.2 under condition (a). □

Proof of Theorem 1.2 for condition (b)

As above, Theorem 1.2 for condition (b) is proved if we can show that

$$\|g(f)\|_{L^p(|x|^\alpha)} \le C\, \|f\|_{L^p(|x|^\alpha)} \tag{3.14}$$

for $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np-2n-p)$ and $2n\delta/(2n+n\delta-2) < p < 2$. Let $\{\Gamma_l\}_{-\infty}^{\infty}$ be as above and

$$(\widehat{X_l f})(\xi) = \Gamma_l(|\xi|)\hat{f}(\xi) \text{ for } \xi \in \mathbf{R}^n.$$

By following a similar argument as above and a change of variable we obtain

$$gf(x) \le \sum_l R_l f(x), \tag{3.15}$$

where

$$R_l f(x) = \left(\sum_j \sum_{k\in\mathbf{Z}} \int_{\Phi(1)}^{\Phi(2)} \left| \int_{\mathbf{S}^{n-1}} \Omega_j(\xi)\,(X_{l+k} f)\,(x - t2^k\xi) d\sigma(\xi) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

As above, we notice that Theorem 1.2 for condition (b) is proved if we can show that

$$\|R_l(f)\|_{p,|x|^\alpha} \le C\lambda^{-\tau|l|}\,\|f\|_{p,|x|^\alpha} \tag{3.16}$$

holds for $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np-2n-p)$ and $2n\delta/(2n+n\delta-2) < p < 2$.

We first compute the L^2 norm of R_l. To this end, for any j, write Ω_j as

$$\Omega_j(x) = \|\Omega_j\|_{L^q(\mathbf{S}^{n-1})}\,\tilde{\Omega}_j(x),$$

where $\tilde{\Omega}_j(x) = \left(\Omega_j(x)/\,\|\Omega_j\|_{L^q(\mathbf{S}^{n-1})}\right)$. Now, we can view $\tilde{\Omega}_j$ as an H^1 atom supported in $\mathbf{S}^{n-1}$ with the L^∞ norm in the definition of an H^1 atom is replaced with the L^q norm. Thus we may assume that the support of $\tilde{\Omega}_j$ is contained in a ball with radius $\rho = 1$. Thus we have

$$R_l f(x) = \left(\sum_j \|\Omega_j\|^2_{L^q(\mathbf{S}^{n-1})} \sum_{k\in\mathbf{Z}} \int_{\Phi(1)}^{\Phi(2)} \left| \int_{\mathbf{S}^{n-1}} \tilde{\Omega}_j(\xi)\,(X_{l+k} f)\,(x - t2^k\xi) d\sigma(\xi) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

By Plancherel's formula, we have

$$\|R_l(f)\|_2^2 = \sum_j \|\Omega_j\|^2_{L^q(\mathbf{S}^{n-1})} \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \Gamma_l(2^{k+l}\,|\xi|) \left(\int_{\Phi(1)}^{\Phi(2)} \left| \int_{\mathbf{S}^{n-1}} \tilde{\Omega}_j(\xi) e^{-it2^k<x,\xi>} d\sigma(\xi) \right|^2 \frac{dt}{t} \right) dx$$

and hence by Lemma 2.6 we get

$$\|R_l(f)\|_2 \le C\lambda^{-\vartheta|l|}\,\|f\|_2 . \tag{3.17}$$

By (3.17) and the same argument as above, (3.10) is proved once we show that

$$\|R_l(f)\|_{p,|x|^\alpha} \le C\,\|f\|_{p,|x|^\alpha} \tag{3.18}$$

for $2n\delta/(2n+n\delta-2) < p < 2$ and $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np-2n-p)$. So, let us turn to the proof of (3.18). By duality there is a function $h = h_{k,l,j}(x,t)$ satisfying $\|h\| \le 1$ and

