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Pullbacks of Crossed Modules and Cat^1- Commutative Algebras

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Abstract

In this paper we first review the definitions of crossed module [10], pullback crossed module and cat^1-object in the category of commutative algebras. We then describe a certain pullback of cat^1- commutative algebras.

Key Words: Crossed modules, Cat^1-Algebra, Pullback, Commutative Algebra.

1. Introduction

The terms of crossed modules over groups and algebras, Cat^1-groups and algebras are very useful in Category theory. Interest in these subjects has been heightened by their exploration via computer. A good example is the program GAP [8] (Groups, Algorithm and Programming)* which is used to calculate crossed modules and cat^1-groups over groups. The applications of crossed modules and cat^1-groups were introduced by Alp and Wensley [3] as a GAP share package known as XMod†. Crossed modules were introduced by J. H. C. Whitehead in [10]. Loday defined cat^1-groups and showed that the category of crossed modules is equivalent to the category of cat^1-groups in [7]. Later, Brown and Wensley defined Pullback crossed module over groups in [5]. Using the equivalence of these two categories, Pullback cat^1-group was defined by Alp [1]. Crossed modules and Pullback crossed module over algebra were presented in [9]. Pullback cat^1-commutative algebra is presented in this paper.

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*www-gap.dcs.st-andrews.ac.uk/˜ gap/

†www-groups.dcs.st-and.ac.uk/˜ gap/Packages/xmod.html

It is hoped this paper will give good motivation for future studies into crossed square and induced crossed module of commutative algebras. The crossed module of commutative algebra and its pullback will constitute a square which will be a crossed square. The defining action of commutative algebras will play very important role in the crossed square case.

2. Crossed modules and Cat^1-commutative Algebra

Fix a commutative ring A (with unit). Recall that a commutative algebra over A is an A-module M with a bilinear map $M \times M \to M,\ (m, m') \mapsto mm'$ satisfying

$$\begin{aligned} mm' &= m'm \\ (mm')m'' &= m(m'm'') \end{aligned}$$

for all $m, m', m'' \in M$. We shall assume all commutative algebras to be over A [6].

Let M and N be commutative algebras. A map $M \times N \to N,\quad (m, n) \mapsto {}^{m}n$ is a commutative action if and only if

$$\begin{array}{ll} \text{COMACT1:} & k({}^{m}n) = {}^{(km)}n = {}^{m}(kn) \\ \text{COMACT2:} & {}^{m}(n + n') = {}^{m}n + {}^{m}n' \\ \text{COMACT3:} & {}^{(m+m')}n = {}^{m}n + {}^{m'}n \\ \text{COMACT4:} & {}^{m}(nn') = ({}^{m}n)n' = n({}^{m}n') \\ \text{COMACT5:} & {}^{(mm')}n = m({}^{m'}n) \end{array}$$

for all $k \in \mathbf{k}, m, m' \in M, n, n' \in N$.

Let M be a $\mathbf{k}$-algebra with identity. A crossed module of commutative algebras is an M-algebra N, together with a commutative action of M on N and an M-algebra morphism $\partial : N \to M$ such that for all $n \in N, m \in M$

$$\begin{array}{lrcl} \text{COMCM1:} & \partial({}^{m}n) & = & m(\partial n) \\ \text{COMCM2:} & {}^{(\partial n)}n' & = & nn'. \end{array}$$

The standard examples of crossed modules are [2] and [9]:

1. Let I be any ideal of a $\mathbf{k}$ algebra M. Consider an inclusion map $\iota : I \to M$ is a crossed module.

2. Let R be an M-module. It can be considered as an M-algebra with zero multiplication, and then $0 : R \to M$ is a crossed M-module.

3. Assume given a simplicial algebra E and a simplicial ideal I. The inclusion $\iota : I \to E$ induces a map $\partial : \pi_0(I) \to \pi_0(E)$ and E acting on I by multiplication an action of $\pi_0(E)$ on $\pi_0(I)$, so ∂ is a crossed module.

4. Any ideal I in P gives an inclusion map, inc : I → P, which is a crossed module. Conversely given an arbitrary crossed P-module $\partial : M \to P$, one easily sees that the Peiffer identity implies that ∂P is an ideal in R.

