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Spectral Problems for Operator Pencils in Non-Separated Root Zones*

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Abstract

Variational principles for real eigenvalues of self-adjoint operator pencils in non-separated root zones are studied.

Key Words: Operator pencils, eigenvalues, variational principles.

1. Introduction

Let $L(\lambda)$ be a function defined on an interval $[a, b] \subset R$, whose values are operators in a Hilbert or Banach space. Such functions are called operator functions or operator pencils. Linear pencils of the form $L(\lambda) = A - \lambda B$ and polynomial pencils of the form $L(\lambda) = \lambda^n A_n + \lambda^{n-1} A_{n-1} + ... + \lambda A_1 + A_0$, where A, B and A_i, $i = 0, 1, ..., n$ are operators, form important subclasses of operator pencils. In general, an operator pencil $L(\lambda)$ may be analytic, smooth or nonsmooth. Polynomial pencils arise mainly from the evolution of equations in abstract spaces (see [10]) but nonpolynomial pencils arise from equations depending on a parameter.

The main concern of this paper is the variational theory of the spectrum for a class of self-adjoint operator pencils. The spectrum of an operator pencil $L(\lambda)$ is defined in the following way:

We say that $\lambda \in \sigma(L)$ if and only if $0 \in \sigma(L(\lambda))$, where $\sigma(L)$ denotes the spectrum of the operator pencil $L(\lambda)$ and $\sigma(L(\lambda))$ denotes the spectrum of the operator $L(\lambda)$ which

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is the value of the operator pencil $L(\lambda)$ at the point λ. The set of eigenvalues $\sigma_e(L)$, the continuous spectrum $\sigma_c(L)$ and other spectral sets are defined analogously. In particular, $\lambda \in [a,b]$ is called an eigenvalue of the pencil $L(\lambda)$ if there exists a vector $x \neq 0$, called an eigenvector such that $L(\lambda)x = 0$. Evidently, if $L(\lambda) = A - \lambda I$ then $\sigma(L) = \sigma(A)$.

It is well known that discrete eigenvalues of a self-adjoint operator A in a Hilbert space H, which lie below or above the essential spectrum of A, can be characterized by three fundamental variational principles, namely by Rayleigh's principle, by the Poincaré-Ritz minimax principle and by the Courant-Fischer-Weyl principle applied to the Rayleigh quotients $p(x) = \frac{(Ax,x)}{(x,x)}$, $x \in H$, $x \neq 0$.

Arrange the eigenvalues below the minimum of the essential spectrum of A as $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ....$ counted according to their multiplicities. In this case the Rayleigh's principle is

$$\lambda_n = \min_{\substack{x \neq 0, (x,x_i)=0 \\ i=1,\dots,n-1}} p(x),$$

where x_i are eigenvectors corresponding to the eigenvalues λ_i and the minimum is attained at the eigenvector x_n. The Poincaré-Ritz principle is defined by

$$\lambda_n = \min_{\substack{L \subset D(A) \\ \dim L = n}} \max_{\substack{x \in L \\ x \neq 0}} p(x).$$

The third variational principle, known as the Courant-Fischer-Weyl principle, can be given in the form

$$\lambda_n = \max_{\substack{L \subset H \\ \dim L = n-1}} \min_{\substack{x \in D(A) \\ x \perp L}} p(x).$$

For λ-nonlinear eigenvalue problems of the form $L(\lambda)x = 0$, the Rayleigh quotient $p(x) = \frac{(Ax,x)}{(x,x)}$, $x \in H$, $x \neq 0$ of a linear problem $Ax = \lambda x$ is replaced by the so called Rayleigh functional p, which is homogeneous, nonlinear functional defined by the equation $(L(p(x))x, x) = 0$, for $x \neq 0$.