$$h_{k,l,j}(x,t)\ \in L^{p'}\left(l^2\left(l^2\left[L^2\left([\Phi(1),\Phi(2)]\,,\frac{dt}{t}\right),k\right],j\right),|x|^{-\alpha p'/p}\,dx\right)$$

such that

$$\begin{aligned}
\|R_l(f)\|_{p,|x|^\alpha} &= \int_{\mathbf{R}^n}\sum_j\sum_{k\in\mathbf{Z}}\int_{\Phi(1)}^{\Phi(2)}\int_{\mathbf{S}^{n-1}}\Omega_j(\xi)\left(X_{l+k}f\right)(x-2^kr\xi)h_{k,l,j}(x,t)d\sigma(\xi)\frac{dt}{t}dx\\
&= \int_{\mathbf{R}^n}\sum_j\sum_{k\in\mathbf{Z}}\int_{\Phi(1)}^{\Phi(2)}\int_{\mathbf{S}^{n-1}}\Omega_j(\xi)\left(X_{l+k}f\right)(x)h_{k,l,j}(x+2^kt\xi,t)d\sigma(\xi)\frac{dt}{t}dx\\
&\le \left\|\left(\sum_{k\in\mathbf{Z}}\left(\sum_j\int_{\Phi(1)}^{\Phi(2)}\int_{\mathbf{S}^{n-1}}\Omega_j(\xi)h_{k,l,j}(\cdot+2^kt\xi,t)d\sigma(\xi)\frac{dt}{t}\right)^2\right)^{1/2}\right\|_{p',|x|^{-\alpha p'/p}}\\
&\quad\times\left\|\left(\sum_{k\in\mathbf{Z}}|X_{l+k}f|^2\right)^{1/2}\right\|_{p,|x|^\alpha}.
\end{aligned}$$

Now set

$$Y(h) = \sum_{k\in\mathbf{Z}}\left(\sum_j\int_{\Phi(1)}^{\Phi(2)}\int_{\mathbf{S}^{n-1}}|\Omega_j(\xi)|\left|h_{k,l,j}(\cdot+2^kt\xi,t)\right|d\sigma(\xi)\frac{dt}{t}\right)^2 .$$

Since $|x|^\alpha \in A_p(\mathbf{R}^n)$ if and only if $-n < \alpha < n(p-1)$, by the weighted Littlewood-Paley theory we have

$$\|R_l(f)\|_{p,|x|^\alpha} \le C_p\,\|f\|_{p,|x|^\alpha}\left\|(Y(h))^{1/2}\right\|_{p',|x|^{-\alpha p'/p}} . \tag{3.19}$$

Since $p'>2$ and

$$\left\|(Y(h))^{1/2}\right\|_{p',|x|^{-\alpha p'/p}}=\|Y(h)\|^{1/2}_{p'/2,|x|^{-\alpha p'/p}},$$

there is a function $b\in L^{(p'/2)'}(\mathbf{R}^n,|x|^{2\alpha/(2-p)})$ such that $\|b\|_{(p'/2)',|x|^{2\alpha/(2-p)}}\leq 1$ and

$$\|Y(h)\|_{p'/2,|x|^{-\alpha p'/p}}=\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\left(\sum_j\int_{\Phi(1)}^{\Phi(2)}\int_{\mathbf{S}^{n-1}}|\Omega_j(\xi)|\left|h_{k,l,j}(x+2^kt\xi,t)\right|d\sigma(\xi)\frac{dt}{t}\right)^2 b(x)dx.$$

