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Uniqueness of Coprimary Decompositions

M. Maani-Shirazi and P. F. Smith

Abstract

Uniqueness properties of coprimary decompositions of modules over non-commutative rings are presented.

Key Words: Coprimary, decomposition, normal decomposition, prime ideal, left Noetherian ring, right Noetherian ring.

1. Introduction

Throughout this paper, R is a ring (not necessarily commutative) with an identity element $1 \neq 0$ and M is a non-zero unital left R-module. For any submodules N, L of M, we define $(N : L) = \{r \in R : rL \subseteq N\}$. Note that $(N : L)$ is an ideal of R. Moreover, $(N : L) = R$ if and only if $L \subseteq N$. Let N be a submodule of M and let A be an ideal of R; we set $(N :_M A) = \{m \in M : Am \subseteq N\}$. Note that $(N :_M A)$ is a submodule of M.

In this paper, by making use of the technique employed in [7], we shall prove uniqueness properties of coprimary decompositions.

Note that, when R is a commutative Noetherian ring, M is coprimary if and only if M is secondary. It is well known that every non-zero injective module over a commutative Noetherian ring has a secondary representation (see [6]). By a similar method to that used in [6], we obtain the following result. For R non-commutative left and right Noetherian we show that if M is injective and if the zero ideal of R is a finite intersection of strongly primary ideals, then M has a coprimary decomposition.

2. Coprimary Decompositions

Definition. *Given a prime ideal P of R, a non-zero module M is called P-coprimary if*

(i) $(N : M) \subseteq P$ for every proper submodule N of M, and

(ii) $P^h \subseteq (0 : M)$ for some positive integer h.

Note that if M is P-coprimary, then $P^h \subseteq (0 : M) \subseteq P$ for some positive integer h. M is called coprimary if it is P-coprimary for some prime ideal P of R.

A non-zero module M has a coprimary decomposition if there exist a positive integer n and submodules $M_i(1 \leq i \leq n)$ of M such that

(i) $M = M_1 + \cdots + M_n$, and

(ii) M_i is coprimary for each $1 \leq i \leq n$.

If M has a coprimary decomposition, then we say that M has a normal coprimary decomposition if there exist a positive integer n, distinct prime ideals $P_i(1 \leq i \leq n)$ of R, and P_i-coprimary submodules $M_i(1 \leq i \leq n)$ of M such that

(i) $M = M_1 + \cdots + M_n$, and

(ii) $M \neq M_1 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n$ for all $1 \leq i \leq n$.

Lemma 2.1. *Let P be a prime ideal of R and let M be a P-coprimary module. Then M/K is a P-coprimary R-module for each proper submodule K of M.*

Proof. This is clear. □

Corollary 2.2. *If M has a coprimary decomposition, then M/K has a coprimary decomposition for every proper submodule K of M.*

Proof. There exist a positive integer n and coprimary submodules $M_i(1 \leq i \leq n)$ of M such that $M = M_1 + \cdots + M_n$. Then $M/K = ((M_1 + K)/K) + \cdots + ((M_n + K)/K)$. Then, for each $1 \leq i \leq n$, $(M_i + K)/K \cong M_i/(M_i \cap K)$ so that $(M_i + K)/K = 0$ or $(M_i + K)/K$ is coprimary by Lemma 2.1. □

Lemma 2.3. *Let P be a prime ideal of R, let n be a positive integer, and let $M_i(1 \leq i \leq n)$ be non-zero left R-modules. Then the R-module $M_1 \oplus \cdots \oplus M_n$ is P-coprimary if and only if M_i is P-coprimary for each $1 \leq i \leq n$.*

Proof.(⇒) This follows from Lemma 2.1.