There are two cases in the variational theory of λ-nonlinear spectral problems:

A) The Rayleigh functional $p(x)$ is defined on the entire space $H \setminus \{0\}$. In this case the λ-nonlinear eigenvalue problem $L(\lambda)u = 0$ is called overdamped. Methods applied in this case depend on whether the operator function $L(\lambda)$, $\lambda \in \Delta = [a,b]$ is polynomial, analytic, and smooth or nonsmooth. On this subject the classical works of R. J. Duffin, R. Turner, E. H. Rogers, B. Werner (see the books [1], and [10]) should be mentioned.

B) The Rayleigh functional $p(x)$ is defined only on a conic subset of $H \setminus \{0\}$ and the conditions given in the above mentioned works are partially fulfilled. Such spectral problems are said to be nonoverdamped.

Notice that investigations in recent years have been concentrated on Case B). The operator pencils of waveguide type form one of the main classes of nonoverdamped operator pencils and have very important applications to physical problems (see [2], [8], [13]). Some classes of nonoverdamped operator pencils were studied in [3], [11] and [12].

A different class of nonoverdamped spectral problems is connected with spectral problems of block operator matrices and λ-rational Sturm-Liouville problems ([4], [5] and [6]). Here the classical overdamped conditions are not satisfied. These conditions are (see also [7]):

a) $L(a)$ is uniformly positive definite, i. e., $L(a) \gg 0$;

b) $L(b) \ll 0$; and

c) the equation $(L(\lambda)x, x) = 0$ has exactly one simple zero $p(x)$ in $[a, b]$ for every $x \in H \setminus \{0\}$.

It follows from condition a) that $\kappa_\alpha := \dim\{E \mid 0 \neq x \in E, \, (L(\alpha)x, x) < 0\} = 0$. The number κ_α defines the index shift and this equality means that for overdamped spectral problems the index shift does not occur. It was shown in [4] (see Theorem 3.5) that if $\kappa_\alpha > 0$ then the classical variational principle is replaced by

$$\lambda_n = \min_{\substack{L \\ \dim L = n + \kappa_\alpha}} \sup_{\substack{x \in L \\ x \neq 0}} p(x), \quad n = 1, 2, \ldots$$

These problems for unbounded operator pencils were studied in [6] (see Theorem 2.1, p. 293). As mentioned above for λ-nonlinear spectral problems the Rayleigh quotients $p(x) = \frac{(Ax,x)}{(x,x)}$, $x \in H$, $x \neq 0$ in the operator theory is replaced by nonlinear functionals $p(x)$ defined by the roots of the equation $\varphi_x(\lambda) := (L(p(x))x, x) = 0, \; x \neq 0$. Moreover, distribution of roots of the equation $\varphi_x(\lambda) = 0$ plays an important role in the variational theory of the spectrum of operator pencils. We now recall some notions.

Definition 1.1 *Let G be a cone in Hilbert space and the functional $p(x)$ is defined on $G \setminus \{0\}$. Then the set $W_p := \{p(x) \Big| x \in G \setminus \{0\}\}$ is called a root zone of pencil L or the numerical range of functional $p(x)$. In addition, we say that $p(x)$ is a root of the first or second kind if the function $\varphi_x(\lambda)$ increases or decreases through $p(x)$, respectively. The other roots are said to be neutral.*

Definition 1.2 *A root zone W_p is said to be separated if it consists of roots of the same kind for $x \in G\backslash\{0\}$ and the set $\overline{W_p}$ does not contain other roots of the equations $\varphi_x(\lambda) = 0$ for $x \notin G$.*

It is known that only the part of the spectrum of the pencil L in W_p can be characterized by the functional $p(x)$. Consequently, to investigate the whole spectrum we have to construct several functionals whose numerical ranges include a part of the spectrum. Note that in all of the cited papers spectral problems in a separated root zone are studied.

In this paper we study eigenvalue problems for a class of nonoverdamped pencils in a non-separated root zone. We now define this class.