By Shwarz inequality and Lemma 2.9, we get

$$\begin{aligned}&\left(\sum_j\int_{\Phi(1)}^{\Phi(2)}\int_{\mathbf{S}^{n-1}}|\Omega_j(\xi)|\left|h_{k,l,j}(x+2^kt\xi,t)\right|d\sigma(\xi)\frac{dt}{t}\right)^2\\
&\leq C\int_{\Phi(1)}^{\Phi(2)}\left(\sum_j\int_{\mathbf{S}^{n-1}}|\Omega_j(\xi)|\left|h_{k,l,j}(x+2^kt\xi,t)\right|d\sigma(\xi)\right)^2\frac{dt}{t}\\
&\leq C\int_{\Phi(1)}^{\Phi(2)}\left(\sum_j\|\Omega_j\|^{\frac{1}{2}\min\{2,q\}}_{L^q(\mathbf{S}^{n-1})}\left(\int_{\mathbf{S}^{n-1}}|\Omega_j(\xi)|^{\max\{0,2-q\}}\left|h_{k,l,j}(x+2^kt\xi,t)\right|^2d\sigma(\xi)\right)^{1/2}\right)^2\frac{dt}{t}\\
&\leq C\left(\sum_j\|\Omega_j\|^{\min\{2,q\}}_{L^q(\mathbf{S}^{n-1})}\right)\int_{\Phi(1)}^{\Phi(2)}\sum_j\left(\int_{\mathbf{S}^{n-1}}|\Omega_j(\xi)|^{\max\{0,2-q\}}\left|h_{k,l,j}(x+2^kt\xi,t)\right|^2d\sigma(\xi)\right)\frac{dt}{t}.\end{aligned}$$

Therefore, by a change of variable, Fubini's theorem, Hölder's inequality, and invoking Lemma 2.12 we get

$$\begin{aligned}&\|Y(h)\|_{p'/2,|x|^{-\alpha p'/p}}\\
\leq\ & C\left(\sum_j\|\Omega_j\|^{\min\{2,q\}}_{L^q(\mathbf{S}^{n-1})}\|\Omega_j\|^{\max\{0,2-q\}}_{L^q(\mathbf{S}^{n-1})}\right)\int_{\mathbf{R}^n}\left(\sum_j\sum_{k\in\mathbf{Z}}\int_{\Phi(1)}^{\Phi(2)}|h_{k,l,j}(x,t)|^2\frac{dt}{t}\right)\times\\
&\left(\mathcal{M}_{Sph}\left(\left|\tilde{b}\right|^{\delta/2}\right)(-x)\right)^{2/\delta}dx,\ \text{where } \tilde{b}(x)=b(-x)\\
\leq\ & C\left(\sum_j\|\Omega_j\|^2_{L^q(\mathbf{S}^{n-1})}\right)\left\|\sum_{k\in\mathbf{Z}}\sum_j\int_{\Phi(1)}^{\Phi(2)}|h_{k,l,j}(\cdot,t)|^2\frac{dt}{t}\right\|_{p'/2,|x|^{-\alpha p'/p}}\times\\
&\left\|\left(\mathcal{M}_{Sph}(\left|\tilde{b}\right|^{\delta/2})\right)^{2/\delta}\right\|_{(p'/2)',|x|^{2\alpha/(2-p)}}.\end{aligned}$$

By the conditions on p and α we have $(2/\delta)(p'/2)' > n/(n-1)$ and $1-n < \frac{2\alpha}{2-p} < (n-1)\left((p'/2)'-1\right)-1$. Thus by the choice of b and invoking Lemma 2.10, we get

$$\|Y(h)\|_{p'/2,|x|^{-\alpha p'/p}} \le C$$

which when combined with (3.19) easily yields (3.18). This completes the proof of Theorem 1.2. □

Proof of Theorem 1.1

We notice that Theorem 1.1 (b) is a special case of Theorem 1.2 (b). So, we only need to prove Theorem 1.1 (a). By the arguments employed in the proof of Theorem 1.2, we notice that Theorem 1.1 for condition (a) is proved if we can show that the inequality

$$\left\|\tilde{R}_l(f)\right\|_{p,\omega} \le C\,\|f\|_{p,\omega} \tag{3.20}$$

holds for $\delta \le p < \infty$ and $\omega \in A_{p/\delta}$, where

$$\tilde{R}_l f(x) \left(\sum_j \sum_{k\in\mathbf{Z}} \int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{S}^{n-1}} \Omega_j(\xi)\left(X_{l+k}f\right)(x-t\xi)d\sigma(\xi)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Notice that $\tilde{R}_l f(x) = R_l f(x)$ if $\Phi(t) \equiv t$. We prove (3.20) by considering two separate cases: $q \ge 2$ and $1 < q < 2$.