(⇐) Let N be a proper submodule of the module $M = M_1 \oplus \cdots \oplus M_n$. There exists $1 \leq i \leq n$ such that $M_i \not\subseteq N$. Then $(M_i + N)/N \cong M_i/(M_i \cap N)$ and $M_i \cap N$ is a proper submodule of M_i so that $(N : M) \subseteq (N : M_i + N) \subseteq P$. There exists a positive integer h such that $P^h \subseteq (0 : M_i)$ for each $1 \leq i \leq n$. Then $P^h \subseteq \bigcap_{i=1}^{n} (0 : M_i) = (0 : M)$. Thus M is P-coprimary. □

Corollary 2.4. *Let P be a prime ideal of R, let n be a positive integer, and let $M_i (1 \leq i \leq n)$ be P-coprimary submodules of M. Then the submodule $M_1 + \cdots + M_n$ of M is a P-coprimary R-module.*

Proof. This follows from Lemmas 2.1 and 2.3. □

Corollary 2.5. *If M has a coprimary decomposition, then M has a normal coprimary decomposition.*

Proof. This follows from Corollary 2.4. □

One can easily prove the following result.

Lemma 2.6. *Let P be a prime ideal of R and let M be a semisimple module. Then the following statements are equivalent.*

(i) M is P-coprimary.

(ii) Every non-zero submodule of M is P-coprimary.

(iii) Every simple submodule of M is P-coprimary.

Corollary 2.7. *Let M be a semisimple module. Then M has a coprimary decomposition if and only if the set $\{(0 : N) : N$ is a simple submodule of $M\}$ is finite.*

Proof. This follows from Lemma 2.6. □

Lemma 2.8. *Let P be a prime ideal of R. Then M is P-coprimary if and only if, for every ideal A of $R, M = AM$ if $A \nsubseteq P$ and there exists a positive integer h such that $A^h M = 0$ if $A \subseteq P$.*

Proof. This is straightforward. □

Lemma 2.9. *If M has a coprimary decomposition, then for each ideal A of R there exists a positive integer h such that $M = AM + (0 :_M A^h)$.*

Proof. There exist a positive integer n, prime ideals $P_i (1 \leq i \leq n)$ of R, and P_i-

coprimary submodules $M_i(1 \leq i \leq n)$ of M such that $M = M_1 + \cdots + M_n$. Let A be an ideal of R. For each $1 \leq i \leq n$, Lemma 2.8 gives $M_i = AM_i$ or $M_i \subseteq (0 :_M A^{h_i})$ for some positive integer h_i. Let $h = \max_{1 \leq i \leq n} h_i$. Then $M_i \subseteq AM + (0 :_M A^h)$ for all $1 \leq i \leq n$. It follows that $M = AM + (0 :_M A^h)$. □

We shall be interested in the following property of a ring R.

(P) For every proper ideal A of R there exists a positive integer n such that $B^n \subseteq A$ for every ideal B of R with $B^h \subseteq A$ for some positive integer h.

Note that any ring which has the ascending chain condition on two-sided ideals or any ring in which prime ideals are finitely generated left ideals satisfies the property (P) (see [3, Lemma 3.1]).

Lemma 2.10. *R satisfies (P) if and only if for every proper ideal A of R, the sum of all nilpotent ideals of the ring R/A is also a nilpotent ideal of R/A.*

Proof. ($\Leftarrow$) This is clear.

($\Rightarrow$) Let C be the ideal of R containing A such that C/A is the sum of all nilpotent ideals of the ring R/A. Let n be the positive integer in the property (P). Let $c_i \in C(1 \leq i \leq n)$. There exist a positive integer h and ideals $B_j(1 \leq j \leq h)$ of R such that $B_j^n \subseteq A \subseteq B_j(1 \leq j \leq h)$ and $c_i \in B_1 + \cdots + B_h(1 \leq i \leq n)$. Note that $(B_1 + \cdots + B_h)^{hn} \subseteq A$ and hence $(B_1 + \cdots + B_h)^n \subseteq A$. This implies that $c_1 \cdots c_n \in A$. Thus $C^n \subseteq A$. □

Lemma 2.11. *Let R satisfy the property (P). Then M is coprimary if and only if for every ideal A of R either $M = AM$ or $A^h M = 0$ for some positive integer h.*

Proof. ($\Rightarrow$) This follows from Lemma 2.8.