Let $L : R \to S(H)$ be a continuously differentiable operator-valued function, where $S(H)$ is the set of bounded self-adjoint operators in a Hilbert space H. We suppose:

I) The Hilbert space H has decomposition into disjoint cones of the form $H = H_+ \bigsqcup H_0 \bigsqcup H_\emptyset$ (let $0 \in H_+$) such that, for all $0 \neq x \in H_+$ the function $\varphi_x(\lambda) = (L(\lambda)x, x)$ has a simple zero $p_+(x)$ in $[a, b]$ and $\varphi_x(\lambda)$ increases through $p_+(x)$ i.e., $\lambda \in [a, b],\ \lambda < p_+(x) \Rightarrow \varphi_x(\lambda) < 0$ and $\lambda \in [a, b]$, $\lambda > p_+(x) \Rightarrow \varphi_x(\lambda) > 0$.

If $x \in H_0$, then the equation $\varphi_x(\lambda) = 0$ has at least one zero in $[a, b]$ and $\varphi_x(\lambda) \geq 0$ for all $x \in H_0$ and $\lambda \in [a, b]$.

Finally, $\varphi_x(\lambda) > 0$ for $x \in H_\emptyset$ and $\lambda \in [a, b]$.

It is clear that in condition **I)** we may assume $b = \sup_{H_+} p_+(x)$ because the interval $[a, \sup_{H_+} p_+(x)]$ has the same properties. It follows from these conditions that the interval $[a, b]$ may include the first kind and neutral zeros of the functions $\varphi_x(\lambda)$. Since an eigenvalue λ is a root of the equation $(L(\lambda)x, x) = 0$, where x is an eigenvector we classify eigenvalues as the roots of the equation $\varphi_x(\lambda) = 0$. The set of eigenvalues of the first kind, second kind and neutral eigenvalues are denoted by $\sigma_e^+(L)$, $\sigma_e^-(L)$ and $\sigma_e^N(L)$, respectively.

Throughout this paper we assume that together with conditions **I)** the condition

II) $b = \sup_{H_+} p_+(x)$ and $[a, b] \cap \sigma_e^N(L) = \emptyset$

is satisfied.

Finally note that in this paper we use a method quite different from the methods used in [4], [5], [6] and [7]. Particularly, we do not extend the functional $p_+(x)$ on whole space as in the above mentioned papers. Our technique is based on properties of the triple $\langle L, p_+(x), H_+\rangle$ which will be studied in detail in the following sections.

2. On the structure of the triple $\langle L, p_+(x), H_+\rangle$

In this section we give some properties of the triple $\langle L, p_+(x), H_+\rangle$, which will be used in next section. The functional $p_+(x)$ and the cone H_+ are defined by condition **I**. An immediate corollary to condition **I** is the following lemma.

Lemma 2.1 *If $\alpha \in [a, b]$ and $x \in H_+$ then $(L(\alpha)x, x) > 0\,(resp. < 0, = 0)$ if and only if $p_+(x) < \alpha\,(resp. > \alpha, = \alpha)$.*

It is clear that the cone H_+ contains all eigenvectors, corresponding to eigenvalues of the first kind in $[a, b]$. Although H_+ is not a subspace, it contains all subspaces spanned by eigenvectors of the first kind in $[a, b]$. We now prove this fact.

Lemma 2.2 *Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues in $[a, b]$ of the first kind and $x_1, x_2, ..., x_n$ be corresponding eigenvectors. Then*

a) $E := span\{x_1, x_2, ..., x_n\} \subset H_+$;

b) $p_+(x) \in [\min_i \lambda_i, \max_i \lambda_i]$ for $\forall x \in E$.