Case $\delta \le p < \infty$, $\omega \in A_{p/\delta}$ and $q \ge 2$: In this case, $2 \le p < \infty$ and $\omega \in A_{p/2}$. First we consider the case $p > 2$. By duality, there is a function $g \in L^{(p/2)'}(\omega^{1-(p/2)'})$ with $\|g\|_{(p/2)',\omega^{1-(p/2)'}} \le 1$ such that

$$\left\|\tilde{R}_l(f)\right\|_{p,\omega}^2 = \sum_{k\in\mathbf{Z}} \sum_j \int_{\mathbf{R}^n} \int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{S}^{n-1}} \Omega_j(\xi)\left(X_{l+k}f\right)(x-t\xi)d\sigma(\xi)\right|^2 \frac{dt}{t}\,|g(x)|\,dx.$$

By Lemma 2.11, we have

$$\left|\int_{\mathbf{S}^{n-1}} \Omega_j(\xi)\left(X_{l+k}f\right)(x-t\xi)d\sigma(\xi)\right|^2 \le C\,\|\Omega_j\|_q^2 \int_{\mathbf{S}^{n-1}} \left|\left(X_{l+k}f\right)(x-t\xi)\right|^2 d\sigma(\xi). \tag{3.21}$$

Thus, by Fubini's theorem and a simple change of variable we get

$$\begin{aligned}
\left\|\tilde{R}_l(f)\right\|_{p,\omega}^2 &\leq C\sum_j \|\Omega_j\|_q^2 \int_{\mathbf{R}^n} \sum_j \sum_{k\in\mathbf{Z}} |X_{l+k}f(x)|^2 \int_{2^k}^{2^{k+1}} \int_{\mathbf{S}^{n-1}} |g(x+t\xi)|\, d\sigma(\xi)\frac{dt}{t}dx \\
&\leq C\sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} |X_{l+k}f(x)|^2 M^*(\tilde{g})(-x)dx \\
&\leq C\left\|\sum_{k\in\mathbf{Z}} |X_{l+k}f|^2\right\|_{p/2,\omega} \|M^*(\tilde{g})\|_{(p/2)',\omega^{1-(p/2)'}} . \qquad (3.22)
\end{aligned}$$

Since $\omega \in A_{p/2}$, by the elementary properties of $A_p(\mathbf{R}^n)$, we have $\omega^{1-(p/2)'} \in A_{(p/2)'}$. Therefore, by the weighted L^p $(1 < p < \infty)$ boundedness of the Hardy-Littlewood maximal operator M^* and the weighted Littlewood-Paley theory [16] we get

$$\left\|\tilde{R}_l(f)\right\|_{p,\omega} \leq C_p \|\Omega_j\|_q \|f\|_{p,\omega} \text{ for } 2 < p < \infty \text{ and } \omega \in A_{p/2}(\mathbf{R}^n). \qquad (3.23)$$

Now, if $p = 2$ and $\omega \in A_1(\mathbf{R}^n)$, by Lemma 2.11 and the definition of A_1 weight we have

$$\begin{aligned}
\left\|\tilde{R}_l(f)\right\|_{2,\omega}^2 &= \sum_j \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{S}^{n-1}} \Omega_j(\xi)\,(X_{l+k}f)\,(x-t\xi)d\sigma(\xi)\right|^2 \frac{dt}{t}\omega(x)dx \\
&\leq C\sum_j \|\Omega_j\|_q^2 \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} |X_{l+k}f(x)|^2 \left(\int_{2^k}^{2^{k+1}} \int_{\mathbf{S}^{n-1}} \omega(x+t\xi)d\sigma(\xi)\frac{dt}{t}\right)dx \\
&\leq C\sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} |X_{l+k}f(x)|^2 M^*(\tilde{\omega})(-x)dx \\
&\leq C\left\|\left(\sum_{k\in\mathbf{Z}} |X_{l+k}(f)|^2\right)^{1/2}\right\|_{2,\omega}^2 .
\end{aligned}$$