($\Leftarrow$) Let P be the ideal of R containing $A = (0 : M)$ such that P/A is the sum of all nilpotent ideals of the ring R/A. By Lemma 2.10, there exists a positive integer n such that $P^n \subseteq A$. Let B, C be ideals of R such that $BC \subseteq P$. If $M = BM$ and $M = CM$, then $M = BM = BCM \subseteq PM$ so that $M = PM = P^2M = \cdots = P^nM = 0$, a contradiction. Thus $M \neq BM$ or $M \neq CM$. By the hypothesis, $B \subseteq P$ or $C \subseteq P$. It follows that P is a prime ideal of R and hence M is P-coprimary by Lemma 2.8.

Next we give an example to show that in Lemma 2.11 the condition on R is necessary.

Example 2.12. *Let p be any prime number, let F be a field of characteristic p, let G be the Prüfer p-group, and let R be the group algebra $F[G]$. (See [2, p.37] for the definition of Prüfer groups). Then R is a commutative ring with unique maximal ideal $J = \sum_{g \in G} R(g-1)$ and J is a nil ideal of R such that $J = J^2$. If A is any ideal of R then A is nilpotent unless $A = J$ or $A = R$. Now let M denote the R-module J. Then, for any ideal A of R, $M = AM$ or $A^k M = 0$ for some positive integer k. However, M is not coprimary because J is the only prime ideal of R and $M = JM$.*

Theorem 2.13. *Let M have a coprimary decomposition. Let $M = K_1 + \cdots + K_s$ and $M = L_1 + \cdots + L_t$ be normal coprimary decompositions of M where K_i is P_i-coprimary for some prime ideal $P_i (1 \leq i \leq s)$ and L_j is Q_j-coprimary for some prime ideal $Q_j (1 \leq j \leq t)$. Then $s = t$ and $\{P_1, \ldots, P_s\} = \{Q_1, \ldots, Q_t\}$.*

Proof. Without loss of generality, we can suppose that P_1 is maximal in the set $\{P_1, ..., P_s\} \cup \{Q_1, ..., Q_t\}$. There exists a positive integer n such that $P_1^n K_1 = 0$. Thus

$$P_1^n M = P_1^n K_1 + \cdots + P_1^n K_s \subseteq K_2 + \cdots + K_s,$$

also

$$P_1^n M = P_1^n L_1 + \cdots + P_1^n L_t.$$

Because $M \neq P_1^n M$, there exists a positive integer j such that $1 \leq j \leq t$ and $L_j \neq P_1^n L_j$ and hence $P_1^n \subseteq Q_j$ by Lemma 2.8. This implies that $P_1 \subseteq Q_j$. Without loss of generality, we can suppose that $j = 1$ and hence $P_1 = Q_1$. We can suppose that $P_1^n K_1 = Q_1^n L_1 = 0$. Then Lemma 2.8 gives

$$P_1^n M = P_1^n K_1 + \cdots + P_1^n K_s = K_2 + \cdots + K_s,$$

and

$$P_1^n M = P_1^n L_1 + \cdots + P_1^n L_t = L_2 + \cdots + L_t.$$

By induction, $s = t$ and $\{P_i : 2 \leq i \leq s\} = \{Q_j : 2 \leq j \leq s\}$. The result follows. □

In view of Theorem 2.13, we call prime ideals $P_i (1 \leq i \leq s)$ of R the coassociated prime ideals of M provided there exists a normal coprimary decomposition $M = K_1 + \cdots + K_s$, where K_i is a P_i-coprimary submodule of M for each $1 \le i \le s$.