5. Given any morphism $\theta : L \to C$ of P-modules we can form the semidirect product $P \ltimes C$ with its usual multiplication

$$(p, c)(p', c') = (pp', pc' + p'c),$$

where $cc' = 0$ by zero multiplication. Giving L the zero multiplication and a $P \ltimes C$ module structure via the projection from $P \ltimes C$ onto P, one obtains a crossed $(P \ltimes C)$-module

$$\hat{\theta} : L \to P \ltimes C,\ \hat{\theta}(l) = (0, \theta(l)).$$

A morphism between two crossed modules from $(\partial : N \to M)$ and $(\partial' : N' \to M')$ is a pair $\langle \theta, \phi \rangle$ of **k**-algebra morphisms such that $\theta({}^{m}n) = {}^{\phi(m)}\theta(n)$ and $\partial'\theta(n) = \phi\partial(n)$.

Given a crossed M-module $\partial : N \to M$ we form the $k-$algebra $R = M \ltimes N$, again the semidirect product algebra with multiplication

$$(m, n)(m', n') = (mm', mn' + m'n + nn').$$

There are two morphism $t, s : R \to M$ given by $t(m, r) = m$ and $s(m, r) = m + \partial r$. There is also the obvious morphism [9] $e : M \to R,\ e(m) = (m, 0)$. These morphisms satisfy the axiom of cat^1-algebra:

COMCAT1: $tes = s$ and $set = t$;
COMCAT2: $\ker t \ker s = 0$.

3. Pullback Crossed Module of Commutative Algebras

Pullback crossed module of commutative algebra was presented in [9]. In that study the verification of crossed modules axioms were not proven. We will re-organize pullback crossed modules presentation here. Let N, M be commutative algebras. Let $\mathcal{X} = (\partial : N \to M)$ be a crossed module of commutative algebras and $\iota : Q \to M$ be a homomorphism. Then

$$\begin{array}{ccc} \iota^{**}N & \longrightarrow & N \\ {\scriptstyle \partial^{**}}\downarrow & & \downarrow{\scriptstyle \partial} \\ Q & \xrightarrow[\iota]{} & M \end{array}$$

$\iota^{**}\mathcal{X} = (\partial^{**} : \iota^{**}N \to Q)$, is the pullback crossed module of commutative algebras by ι, where $\iota^{**}N = \{(q, n) \in Q \times N \mid \iota q = \partial n,\ q \in Q, n \in N\}$ and $\partial^{**}(q, n) = q$. The action of Q on $\iota^{**}N$ is given by ${}^{q}(q_1, n) = (qq_1, {}^{\iota q}n)$.

Proposition 3.1 $\iota^{**}N$ is a commutative algebra in which scalar multiplication $k(q, n) = (kq, kn)$, addition is $(q_1, n_1)+(q_2, n_2) = (q_1+q_2, n_1+n_2)$ and multiplication is $(q_1, n_1)(q_2, n_2) = (q_1q_2, n_1n_2)$.

Proposition 3.2 The map is a commutative algebra action of Q on $\iota^{**}N$.

Proof. To complete proof we must show that the conditions of commutative algebra action are satisfied.

COMACT1:

$$\begin{aligned} k({}^{q}(q_1, n_1) &= k(qq_1, {}^{\iota q}n_1) \\ &= ((kq)q_1, k({}^{\iota q}n_1)) \\ &= ({}^{kq}q_1, {}^{k\iota q}n_1) \\ &= ({}^{qk}q_1, {}^{\iota qk}n_1) \\ &= {}^{q}(kq_1, kn_1). \end{aligned}$$

COMACT2:

$$\begin{aligned} {}^{q}((q_1, n_1) + (q_2, n_2)) &= {}^{q}(q_1 + q_2, n_1 + n_2) \\ &= (q(q_1 + q_2), {}^{\iota q}(n_1 + n_2)) \\ &= (qq_1 + qq_2, {}^{\iota q}n_1 + {}^{\iota q}n_2) \\ &= (qq_1, {}^{\iota q}n_1) + (qq_2, {}^{\iota q}n_2) \\ &= {}^{q}(q_1, n_1) + {}^{q}(q_2, n_2). \end{aligned}$$