Thus, by the weighted Littlewood-Paley theory we get

$$\left\|\tilde{R}_l(f)\right\|_{2,\omega} \leq \|\Omega_j\|_q \|f\|_{2,\omega} \text{ for } \omega \in A_1(\mathbf{R}^n). \qquad (3.24)$$

Case $\delta \leq p < \infty$, $\omega \in A_{p/\delta}$ and $1 < q < 2$: It is clear that $q' \leq p < \infty$, $\omega \in A_{p/q'}$ and $p > 2$.

As above, by duality, there is a function $g \in L^{(p/2)'}(\omega^{1-(p/2)'})$ and satisfies $\|g\|_{(p/2)',\omega^{1-(p/2)'}} \leq 1$ such that

$$\left\|\tilde{R}_l(f)\right\|_{p,\omega}^2 = \sum_j \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{S}^{n-1}} \Omega_j(\xi) \left(X_{l+k} f\right)(x-t\xi) d\sigma(\xi)\right|^2 \frac{dt}{t} \left|g(x)\right| dx.$$

By Lemma 2.11, Fubini's theorem, the weighted Littlewood-Paley theory, and a change of variable we obtain

$$\begin{aligned}
&\left\|\tilde{R}_l(f)\right\|_{p,\omega}^2 \\
&\leq C \sum_j \|\Omega_j\|_q^q \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{2^k}^{2^{k+1}} \int_{\mathbf{S}^{n-1}} \left|\Omega_j(\xi)\right|^{2-q} \left|\left(X_{l+k} f\right)(x-t\xi)\right|^2 d\sigma(\xi) \frac{dt}{t} \left|g(x)\right| dx \\
&\leq C \sum_j \|\Omega_j\|_q^q \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \left|X_{l+k} f(x)\right|^2 M^*_{\Omega_j^{(2-q)}}(\tilde{g})(-x) dx \\
&\leq C \sum_j \|\Omega_j\|_q^q \left\|\left(\sum_{k\in\mathbf{Z}} \left|X_{l+k} f\right|^2\right)^{1/2}\right\|_{p,\omega}^2 \left\|M^*_{\Omega_j^{(2-q)}}(\tilde{g})\right\|_{(p/2)',\omega^{1-(p/2)'}} \qquad (3.25) \\
&\leq C \sum_j \|\Omega_j\|_q^q \|f\|_{p,\omega}^2 \left\|M^*_{\Omega_j^{(2-q)}}(\tilde{g})\right\|_{(p/2)',\omega^{1-(p/2)'}}. \qquad (3.26)
\end{aligned}$$

By applying Lemma 2.9, we now show that

$$\left\|M^*_{\Omega_j^{(2-q)}}(\tilde{g})\right\|_{(p/2)',\omega^{1-(p/2)'}} \leq C \|\Omega\|_q^{2-q} \|g\|_{(p/2)',\omega^{1-(p/2)'}} \qquad (3.27)$$

$q' \leq p < \infty$ and $\omega \in A_{p/q'}$. To see this, let $d = q/(2-q)$. Then we notice that $|\Omega_j|^{2-q} \in L^d(\mathbf{S}^{n-1})$, $d' = q'/2$, $\left(\omega^{1-(p/2)'}\right)^{1-(p/2)} = \omega \in A_{p/q'} = A_{(p/2)/d'}$ and $(p/2)' < d$. Therefore, d, $(p/2)'$ and $\omega^{1-(p/2)'}$ satisfy condition (b) in Lemma 2.9 and hence (3.27) holds (see also [8] and [1]). The proof of Theorem 1.1 is complete. □