Theorem 2.14. *Let M have a coprimary decomposition and let $P_i (1 \le i \le n)$ be the coassociated prime ideals of M, for some positive integer n. Suppose that there exists $1 \le k \le n$ such that for all $1 \le i \le k$ and all $k+1 \le j \le n$, $P_j \not\subseteq P_i$. Let $M = M_1 + \cdots + M_n$ and $M = L_1 + \cdots + L_n$ be normal coprimary decompositions of M in terms of P_i-coprimary submodules M_i and $L_i (1 \le i \le n)$. Then $M_1 + \cdots + M_k = L_1 + \cdots + L_k$.*

Proof. There exists a positive integer s such that $P_j^s M_j = P_j^s L_j = 0 (k+1 \le j \le n)$. Let $A = P_{k+1}^s \cdots P_n^s$. Then for all $1 \le i \le k, A \not\subseteq P_i$ so that $M_i = AM_i$ and $L_i = AL_i$. Now we have

$$AM = AM_1 + \cdots + AM_k + AM_{k+1} + \cdots + AM_n = M_1 + \cdots + M_k,$$

and

$$AM = AL_1 + \cdots + AL_k + AL_{k+1} + \cdots + AL_n = L_1 + \cdots + L_k.$$

Thus $M_1 + \cdots + M_k = L_1 + \cdots + L_k$. □

Let P be a prime ideal of R. M^P is defined to be the intersection $\cap AM$, where A runs over the ideals of R not contained in P.

Remark 2.15. *Let $M = M_1 + \cdots + M_n$ and $M = L_1 + \cdots + L_n$ be normal coprimary decompositions of M where n is a positive integer and, for each $1 \le i \le n, M_i$ and L_i are P_i-coprimary submodules of M for some prime ideal P_i of R. If P_j is minimal in the set $\{P_1, ..., P_n\}$, then $M_j = L_j$ by Theorem 2.14. Moreover, we have also $M_j = L_j = M^{P_j}$ (see [5]).*

Next, we give a characterization of the coassociated prime ideals of M with coprimary decomposition.

Theorem 2.16. *Let P be a prime ideal of R and let M have a coprimary decomposition. Then P is a coassociated prime ideal of M if and only if $P = (K : M)$ for some proper submodule K of M.*

Proof. Let $M = M_1 + \cdots + M_n$ be a normal coprimary decomposition of M where n is a positive integer and, for each $1 \le i \le n, M_i$ is a P_i-coprimary submodule of M for some prime ideal P_i of R. Let P be a coassociated prime ideal of M. Without loss of generality, we can suppose that $P = P_1$. There exists a positive integer k such that $P^k M_1 = 0$. Thus

$M = M_1 + M_2 + \cdots + M_n$ but $M \neq P^k M_1 + M_2 + \cdots + M_n$. There exists $1 \leq j \leq k$ such that $M = P^{j-1} M_1 + M_2 + \cdots + M_n$ but $M \neq P^j M_1 + M_2 + \cdots + M_n$. Let K denote the proper submodule $P^j M_1 + M_2 + \cdots + M_n$.

Let $A = (K : M)$. Clearly $PM \subseteq K$ gives $P \subseteq A$. If $P \neq A$, then $M_1 = AM_1$ and hence $M_1 \subseteq AM \subseteq K$ so that $K = M$, a contradiction. Thus $P = A$.

Conversely, let Q be a prime ideal of R such that $Q = (N : M)$ for some proper submodule N of M. There exists $1 \leq i \leq n$ such that $M_i \nsubseteq N$. Without loss of generality, we can suppose that there exists $1 \leq t \leq n$ such that $M_i \nsubseteq N$ for all $1 \leq i \leq t$ but $M_i \subseteq N$ for all $t+1 \leq i \leq n$. For each $1 \leq i \leq t$, $M_i \cap N$ is a proper submodule of M_i and $QM_i \subseteq M_i \cap N$ so that $Q \subseteq P_i$. There exists a positive integer s such that $P_i^s M_i = 0 (1 \leq i \leq t)$. Now $M = M_1 + \cdots + M_n = M_1 + \cdots + M_t + N$ and hence $(P_1^s \cdots P_t^s) M \subseteq N$ so that $P_1^s \cdots P_t^s \subseteq Q$. It follows that there exists $1 \leq j \leq t$ such that $P_j \subseteq Q$ and hence $P_j = Q$. Therefore Q is a coassociated prime ideal of R. □