COMACT3:

$$\begin{aligned} {}^{(q_1+q_2)}(q,n) &= ({}^{q_1+q_2}q, {}^{\iota(q_1+q_2)}n) \\ &= (q_1q+q_2q, {}^{\iota q_1+\iota q_2}n) \\ &= (q_1q, {}^{\iota q_1}n)+(q_2q, {}^{\iota q_2}n) \\ &= {}^{q_1}(q,n)+{}^{q_2}(q,n). \end{aligned}$$

COMACT4:

$$\begin{aligned} {}^{q}(q_1,n_1)(q_2,n_2) &= {}^{q}(q_1q_2,n_1n_2) \\ &= (qq_1q_2, {}^{\iota q}(n_1n_2)) \\ &= (qq_1q_2, nn_1n_2) \\ &= ({}^{q}(q_1,n_1))(q_2,n_2). \end{aligned}$$

And since multiplication is commutative $qq_1 = q_1q$, then

$$\begin{aligned} (qq_1q_2, nn_1n_2) &= (q_1qq_2, n_1nn_2) \\ &= (q_1,n_1){}^{q}(q_2,n_2). \end{aligned}$$

Finally,

COMACT5:

$$\begin{aligned} {}^{q_1q_2}(q,n) &= (q_1q_2q, {}^{\iota q_1q_2}n_2) \\ &= {}^{q_1}(q_2q, {}^{\iota q_2}q) \\ &= {}^{q_1}({}^{q_2}(q,n)). \end{aligned}$$

□

Theorem 3.3 The homomorphism $\partial^{**} : \iota^{**}N \to Q$ has the structure of a crossed module.

Proof. Boundary homomorphism $\partial^{**}(q,n) = q$ and commutative algebra action of Q on $\iota^{**}N$, ${}^{q}(q_1,n_1) = (qq_1, {}^{\iota q}n_1)$ satisfy the COMCM1 and COMCM2 conditions:

COMCM1:

$$\begin{aligned} \partial^{**}({}^{q}(q_1,n_1)) &= \partial^{**}(qq_1, {}^{\iota q}n_1) \\ &= qq_1 \\ &= q\partial^{**}(q_1,n_1) \end{aligned}$$

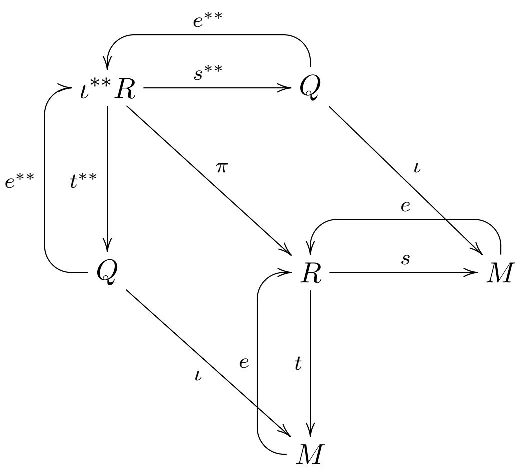
COMCM2:

$$\begin{aligned} {}^{\partial^{**}(q,n)}(q_1,n_1) &= {}^{q}(q_1,n_1) \\ &= (qq_1, {}^{\iota q}n_1) \\ &= (qq_1, nn_1) \text{ since } {}^{\iota q}n_1 = {}^{\partial n}n_1 \\ &= (q,n)(q_1,n_1) \end{aligned}$$

Thus the axioms of crossed module are satisfied. □

4. Pullback Cat^1-Commutative Algebra

A Pullback Cat^1-commutative Algebra is defined as



Let $\mathcal{CL} = (e; t, s : R \to M)$ be a cat^1-commutative algebra and let $\iota : Q \to M$ be a homomorphism. Define $\iota^{**}\mathcal{R} = (e^{**}; t^{**}, s^{**} : \iota^{**}R \to Q)$ to be the pullback of R, where

$$\iota^{**}R = \{(q_1, r, q_2) \in Q \times R \times Q \mid \iota q_1 = tr, \iota q_2 = sr\},$$