Proof. It can be assumed that $\lambda_1 > \lambda_2 > ... > \lambda_n$. We consider only the case $n = 2$. The general case can be proved by induction. Set $x = c_1x_1 + c_2x_2$ (c_1, $c_2 \neq 0$). Since $(L(\lambda_1)x_1, x_1) = 0$ and $x_1 \in H_+$ we have $(L(\lambda_2)x_1, x_1) < 0$. Consequently, $(L(\lambda_2)(c_1x_1 + c_2x_2), c_1x_1 + c_2x_2) = |c_1|^2(L(\lambda_2)x_1, x_1) < 0$. Thus, $(L(\lambda_2)x, x) < 0$. On the other hand, because of $(L(\lambda_2)x_2, x_2) = 0$ and $x_2 \in H_+$, we have $(L(\lambda_1)x_2, x_2) > 0$. Hence $(L(\lambda_1)x, x) = |c_2|^2(L(\lambda_1)x_2, x_2) > 0$. Now from the inequalities $(L(\lambda_2)x, x) < 0$ and $(L(\lambda_1)x, x) > 0$, it follows that $x \in H_+$ and, by Lemma 2.1, $p_+(x) \in (\lambda_2, \lambda_1)$. $\square$

Now we consider an extremal problem for $p_+(x)$ on subspaces in H. Let $\lambda_1, \lambda_2, ..., \lambda_n, ...$ be eigenvalues of the first kind of the pencil L in $[a, b]$. Arrange these in decreasing order

$$\lambda_1 \geq \lambda_2 \geq ....\lambda_{\underline{n}-1} > \lambda_{\underline{n}} = ... = \lambda_n = ...\lambda_{\overline{n}} > \lambda_{\overline{n}+1} \geq ...,$$

where $\underline{n} = min\{i|\,\theta_i = \theta_n\}$ and $\overline{n} = max\{i|\,\theta_i = \theta_n\}$.

We denote by $E^i(resp., E_i)$ the set of all subspaces of codimension i (resp., dimension i).

Lemma 2.3 *If $E \in E^i$, $1 \leq i \leq \overline{n} - 1$ then*

$p_+(E \cap H_+) := \sup_{E\cap H_+} p_+(x) \geq \lambda_n$.

Proof. Let $X_{\overline{n}}$ be the linear span of eigenvectors, corresponding to $\lambda_1, \lambda_2, ..., \lambda_{\overline{n}}$. Since $dim X_{\overline{n}} = \overline{n}$ and $i < \overline{n} - 1$, it follows that $X_{\overline{n}} \cap E \neq \{0\}$. By Lemma 2.2 $X_{\overline{n}} \subset H_+$. Consequently, $X_{\overline{n}} \cap E \cap H_+ \neq \{0\}$. Since $p_+(x) \in [\lambda_n, \lambda_1]$ we have

$$p_+(x) \geq \lambda_n, \tag{2.1}$$

for all $x \in X_{\overline{n}}$. Let y be a point from $X_{\overline{n}} \cap E \cap H_+$. It follows from (2.1) that $p_+(y) \geq \lambda_n$. It means that $p_+(E \cap H_+) = \sup_{E \cap H_+} p_+(x) \geq p_+(y) \geq \lambda_n$. □

In what follows we shall denote by $\sigma(L)$, $\sigma_R(L)$, and $\sigma_{ess}(L)$ the spectrum, the real spectrum and the essential spectrum of the pencil L, respectively. By definition

$\sigma_{ess}(L) = \{\lambda \big| \exists \{x_n\},\ \|x_n\| = 1,\ x_n \to 0\,(weakly),\ L(\lambda)x_n \to 0\}$.

We use also the notation $\sigma^R_{ess}(L) := \sigma_{ess}(L) \cap R$. It is known that the spectrum of L is discrete in $\sigma_R(L) \setminus \sigma^R_{ess}(L)$ (see [1] and [9]). In the following Lemma we give a property of the boundary points of the set W_{p_+}.

Lemma 2.4 $b = \sup_{H_+} p_+(x) \in \sigma_R(L)$.