4. Further Results

Our chief concern in this section is to present some applications of Corollaries 1 and 2 and obtain several results concerning the weighted L^p boundedness of singular integral operators $S_{\Omega,h}$ and the Marcinkiewicz integral operators $\mu_{\Omega,h}$ (see the definitions below). The weighted L^p boundedness of these operators have been studied extensively by a number of authors under various conditions on Ω and h (see, for example [8], [10], [15], [19], [20]). We shall extend or strengthen the existing results by establishing weighted L^p bounds of these operators which are either that these results were previously unavailable or were proved under strong conditions on Ω or h. Let us now describe our results. For simplicity, let us only deal with the case $\Phi(t) \equiv t$. The operators $S_{\Omega,h}$ and $\mu_{\Omega,h}$ are defined as follows:

$$S_{\Omega,h} f(x) = \lim_{\varepsilon \to 0} S^{(\varepsilon)} f(x) \tag{4.1}$$

and

$$\mu_{\Omega,h} f(x) = \left(\int_0^\infty \left| \int_{|y| \le t} f(x-y) \frac{\Omega(y')}{|y|^{n-1}} h(|y|) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}, \tag{4.2}$$

where

$$S^{(\varepsilon)} f(x) = \int_{|y|>\varepsilon} f(x-y) \frac{\Omega(y')}{|y|^n} h(|y|) dy,$$

$\Omega \in L^1(\mathbf{S}^{n-1})$ and satisfies the vanishing condition (1.1) and h is a measurable function on $(0,\infty)$.

For $\gamma > 1$, let $\Delta_\gamma(\mathbf{R}_+)$ denote the set of all measurable functions h on $\mathbf{R}_+$ such that

$$\sup_{R>0} \left(\frac{1}{R} \int_0^R |h(t)|^\gamma \, dt \right)^{1/\gamma} < \infty.$$

It is easy to verify that

$$L^\infty(\mathbf{R}^+) \subset \Delta_{\gamma_2}(\mathbf{R}^+) \subset \Delta_{\gamma_1}(\mathbf{R}^+) \text{ for } \gamma_1 < \gamma_2 \tag{4.3}$$

and

$$\mathcal{R}(\mathbf{R}_+) \subsetneq \Delta_\gamma(\mathbf{R}_+) \text{ for } 1 < \gamma \le 2. \tag{4.4}$$

We notice that the maximal operator $\mathcal{M}_\Omega$ is closely related to the singular integral operator $S_{\Omega,h}$. Let us now recall some known results. We start with some results on singular integrals.

Theorem C [10] *Suppose that $h \in L^\infty(\mathbf{R}^+)$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then $S_{\Omega,h}$ is bounded on $L^p(\omega)$ if $q' \le p < \infty$, $p \ne 1$ and $\omega \in A_{p/q'}$.*

For a special class of radial weights $\tilde{A}_p(\mathbf{R}_+)$, Duoandikoetxea used the method of rotations and proved the following sharper result:

Theorem D [10] *If $\omega \in \tilde{A}_p(\mathbf{R}_+)$ for $1 < p < \infty$, then $S_{\Omega,1}$ is bounded on $L^p(\omega)$ provided that $\Omega \in L\log L\left(\mathbf{S}^{n-1}\right)$.*

In 1999, Fan-Pan-Yang improved the result in Theorem D and obtained the following:

Theorem E [15] *If $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma \ge 2$ and $\Omega \in H^1\left(\mathbf{S}^{n-1}\right)$, then $S_{\Omega,h}$ is bounded on $L^p(\omega)$ for $\gamma' \le p < \infty$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$.*

Theorem F *Let $1 < q \le \infty$, $1 < p < \infty$ and $S_{\Omega,h}$ be the operator defined by (4.1) with $h \equiv 1$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ satisfying (1.2). Then $S_{\Omega,1}$ is bounded on $L^p(|x|^\alpha)$ if*

$$\max(-n, -1-(n-1)p/q') < \alpha < \min(n(p-1), p-1+(n-1)p/q'). \qquad (4.5)$$

Moreover, the range (4.5) is optimal.