Lemma 2.17. *If M has a coprimary decomposition, then every minimal prime ideal over $A = (0 : M)$ is a coassociated prime ideal of M.*

Proof. Let $M = M_1 + \cdots + M_n$ be a normal coprimary decomposition of M where n is a positive integer and, for each $1 \leq i \leq n, M_i$ is a P_i-coprimary submodule of M for some prime ideal P_i of R. Then $A = \bigcap_{i=1}^{n} (0 : M_i)$. Suppose Q is a minimal prime ideal of A. There exists $1 \leq i \leq n$ such that $A \subseteq (0 : M_i) \subseteq Q$. So $Q = P_i$. □

Lemma 2.18. *Let R be a prime left or right Noetherian ring and let $M = PM$ for all non-zero prime ideals P of left R-module. Then M is 0-coprimary.*

Proof. Let A be a non-zero ideal of R. There exist a positive integer n, prime ideals $P_i (1 \leq i \leq n)$ of R such that $P_1 \cdots P_n \subseteq A \subseteq P_1 \cap \cdots \cap P_n$. But $M = P_i M$ for all $1 \leq i \leq n$. So $M = P_n M = \cdots = P_1 \cdots P_n M \subseteq AM$ and hence $M = AM$. Lemma 2.8 yields that M is 0-coprimary. □

Remark 2.19. *Let R be left or right Noetherian. Then there exists a prime ideal P of R such that $PM \neq M$. For, suppose that $QM = M$ for all prime ideals Q of R. There exist a positive integer n and prime ideals $P_i (1 \leq i \leq n)$ of R such that $0 = P_1 \cdots P_n$.*

Then we have $M = P_nM = \cdots = P_1 \cdots P_nM = 0$, a contradiction.

Corollary 2.20. *Let R be left or right Noetherian. Then M has a coprimary quotient R-module.*

Proof. Since R is left or right Noetherian and by Remark 2.19, there exists a prime ideal P of R such that $PM \neq M$ but $QM = M$ for all prime ideals Q of R properly containing P. Then M/PM is a 0-coprimary (R/P)-module by Lemma 2.18 which implies that M/PM is a P-coprimary R-module. □

For any non-empty set $I, M^{(I)}$ is the direct sum $\bigoplus_{i \in I} M_i$, where $M_i = M(i \in I)$.

The prime radical, $\sqrt{A}$, of an ideal A of R is defined to be the intersection of all prime ideals which contain A.

Lemma 2.21. *Let P be a prime ideal of R and let M be P-coprimary. Then $M^{(I)}$ is a P-coprimary R-module for every non-empty set I.*

Proof. We have $\sqrt{(0 : M^{(I)})} = \sqrt{(0 : M)} = P$. There exists a positive integer n such that $P^nM = 0$. Let A be an ideal of R. If $A \subseteq P$ then $A^nM^{(I)} = (A^nM)^{(I)} \subseteq (P^nM)^{(I)} = 0$. Now suppose that $A \nsubseteq P$. Then $AM = M$ and so $AM^{(I)} = (AM)^{(I)} = M^{(I)}$. By Lemma 2.8, $M^{(I)}$ is P-coprimary. □

Recall that any left R-module is M-generated if it is a quotient module of $M^{(I)}$ for some non-empty set I.

Corollary 2.22. *If M has a coprimary decomposition, then any non-zero M-generated R-module has a coprimary decomposition.*

Proof. Let $M = M_1 + \cdots + M_n$ be a coprimary decomposition of M where n is a positive integer and, for each $1 \leq i \leq n, M_i$ is a P_i-coprimary submodule of M for some prime ideal P_i of R. Let I be a non-empty set. Then we have $M^{(I)} = M_1^{(I)} + \cdots + M_n^{(I)}$. Lemma 2.21 yields that $M_i^{(I)}$ is P_i-coprimary for each $1 \leq i \leq n$. Hence $M^{(I)}$ has a coprimary decomposition. Corollary 2.2 completes the proof. □

Remark 2.23. *Let P be a maximal ideal of R. Then M is P-coprimary if and only if $P^nM = 0$ for some positive integer n. In this case, every non-zero submodule of M is* *also P-coprimary.*

Theorem 2.24. *The following statements are equivalent.*

(i) Every non-zero left R-module has a coprimary decomposition.