Proof. Since $b = \sup_{H_+} p_+(x)$ we have $\exists \{x_n\} \in H_+$, $\|x_n\| = 1$, $p_+(x_n) \to b$. By condition **I,** $L(b) \geq 0$. Consequently,

$$\|L(b)x_n\|^2 \leq \|L(b)\| \big|(L(b)x_n, x_n)\big| . \tag{2.2}$$

On the other hand $\big|(L(b)x_n, x_n)\big| = \big|((L(b)x_n - L(p_+(x_n))x_n, x_n)\big| \leq \|L(b) - L(p_+(x_n))\| \to 0$ as $n \to \infty$.

By using (2.2) this gives us $L(b)x_n \to 0$. Thus $\exists\, \{x_n\}$, $\|x_n\| = 1$ and $L(b)x_n \to 0$. It means that $b \in \sigma_R(L)$. □

An important Corollary of Lemma 2.4 is

Corollary 2.5 *If* $b \notin \sigma^R_{ess}(L)$ *then* $b \in \sigma^+_e(L)$.

3. Variational principles for eigenvalues of the first kind

In this section we establish variational principles for eigenvalues in $\sigma^+_e(L)$. The class of operator pencils considered in this paper does not generate a Rayleigh system in the entire space but the triple $\langle L, p_+(x), H_+\rangle$ has some properties similar to properties of

Rayleigh systems. We use these properties to establish variational principles. Here the set H_+ is a nonconvex homogeneous set, which will be used instead of the Hilbert space H.

Let $\lambda_1, \lambda_2, ..., \lambda_n, ...$ be eigenvalues of the first kind and $x_1, x_2, ..., x_n, ...$ be eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n, ...$, respectively. With the elements $x_1, x_2, ..., x_n, ...$ we associate the following vector-valued functions:

$$x_i(\alpha) = \frac{L(\alpha) - L(\lambda_i)}{\alpha - \lambda_i} x_i, \quad \alpha \in [a, b], \quad i = 1, 2, ..., n, ...$$

We use the notation $X_n(\alpha) = span\{x_i(\alpha)\}_{i=1}^n$, $Y^n(\alpha) = X_n^\perp(\alpha)$ and $Y^n = X_n^\perp$, where $X_n = span\{x_i\}_{i=1}^n$. We denote $Y_+^n(\alpha) := Y^n(\alpha) \cap H_+$. Define the function

$$\gamma_n(\alpha) = \sup_{Y_+^n(\alpha)} p_+(x).$$

The following theorem was proved in [1] for smooth Rayleigh systems. The proof in our case is not different from that given in [1]. For this reason we give it without proof.

Theorem 3.1 *If $\alpha \in [a, b]$, $\alpha \leq \lambda_n$ and $\{\lambda_i\}_{i=1}^\infty \subset \sigma_e^+(L)$ then*

1. $dim X_n(\alpha) = n$, $H = X_n + Y^n(\alpha)$,

2. $L(\alpha) : Y^n(\alpha) \to Y^n$,

3. The function $\gamma_n(\alpha)$ is continuous.

Now we establish a connection between the spectrum of the operator pencil L and the spectrum of the restriction $L|_E$ to a subspace E. Here $L|_E = Q_E L$, where Q_E is the orthogonal projection of H onto E.

Theorem 3.2 *If $\alpha \in [a, b]$, $\alpha \leq \lambda_n$ and $codim E < +\infty$, then*

1) $\sigma_{ess}(L|_E) \subset \sigma_{ess}(L)$;

2) If $\alpha \in \sigma_e(L|_{Y^n(\alpha)})$ then $\alpha \in \sigma_e(L)$.