$t^{**}(q_1, r, q_2) = q_1$, $s^{**}(q_1, r, q_2) = q_2$ and $e^{**}(q) = (q, e\iota q, q)$. Multiplication in $\iota^{**}R$ is componentwise. Let's show that COMCAT1 and COMCAT2 are satisfied:

COMCAT1:

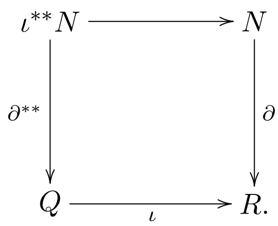
$$
\begin{aligned}
t^{**}e^{**}s^{**}(q_1,r,q_2) &= t^{**}e^{**}(q_2) = t^{**}(q_2, e\iota q_2, q_2)\\
&= q_2\\
&= s^{**}(q_1,r,q_2);
\end{aligned}
$$

$$
\begin{aligned}
s^{**}e^{**}t^{**}(q_1,r,q_2) &= s^{**}e^{**}(q_1) = s^{**}(q_1, e\iota q_1, q_1)\\
&= q_1\\
&= t^{**}(q_1,r,q_2).
\end{aligned}
$$

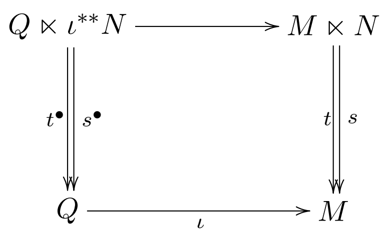
To prove COMCAT2, suppose $a = (q_1', r_1, q_1) \in \ker t^{**}$, $b = (q_2, r_2, q_2') \in \ker s^{**}$. Then $q_1' = q_2' = 0$; so, by the definition of ι^{**}, we have $r_1 \in \ker t$, $r_2 \in \ker s$. Then $[a,b] = (0, [r_1, r_2], 0) = (0, 0_G, 0)$ and $[ab] = 0$ so that COMCAT2 is satisfied. It is easily verified that t^{**} and s^{**} are homomorphisms.

Proposition 4.1 If $\iota^{**}\mathcal{X}$ is the pullback of the crossed module $\mathcal{X}$ over $\iota : Q \to M$ and if $\mathcal{R}, \mathcal{D}$ are the cat^1-commutative algebras obtained from $\mathcal{X}, \iota^{**}\mathcal{X}$, respectively, then $\mathcal{D} \cong \iota^{**}\mathcal{R}$.

Proof.



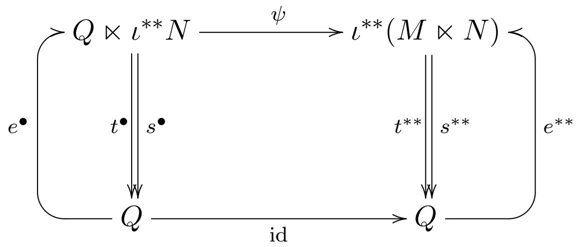
Starting with the pullback crossed module $\iota^{**}\mathcal{X} = (\partial^\bullet : \iota^{**}N \to Q)$, the source algebra of $\mathcal{D}$ is defined as the semi-direct product $Q \ltimes \iota^{**}N$.



The target, source and embedding of $\mathcal{D}$ are respectively given by

$$\begin{aligned} t^{\bullet}(q',(q,n)) &= q' \\ s^{\bullet}(q',(q,n)) &= q'\partial^{**}(q,n) \\ &= q'q \\ e^{\bullet}(q) &= (q,(1_Q,1_N)). \end{aligned}$$

We then define an isomorphism of cat^1-commutative algebra $(\psi,\mathrm{id}_Q):\mathcal{D}\to\iota^{**}\mathcal{C}$ as