(ii) The left R-module R has a coprimary decomposition.

(iii) There exist positive integers n, h and maximal ideals $P_i (1 \leq i \leq n)$ of R such that $P_1^h \cap \cdots \cap P_n^h = 0$.

Proof. $(i) \Rightarrow (ii)$ This is clear.

$(ii) \Rightarrow (i)$ This follows from Corollary 2.22.

$(ii) \Rightarrow (iii)$ Let $R = A_1 + \cdots + A_n$ be a coprimary decomposition of the left R-module R where n is a positive integer and, for each $1 \leq i \leq n$, A_i is a P_i-coprimary submodule of the left R-module R for some prime ideal P_i of R. There exists a positive integer h such that $P_i^h A_i = 0$ for each $1 \leq i \leq n$ and so $P_1^h \cap \cdots \cap P_n^h = 0$. Note that any prime ring which has a coprimary decomposition as a left module over itself has only two ideals. Let P be a prime ideal of R. Then the ring R/P is prime with coprimary decomposition as a module over itself. So R/P has only two ideals, i.e., P is maximal. We have shown that every prime ideal of R is maximal. The result follows.

$(iii) \Rightarrow (ii)$ We have $R \cong R/P_1^h \bigoplus \cdots \bigoplus R/P_n^h$ as left R-modules. It follows that R has a coprimary decomposition as a left R-module by Remark 2.23. □

There are modules in which every non-zero submodule has a coprimary decomposition (see [5]). In certain situations it is possible to write down explicitly a normal coprimary decomposition for a non-zero module once its coassociated prime ideals are known, as we show next.

Theorem 2.25. *Suppose that every non-zero submodule of M has a coprimary decomposition. Let N be a non-zero submodule of M and let $P_i (1 \leq i \leq k)$ be the coassociated prime ideals of N. Then there exists a positive integer h such that $N = (0 :_N P_1^h)^{P_1} + \cdots + (0 :_N P_k^h)^{P_k}$ is a normal coprimary decomposition of N.*

Proof. Let $N = N_1 + \cdots + N_k$ be a normal coprimary decomposition of N where k is a positive integer and, for each $1 \leq i \leq k$, N_i is a P_i-coprimary submodule of N for some prime ideal P_i of $R (1 \leq i \leq k)$. There exists a positive integer h such that $P_i^h N_i = 0 (1 \leq i \leq k)$. Then, for each $1 \leq i \leq k$, we have

$$N_i = N_i^{P_i} \subseteq (0 :_N P_i^h)^{P_i} \subseteq (0 :_N P_i^h) \subseteq N$$

and so

$$P_i^h \subseteq (0 : (0 :_N \ P_i^h)) \subseteq (0 : (0 :_N \ P_i^h)^{P_i}) \subseteq (0 : N_i) \subseteq P_i.$$

For each $1 \leq i \leq k$, $\sqrt{(0 : (0 :_N \ P_i^h)^{P_i})} = \sqrt{(0 : (0 :_N \ P_i^h))} = P_i$ and by the hypothesis $(0 :_N P_i^h)$ has a coprimary decomposition so that P_i is the only minimal member in the set of coassociated prime ideals of $(0 :_N \ P_i^h)$. Hence by Remark 2.15, $(0 :_N \ P_i^h)^{P_i}$ is P_i-coprimary for each $1 \leq i \leq n$ so that $N = (0 :_N P_1^h)^{P_1} + \cdots + (0 :_N P_k^h)^{P_k}$ is a normal coprimary decomposition of N. □