Proof. 1) Let $\alpha \in \sigma_{ess}(L|_E)$. Then there exists $\{x_n\}_{n=1}^\infty \subset E$, $\|x_n\| = 1$, $x_n \to 0\,(weakly)$ such that $Q_E L(\alpha) x_n \to 0$. We have

$$L(\alpha) x_n = Q_E L(\alpha) x_n + Q_{E^\perp} L(\alpha) x_n. \tag{3.1}$$

Since $dim E^\perp < +\infty$ the operator $Q_{E^\perp}$ is compact. Consequently, it follows from $x_n \to 0$ (weakly) that $Q_{E^\perp} L(\alpha) x_n \to 0$. Then by taking the limit in (3.1) we obtain $L(\alpha) x_n \to 0$, so that $\alpha \in \sigma_{ess}(L)$.

2) To prove this connection we take a point $\alpha \in \sigma_e(L|_{Y^n(\alpha)})$ and show that $\alpha \in \sigma_e(L)$. Since $\alpha \in \sigma_e(L|_{Y^n(\alpha)})$ there exists $0 \neq y \in Y^n(\alpha)$ satisfying $Q_{Y^n(\alpha)}L(\alpha)y = 0$. By Theorem 3.1 $H = X_n + Y^n(\alpha)$. Therefore for all $z \in H$ we can write $z = u + v$, where $u \in X_n$ and $v \in Y^n(\alpha)$. But then

$(L(\alpha)y, v) = (L(\alpha)y, Q_{Y^n(\alpha)}v) = (Q_{Y^n(\alpha)}L(\alpha)y, v) = 0.$

Using the condition $\alpha \leq \lambda_n$ by Theorem 3.1 we obtain $L(\alpha) : Y^n(\alpha) \to Y^n$. Now it follows from this that $(L(\alpha)y, u) = 0$. Thus $(L(\alpha)y, z) = 0$ for all $z \in H$. This implies $L(\alpha)y = 0$, $y \in Y^n(\alpha) \setminus \{0\}$, i.e., $\alpha \in \sigma_e(L)$. □

Now we prove a fixed point theorem for the functions, $\gamma_n(\alpha)$, $n = 0, 1, 2, ...$ which is the main step for establishing variational principles.

Theorem 3.3 *Let $(c, b] \cap \sigma_{ess}(L) = \emptyset$ for $c \geq a$ and $\lambda_1, \lambda_2, ..., \lambda_n, ...$ be the eigenvalues of L, arranged in decreasing order counting multiplicity:*

$$\lambda_1 \geq \lambda_2 \geq ....\lambda_{\underline{n}-1} > \lambda_{\underline{n}} = ... = \lambda_n = ...\lambda_{\overline{n}} > \lambda_{\overline{n}+1} \geq ......$$

Then $\gamma_n(\lambda_{n+1}) = \lambda_{n+1}$, $n = 0, 1, 2, ...$

Proof. By conditions **I** and **II** all eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ are eigenvalues of the first kind. Consequently, Theorem 3.1 and Theorem 3.2 are satisfied. Let $n = 0$. Setting $Y^0(\alpha) = H$ we have

$$\gamma_0(\lambda_1) = \max_{H_+} p_+(x) = b,$$

and by Corollary 2.5, $b = \lambda_1$. Thus $\gamma_0(\lambda_1) = \lambda_1$. Now we suppose

$$\gamma_k(\lambda_{k+1}) = \lambda_{k+1}, \quad k = 0, 1, 2, ..., n-1. \tag{3.2}$$

Let us prove (3.2) for $k = n$. For $k = n - 1$ it follows from (3.2) that

$$\lambda_n = \gamma_{n-1}(\lambda_n) = \max_{Y_+^{n-1}(\lambda_n)} p_+(x). \tag{3.3}$$

Two cases are possible.