This result was proved by Muckenhoupt and Wheeden in [20] and by Duoandikoetxea [10] with a different proof. We notice that in the limit case $q = 1$, the range in (4.5) becomes $\alpha \in (-1, p-1)$. It is well-known that the theorem fails for some $\Omega \in L^1(\mathbf{S}^{n-1})$, even in the unweighted case $\alpha = 0$. However Theorem F remains true if $\alpha \in (-1, p-1)$ and $\Omega \in L\log L$ as pointed out in ([10], p. 880). In the ensuing development of this result, an improvement was obtained by Fan-Pan-Yang in [15] as described in the following result:

Theorem G *If $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma \ge 2$ and $\Omega \in H^1\left(\mathbf{S}^{n-1}\right)$, then $S_{\Omega,h}$ is bounded on $L^p(|x|^\alpha)$ for $\gamma' < p < \infty$* and $\alpha \in (-1, p/\gamma' - 1)$.

In view of the above results, one is naturally led to the following question:

Question 1 *Under the same condition on Ω and under a similar (or the same) condition on ω in Theorems C-F, does the $L^p(\omega)$ boundedness of the operator $S_{\Omega,h}$ still hold if*

$h \in \Delta_\gamma(\mathbf{R}^+)$ for some $1 < \gamma < \infty$ in Theorems C and F and for some $1 < \gamma < 2$ in Theorems E and G?

By applying Corollaries 1 and 2, we are able to get some progress in answering the above question as described in the following results:

Theorem 4.1 *Suppose that $h \in \mathcal{R}(\mathbf{R}_+)$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then $S_{\Omega,h}$ is bounded on $L^p(\omega)$ if p and ω satisfy the same conditions as in Theorem 1.1.*

Theorem 4.2 *If $h \in \mathcal{R}(\mathbf{R}_+)$ and $\Omega \in H^1\left(\mathbf{S}^{n-1}\right)$, then $S_{\Omega,h}$ is bounded on $L^p(\omega)$ for $2 \le p < \infty$ and $\omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+)$.*

Theorem 4.3 Let $1 < p < \infty$. *Suppose that $h \in \mathcal{R}(\mathbf{R}_+)$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then $S_{\Omega,h}$ is bounded on $L^p(|x|^\alpha)$ if*

$$\max(-n, -1-(n-1)p/q') < \alpha < \min(n(p-1), p-1+(n-1)p/q'). \tag{4.6}$$

Theorem 4.4 *If $h \in \mathcal{R}(\mathbf{R}_+)$ and $\Omega \in H^1\left(\mathbf{S}^{n-1}\right)$, then $S_{\Omega,h}$ is bounded on $L^p(|x|^\alpha)$ for $2 < p < \infty$ and $\alpha \in (-1, p/2-1)$.*

Proof of Theorems 4.1–4.4

There is no loss of generality to assume that $h \in \mathcal{R}(\mathbf{R}_+)$ with $\|h\|_{L^2(\mathbf{R}_+,dr/r)} = 1$. Then we notice that

$$S^{(\varepsilon)}f(x) = \int_{\mathbf{R}^n} f(x-y)\frac{\Omega(y')}{|y|^n}\tilde{h}(|y|)dy,$$

where $\tilde{h}(|y|) = h(|y|)\chi_\varepsilon(|y|)$ is the characteristic function on the set $\{y \in \mathbf{R}^n : |y| > \varepsilon\}$. Since

$$\left\|\tilde{h}\right\|_{L^2(\mathbf{R}_+,dr/r)} \le \|h\|_{L^2(\mathbf{R}_+,dr/r)} = 1,$$