Let A be an ideal of R. Then A is said to be left primary if, given any two ideals B and C of R such that $BC \subseteq A$, then either $C \subseteq A$ or $B^n \subseteq A$ for some positive integer n. In a similar way we can define right primary. A is said to be primary if it is both left and right primary. If R is left and right Noetherian and if A is a proper ideal of R, which is primary, then $P = \sqrt{A}$ is prime such that $P^n \subseteq A$ for some positive integer n. In this case, A is called P-primary. If A is a proper ideal of R, then $C(A)$ will denote the set of elements c in R such that $c + A$ is a non-zero-divisor in R/A. Clearly, $c \in C(A)$ if and only if, for any $r \in R, cr \in A$ or $rc \in A$ implies $r \in A$. The ideal A of R is said to be strongly primary if A is primary and $C(A) = C(\sqrt{A})$.

In [6], it is proved that every non-zero injective module over a commutative Noetherian ring has a secondary representation. By a similar method, we prove Theorem 2.28.

Lemma 2.26. *Let R be left and right Noetherian, let A be a strongly P-primary ideal of R, and let M be an injective R-module. Then $N = (0 :_M A)$ is zero or coprimary.*

Proof. Suppose N is a non-zero submodule of M. Let B be an ideal of R. If $B \subseteq P$, then $B^h N \subseteq P^h N \subseteq AN = 0$ for some positive integer h. Now suppose $B \not\subseteq P$. Clearly $BN \subseteq N$. Let $n \in N$. There is a left R-module homomorphism $\varphi : R/A \to M$ for which $\varphi(r + A) = rn$ for all $r \in R$. Since $B \not\subseteq P, (B + P)/P$ is a non-zero ideal of the prime left and right Noetherian ring R/P. By Goldie's Theorem, there exists an element $b \in B \cap C(P)$. Define a mapping $\theta : R/A \to R/A$ by $\theta(r + A) = rb + A$ for all $r \in R$. Since $b \in B \cap C(P)$ and A is strongly P-primary, θ is a left R-module monomorphism. As the diagram

$$\begin{array}{ccc} 0 \to & R/A \xrightarrow{\theta} R/A \\ & \varphi \downarrow \quad\quad \\ & M \quad\quad \end{array}$$

has exact row, it can be completed with a left R-module homomorphism $\psi : R/A \to M$ which makes the extended diagram commute. Thus $n = \varphi(1 + A) = \psi\theta(1 + A) = \psi(b + A) = b\psi(1 + A)$. Since $\psi(1 + A) \in N, n \in bN \subseteq BN$. We have shown that $N \subseteq BN$. The result follows. □

For the proof of the next result see [6, Lemma 2.2].

Lemma 2.27. *Let M be injective, let n be a positive integer, and let $A_i (1 \leq i \leq n)$ be ideals of R. Then* $\sum_{i=1}^{n}(0 :_M A_i) = (0 :_M \bigcap_{i=1}^{n} A_i)$.

Theorem 2.28. *Let R be left and right Noetherian such that the zero ideal of R is a finite intersection of strongly primary ideals. Then every non-zero injective R-module has a coprimary decomposition.*

Proof. Let n be a positive integer and let $A_i (1 \leq i \leq n)$ be strongly primary ideals of R such that $0 = \bigcap_{i=1}^{n} A_i$. Let M be a non-zero injective R-module. Then $M = (0 :_M 0) = (0 :_M \bigcap_{i=1}^{n} A_i) = \sum_{i=1}^{n}(0 :_M A_i)$, where $(0 :_M A_i)$ is zero or coprimary for all $1 \leq i \leq n$. □

Note that the condition on R in Theorem 2.28 is satisfied if R is the universal enveloping algebra of a finite-dimensional nilpotent Lie algebra (see [1, p.78] and [4]).

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M. MAANI-SHIRAZI
Department of Mathematics,
College of Sciences, Shiraz University,
Shiraz 71454, IRAN
e-mail: liliumsor7@yahoo.com

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P. F. SMITH
Department of Mathematics,
University of Glasgow,
Glasgow, G12 8QW, Scotland, UK
e-mail: pfs@maths.gla.ac.uk