1) $\lambda_{n+1} = \lambda_n$, i.e., $n < \overline{n}$. Using $Y^n(\lambda_n) \cap X_{\overline{n}} \neq \{0\}$ and $X_{\overline{n}} \subset H_+$ (Lemma 2.2) it is not difficult to prove that there exists a vector $x \in Y_+^n(\lambda_n)$ such that $L(\lambda_n)x = 0$ (see [1], [6]). From (3.2), taking into account the inclusion $Y_+^n(\lambda_n) \subset Y_+^{n-1}(\lambda_n)$ we can write

$$\lambda_{n+1} = \lambda_n = p_+(x) \leq \gamma_n(\lambda_n) \leq \gamma_{n-1}(\lambda_n) = \lambda_n.$$

Consequently, $\gamma_n(\lambda_n) = \gamma_n(\lambda_{n+1}) = \lambda_{n+1}$.

2) Let $n = \overline{n}$, i.e., $\lambda_{n+1} < \lambda_n$. By Theorem 3.1 $codim\, Y^n(\alpha) = n$ for $\alpha \leq \lambda_n$. Then it follows from Lemma 2.3 that $\gamma_n(\alpha) \geq \lambda_{n+1}$ for $\alpha \leq \lambda_n$. In particular,

$$\gamma_n(\lambda_{n+1}) \geq \lambda_{n+1}. \tag{3.4}$$

At the same time since $Y^n(\lambda_n) \subset Y^{n-1}(\lambda_n)$ then by (3.3)

$$\gamma_n(\lambda_n) \leq \gamma_{n-1}(\lambda_n) = \lambda_n. \tag{3.5}$$

By Theorem 3.1 the function $\gamma_n(\alpha)$ is continuous on $[a, \lambda_n]$. Let us define the function $F(\alpha) = \alpha - \gamma_n(\alpha)$. Using (3.4) and (3.5) we obtain $F(\lambda_n) \geq 0$ and $F(\lambda_{n+1}) \leq 0$. The function $F(\alpha)$ is continuous on $[\lambda_{n+1}, \lambda_n]$. Consequently, there exists a point $\theta \in [\lambda_{n+1}, \lambda_n]$ such that $\gamma_n(\theta) = \theta$. It remains to show $\theta = \lambda_{n+1}$. By Theorem 3.2 we have $\theta \notin \sigma_{ess}(L|_{Y^n(\theta)})$, and since the restricted operator function $L|_E$ satisfies condition **I,** by Corollary 2.5 we have $\theta \in \sigma_e(L|_{Y^n(\theta)})$, i.e., $Q_{Y^n(\theta)}\, L(\theta)y = 0$ for same $y \in Y^n(\theta) \setminus \{0\}$. Then by the assertion 2) of Theorem 3.2, $\theta \in \sigma_e(L) = \sigma_e^+(L)$ in $[a, b]$. This implies $y \in Y_+^n(\theta)$ and $L(\theta)y = 0$. Since $\theta \in [\theta_{n+1}, \theta_n]$ two cases are possible. Let $\theta = \theta_n$. Then $y \in X_n$. But $X_n \cap Y_+^n(\theta_n) = \emptyset$. We now have a contradiction: $y = 0$. Therefore $\theta = \theta_{n+1}$ and $\gamma_n(\theta_{n+1}) = \theta_{n+1}$. □

Finally, we give a variational principle for eigenvalues of the first kind.

Theorem 3.4 *Under conditions of Theorem 3.3*

$$\lambda_{n+1} = \min_{E \in E^n} \sup_{E \cap H_+} p_+(x). \tag{3.6}$$

Proof. By Lemma 2.3 we have

$$\sup_{E \cap H_+} p_+(x) \geq \lambda_{n+1} \quad \text{for} \quad E \in E^n. \tag{3.7}$$

On the other hand, it follows from Theorem 3.3 that there is a subspace E with $codim E = n$ satisfying

$$\sup_{E \cap H_+} p_+(x) = \lambda_{n+1}. \tag{3.8}$$

Indeed, setting $E = Y^n(\theta_{n+1})$ we have $\sup_{E \cap H_+} p(x) = \lambda_{n+1}$.

Now (3.6) follows from (3.7) and (3.8). □

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