by Corollaries 1 and 2 we get

$$\left\|S^{(\varepsilon)}f\right\|_{p,\omega} \le \left\|\sup_h |S_{\Omega,h}f|\right\|_{p,\omega} = \|\mathcal{M}_\Omega f\|_{p,\omega} \le C_p\|f\|_{p,\omega}$$

with a C_p independent of ε. Passing to the limit as $\varepsilon \to 0$, Fatou's lemma gives

$$\|S_{\Omega,h} f\|_{p,\omega} \leq C_p \|f\|_{p,\omega}$$

which completes the proofs of Theorems 4.1–4.4.

Now, let us see how we can apply Corollaries 1 and 2 to improve some results on the Marcinkiewicz integral operator $\mu_{\Omega,h}$. Let us first recall some known results. We start with the following theorem due to Ding-Fan-Pan.

Theorem H [8] *Suppose that $h \in L^\infty(\mathbf{R}^+)$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then $\mu_{\Omega,h}$ is bounded on $L^p(\omega)$ if $q' < p < \infty$, and $\omega \in A_{p/q'}$.*

Theorem I [19] *If $h \in L^\infty(\mathbf{R}^+)$ and $\Omega \in H^1(\mathbf{S}^{n-1})$, then $\mu_{\Omega,h}$ is bounded on $L^p(\omega)$ for $1 < p < \infty$ and $\omega \in \tilde{A}_p^I(\mathbf{R}_+)$.*

Theorem J [12] *Let $1 < q \leq \infty$, $1 < p < \infty$ and $\mu_{\Omega,h}$ be the operator defined by (4.2) with $h \equiv 1$ and $\Omega \in L^q(\mathbf{S}^{n-1})$. Then $\mu_{\Omega,1}$ is bounded on $L^p(|x|^\alpha)$ if*

$$\max(-n, -1-(n-1)p/q') < \alpha < \min(n(p-1), p-1+(n-1)p/q'). \qquad (4.7)$$

In view of the above results, one is naturally led to the following question:

Question 2 *Under the same condition on Ω and under a similar (or the same) condition on ω in Theorems H-J, does the $L^p(\omega)$ boundedness of the operator $\mu_{\Omega,h}$ still hold if $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $1 < \gamma < \infty$ in Theorems H-J?*

By applying Corollaries 1 and 2, we are able to obtain some progress in answering Question 2 as described in the following results:

Theorem 4.5 *Suppose that $h \in \mathcal{R}(\mathbf{R}_+)$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then $\mu_{\Omega,h}$ is bounded on $L^p(\omega)$ if p and ω satisfy the same conditions as in Theorem 1.1.*

Theorem 4.6 *If $h \in \mathcal{R}(\mathbf{R}_+)$ and $\Omega \in H^1(\mathbf{S}^{n-1})$, then $\mu_{\Omega,h}$ is bounded on $L^p(\omega)$ for $2 \leq p < \infty$ and $\omega \in \tilde{A}_{p/2}^I(\mathbf{R}_+)$.*

Theorem 4.7 *If $h \in \mathcal{R}(\mathbf{R}_+)$ and $\Omega \in H^1(\mathbf{S}^{n-1})$, then $\mu_{\Omega,h}$ is bounded on $L^p(|x|^\alpha)$ for* $2 < p < \infty$ and $\alpha \in (-1, p/2-1)$.

Proof of Theorems 4.5-4.7 By the argument in [2], we get

$$\mu_{\Omega,h} f(x) \leq \frac{1}{\sqrt{2}} \mathcal{M}_\Omega f(x).$$

Thus all the weighted results which hold to $\mathcal{M}_\Omega$ also hold for $\mu_{\Omega,h}$. □

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H. M. AL-QASSEM
Department of Mathematics
Yarmouk University
Irbid-JORDAN
e-mail: husseink@yu.edu.jo

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