where

$$\psi(q',(q,n)) = (q',(\iota q',n),q'q).$$

First note that $\psi(q',(q,n)) \in \iota^{**}(M \ltimes N)$ because

$$t(\iota q',n) = \iota q'$$

and

$$s(\iota q',n) = (\iota q')(\partial n) = (\iota q')(\iota q) = \iota(q'q).$$

We verify that ψ is a homomorphism

$$\begin{aligned} \psi((q_1',(q_1,n_1))(q_2',(q_2,n_2))) &= \psi(q_1'q_2',(q_1^{q_2'}q_2,n_1^{\iota q_2'}n_2)) \\ &= (q_1'q_2',(\iota(q_1'q_2'),n_1^{\iota q_2'}n_2),q_1'q_1q_2'q_2) \\ \psi(q_1',(q_1,n_1))\psi(q_2',(q_2,n_2)) &= (q_1',(\iota q_1',n_1),q_1'q_1)(q_2',(\iota q_2',n_2),q_2'q_2) \\ &= (q_1'q_2',(\iota q_1',n_1)(\iota q_2',n_2),q_1'q_1q_2'q_2) \\ &= (q_1'q_2',((\iota q_1')(\iota q_2'),n_1^{\iota q_2'}n_2),q_1'q_1q_2'q_2). \end{aligned}$$

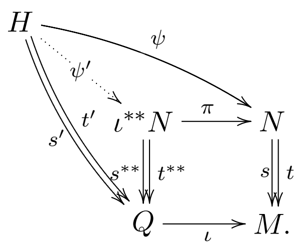
The inverse of ψ is given by $\psi^{-1}(q_1,(m,n),q_2)=(q_1,(q_1^{-1}q_2,n))$.

Then

$$\begin{aligned}
t^{**}\psi(q',(q,n)) &= t^{**}(q',(\iota q',n),q'q)\\
&= q'\\
&= t^{\bullet}(q',(q,n)),\\
s^{**}\psi(q',(q,n)) &= s^{**}(q',(\iota q',n),q'q)\\
&= q'q\\
&= s^{\bullet}(q',(q,n)),\\
\psi e^{\bullet}(q) &= \psi(q,(1_Q,1_N))\\
&= (q,(\iota q,1_n),q)\\
&= e^{**}(q),
\end{aligned}$$

so the diagram commutes and the proof is complete. □

The universal property of induced cat^1-commutative algebra is the following. Let $\mathcal{C}=(e;t,s:R\to M)$ be a cat^1-commutative algebra and let $\iota^{**}\mathcal{C}=(e^{**};t^{**},s^{**}:\iota^{**}N\to Q)$ be induced by the homomorphism $\iota:Q\to M$ as given by the diagram



The pair (π,ι) is a morphism of cat^1-commutative algebra such that, for any cat^1-commutative algebra $\mathcal{H}=(e';t',s':H\to Q)$ and any morphism of cat^1-commutative algebra $(\psi,\iota):\mathcal{C}\to\mathcal{H}$, there is a unique morphism $((\psi',1):\iota^{**}\mathcal{C}\to\mathcal{H}))$ of cat^1-commutative algebra such that $\pi\psi'=\psi$.

References

[1] Alp, M., Left adjoint of pullback cat^1-groups, *Turkish journal of Mathematics,* Vol. 23 No. 2 243-249 (1999).

[2] Arvasi, Z., Applications in Commutative Algebra of the Moore Complex of a simplicial algebra, *Ph.D thesis, Univ. of Wales, Bangor,* (1994).

[3] Alp, M., and Wensley, C. D., XMOD, Crossed modules and cat1-groups in GAP, version 1.3 *Manual for the XMOD share package,* 1-80, (1997).

[4] Alp, M., and Wensley, C. D., Enumeration of cat^1-group of low order, *International Journal of Algebra and Computation,* 10, 4, 407- 424 (2000).

[5] Brown, R. and Wensley, C.D., On finite induced crossed modules, and the homotopy 2-type of mapping cones, *Theory and Applications of Categories,* 1, 3, 54-71, (1995).

[6] Ellis, G. J., Crossed modules and their higher dimensional analogues, *Ph.D thesis, Univ. of Wales, Bangor,* (1984).

[7] Loday, J. L., Spaces with finitely many non-trivial homotopy groups, *J.App.Algebra,* 24, 179-202, (1982).

[8] Schönert, M. et al, GAP: Groups, Algorithms, and Programming, Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, third edition, (1993).

[9] Porter, T., Some Categorical Results in the Theory of Crossed Modules in Commutative Algebra, *Journal of Algebra,* 109, 2, 415-429, (1987).

[10] Whitehead, J. H. C., On adding relations to homotopy groups, *Ann. Math.,* 47, 806-810, (1946).

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